Morphism-driven design of graph colouring institutions

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references

G. Scollo, Graph colouring institutions,
Proc. 7th Int'l Seminar on Relational Methods in Computer Science (RelMiCS 7),
Malente, Germany, 12–17 May 2003.
At URL: http://www.informatik.uni-kiel.de/˜relmics7

and

G. Scollo, Morphism-driven design of graph colouring institutions,
U. Verona, Dipartimento di Informatica, RR 03/2003, March 2003
At URL: http://www.di.univr.it/˜scollo/papers/dirr0303.pdf

the problem

Given:

- two institutions which formalize graph colouring problems, viz.:
  $\mathcal{V}$: proper 4-colouring of vertices
  $\mathcal{E}$: proper 3-colouring of edges
  for maximal planar graphs (triangulations of the sphere)
- algorithms (Tait, 1880) which prove those problems equivalent

Goal: use Tait's equivalence to design an institution isomorphism

a solution

Redesign the given institutions!

$$\mathcal{V} \leftarrow \mathcal{V}' \longleftrightarrow \mathcal{E}'' \rightarrow \mathcal{E}' \ldots \mathcal{E}$$

or, if using *comorphisms* rather than (forgetful) *morphisms*:

$$\mathcal{V} \rightarrow \mathcal{V}' \longleftrightarrow \mathcal{E}'' \leftarrow \mathcal{E}' \ldots \mathcal{E}$$

plan of seminar (I)

1. graph colouring preliminaries
2. category theory: basic concepts
3. institutions and their morphisms
4. the vertex colouring institution $\mathcal{V}$
5. the edge colouring institution $\mathcal{E}$

plan of seminar (II)

6. Tait's equivalence, algebraically
7. morphism-driven institution redesign: $\mathcal{E}'$
8. natural fine tuning of institution design: $\mathcal{V}', \mathcal{E}''$
9. graph colouring institution morphisms
10. conclusions, research questions

map colouring

a 151-year old, controversial problem (F. Guthrie): map colouring

- Kempe (1879), Heawood (1890), Tait (1880), Tutte (1946),
- Birkhoff (1913), Heesch (1969),
- Appel, Haken & Koch (1976, 1989),
- Robertson, Sanders, Seymour & Thomas (1995), . . .

curious? cf.: R. Fritsch & G. Fritsch, *The Four Colour Theorem* (1998)

equivalent problems:

(cubic) map colouring $\simeq$ (maximal) planar graph colouring

graph colouring

maximal planar graphs $\simeq$ triangulations of sphere

notation (convenient for combinatorial graph abstraction):

$\underline{\mathbf{n}}$: finite ordinal with n elements