Recurrence and attractors in state transition dynamics

A relational view

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Overview

relation algebra: standard concepts and notation

- **relation algebra:** a *complete, atomic* boolean algebra enriched with
  unary **converse**: $r^{\smile}$
  binary associative **composition**: $p \,;\, (q \,;\, r) = (p \,;\, q) \,;\, r$
  an **identity** constant: $1' ; r = r \,;\, 1' = r$

  that satisfy some further **laws:**

  Schröder equivalences: $p \,;\, q \leq r \;\Leftrightarrow\; p^{\smile} ; r^{-1} \leq q^{-1} \;\Leftrightarrow\; r^{-1} ; q^{\smile} \leq p^{-1}$

  Tarski rule: $r \neq 0 \Rightarrow 1; r \,;\, 1 = 1$
- **provable laws** (among many others):
  $r^{\smile\smile} = r$
  $(q \,;\, r)^{\smile} = r^{\smile} \,;\, q^{\smile}$
  $p \,;\, (q + r) = (p \,;\, q) + (p \,;\, r)$, $\quad (p + q) \,;\, r = (p \,;\, r) + (q \,;\, r)$
- a relation algebra is *representable* if
  it is isomorphic to a boolean algebra of binary relations,
  with set-theoretic interpretation of converse, composition and identity
- while every boolean algebra is representable (Stone),
  not every relation algebra so is (Lyndon)

concepts and notation for state transition dynamics

- **iteration:** $q^0 = 1'$, $q^{i+1} = q \,;\, q^i$
- **Kleene closure** and the like:

$$q^* = \sum_{i \in \mathbb{N}} q^i, \quad q^+ = \sum_{i > 0} q^i, \quad q^{\geq n} = \sum_{i \geq n} q^i$$

  note: $q^+ = q^{\geq 1}$ and $q^* = q^{\geq 0}$.
- **monotypes:** subrelations of the identity, viz. $x \leq 1'$, such as
  domain of q: $\mathrm{dom}\ q = 1' \cdot (q \,;\, 1)$, **image** of q: $\mathrm{img}\ q = 1' \cdot (1 \,;\, q)$
- **atomic monotypes:** characterized by the quasiequations
  $x \leq 1', \quad 1 \,;\, x \,;\, 1 = 1, \quad \text{and} \quad y \leq x \wedge 1 \,;\, y \,;\, 1 = 1 \Rightarrow y = x$
- **notation:** $x \lessdot y$ means that x is an atomic monotype and $x \leq y$
- useful *higher-order binary relations on monotypes*: if q is a binary relation and x, y are monotypes in a relation algebra, **define:**
  - $x \leq_q^{(n)} y \quad \text{if} \quad \mathrm{img}(x \,;\, q^{\geq n}) \leq y$
  - the **eventually below under q-iteration** relation on monotypes:
    $\precsim_q = \sum_{n \in \mathbb{N}} \leq_q^{(n)}$

state transition dynamics: basic concepts

- **state transition (ST) dynamics**: (S, q)
  S : (discrete) **state space**
  $q \subseteq S \times S$: [total] transition **dynamics** [dom $q = 1'_S$]
- straightforward extension of q to **quasistates**, represented by nonzero monotypes $x \leq 1'_S$—the atomic ones represent **individual states**: $x \lessdot 1'_S$
- **fixed point** of q: a state x such that img$(x ; q) = x$
- **orbit** (of *origin* x_0) : $(x_i \mid i \in \mathbb{N})$ such that $x_{i+1} = \text{img}(x_i ; q)$
- **eventually periodic** orbit:
  $\exists k \geq 0 \exists n > 0 : \text{img}(x_0 ; q^{k+n}) = \text{img}(x_0 ; q^k)$ (**periodic** orbit if $k = 0$)
- orbit $(x_i \mid i \in \mathbb{N})$ **(eventually) included** in $(y_i \mid i \in \mathbb{N})$:
  $(\exists j \geq 0)\ \forall i (\geq j)\ x_i \leq y_i$
- **basin** b: $0 \neq b \leq 1'_S$ s.t. img$(b ; q) \leq b$
- x**-trajectory** : function $\xi : \mathbb{N} \to 1'_S$ s.t. $\xi_0 = x \lessdot 1'_S$ and $\xi_{n+1} \lessdot \text{img}(\xi_n ; q)$
- x**-flight** : an **injective** x**-trajectory**
- x**-antiflight** : an x-flight in the converse $q\breve{}$-dynamics

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basic concepts
a contrived example

state transition dynamics: a contrived example

a simple model of epidemic propagation

as well as of unstable catalythic reaction:

$CG \to GG$

$C \to C$

$G \to \lambda$

$G \to K$

$G \to G$

- **instability** of agent G: it either dies or recovers (becoming immune: K)
- **states** : $(|C|+|K|, |G|)$
- **transitions** can be determined by equipping rules with a measure of **relative strength**: see
  [(Bianco *et al.*, 2006a), (Bianco *et al.*, 2006b), (Manca, 2006)]

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recurrence and attractors: basic definitions

Because of nondeterminism of the transition relation, concepts of **attracting set, recurrence, attractor** take *two distinct modal flavours*.
Let b be a basin and $0 \neq a \leq b$:

- a is an *unavoidable attracting set* of b, a **□-attracts** b, if

$$b \leq \sum_{r \prec\!\!\leq_q a} r$$

- a is a *potential attracting set* of b, a **◊-attracts** b, if

$$b \leq \mathrm{dom}(q^* \,;\, \sum_{r \prec\!\!\leq_q a} r)$$

- the (unavoidable | potential) **attractor** of a basin b is a minimal (unavoidable | potential) attracting set of b, under boolean ordering
- x is **recurrent** in basin b, x **◊-rec** b, if $x \leq b$ and $x \leq \mathrm{img}(x\,;q^+)$
- x is **eternally recurrent** in basin b, x **□-rec** b, if $x \leq b$ and $x\,;q^* \leq x\,;q^{\smile *}$

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flights in absence of eternal recurrence
characterization of recurrence and attractors

recurrence and attractors: basic facts

let b be a basin in the q-dynamics:

- b **□-attracts** b
- a **□-attracts** $b \Rightarrow a$ **◊-attracts** b
- $a_{\Box}$ is the *unique* unavoidable attractor of b, when it exists, otherwise we write $a_{\Box} = 0$
- $a_{\Diamond}$ is the *unique* potential attractor of b, when it exists, otherwise we write $a_{\Diamond} = 0$
- **recurrence sets** in b: let

$$r_{\Diamond} = \sum_{x\ \Diamond\text{-rec}\ b} x\,, \quad r_{\Box} = \sum_{x\ \Box\text{-rec}\ b} x$$

then: $r_{\Box} \leq r_{\Diamond}$ and $\mathrm{img}(r_{\Box}\,;q^*) = r_{\Box}$

- **Definition:**
  - flight ξ is **recurrent in** b if $\xi_n \leq \mathrm{img}(r_{\Diamond}\,;q^*)$ for some $n \in \mathbb{N}$
  - flight ξ is **eternally recurrent in** b if $\xi_{\mathbb{N}} \leq \mathrm{img}(r_{\Box}\,;q^{\smile *})$

Lemma 1: existence of nonrecurrent flights

may hamper existence of the unavoidable attractor (see later)

Definition The q-dynamics is **finitary** if the q relation is image-finite on individual states, *i.e.* img $(x\,;q)$ represents a finite state whenever x represents an individual state

Lemma 1 Let b be a basin in a finitary q-dynamics. If there exists $x \lesssim b$ such that for no $n \in \mathbb{N}$ $\mathrm{img}(x\,;q^{\geq n}) \leq \mathrm{img}(r_\diamond\,;q^*)$, then there is a nonrecurrent x-flight in b

Proof idea:

- arrange the x-orbit in a x-rooted tree:
  - nodes are labeled by individual states in the x-orbit
  - children of y are its successor states: $\mathrm{img}(y\,;q)$
- prune the nodes labeled by states in $\mathrm{img}(r_\diamond\,;q^*)$
- use König's Lemma

Remark the q-finitarity hypothesis is fairly essential: consider an antiflight ξ, with ξ_0 the only fixed point in basin b, and an additional individual state $x \lesssim b$ with $\mathrm{img}(x\,;q) = \xi_{\mathbb{N}}$

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Lemma 2: nonexistence of the unavoidable attractor

here is a **sufficient condition**

Definition A flight ξ is **antiflight-free** if no individual state in $\xi_{\mathbb{N}}$ is the target of an antiflight

Lemma 2 For any basin b in the q-dynamics, $a_\Box = 0$ if

(i) the converse $q^\smile$-dynamics is finitary, and

(ii) there is a nonrecurrent antiflight-free flight in b, under the q-dynamics

Proof sketch:

- if ξ is a nonrecurrent antiflight-free flight, in b, then:
  - (i) every individual state in $\xi_{\mathbb{N}}$ is removable from the unavoidable attracting set b, viz. for no $x \lesssim b$ may any ξ_i occur infinitely often in the x-orbit, whereas
  - (ii) no infinite subset of $\xi_{\mathbb{N}}$ is removable from b without losing the unavoidable attracting set property
- it's enough to prove (i) for ξ_0; much like in the proof of Lemma 1, arrange the ξ_0-orbit (under $q^\smile$) in a finitely branching tree, where individual states may only occur once in any given path, by nonrecurrence of ξ; then
- by contradiction, assume ξ_0 occurs infinitely often in the x-orbit; then the set of path lengths in the tree would be unbounded, so the tree should be infinite, thus having an infinite path by König's Lemma, which entails that ξ_0 is the target of an antiflight, against the hypotesis

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Lemma 3: flights in absence of eternal recurrence

start everywhere

Lemma 3 If b is a basin in the q-dynamics with no eternally recurrent states, then every $x \leq b$ is the origin of a flight

Proof idea:

- for each x, find $x' \leq \mathrm{img}(x\,;q^{+}) \setminus \mathrm{img}(x\,;\breve{q}^{*})$ and a finite sequence of $n{+}2$ individual states $(\xi_i \mid 0 \leq i \leq n+1)$, for some $n \geq 0$, that satisfies the following requirements:
  - (i) $\xi_0 = x$, $\xi_{n+1} = x'$, $\xi_{i+1} \leq \mathrm{img}(\xi_i\,;q)$, for $0 \leq i \leq n$;
  - (ii) $\xi_i \leq \mathrm{img}(x\,;\breve{q}^{*})$, for $0 < i \leq n$;
  - (iii) $\xi_i = \xi_j \Leftrightarrow i = j$, for $0 \leq i, j \leq n+1$.
- iterate the previous procedure and show injectivity of the resulting flight map

Theorem: recurrence and attractors

characterizes both *existence* and *extent* of attractors

Theorem In any basin b with the q-dynamics:

(i) $a_{\Box} = \mathrm{img}(r_{\Diamond}\,;q^{*})$
if the q-dynamics is finitary and every flight is recurrent, otherwise $a_{\Box} = 0$ if the converse $\breve{q}$-dynamics is finitary and if there is a nonrecurrent antiflight-free flight, under the q-dynamics

(ii) $a_{\Diamond} = r_{\Box}$ if every flight is eternally recurrent, otherwise $a_{\Diamond} = 0$

Remarks:

- finitarity assumptions are only needed for the characterization of the unavoidable attractor
- despite the structural difference, a certain analogy with Poincaré Recurrence Theorem surfaces, with boundedness and invariance replaced by finitarity and flight recurrence hypotheses

research questions and perspectives

Moving from structureless to topological state spaces:

- definability of **weaker** notions of **recurrence**

  that is: replace **exact** occurrence of a state in its own trajectory with **approximate** occurrence
- definability of **weaker** notions of **attraction**

  that is: replace **exact** inclusion of orbits in the attracting set with **approximate** inclusion
- **generalization** of the characterization results presented here, linking recurrence and attractors, under the aforementioned weakenings

  at least for the **deterministic case**

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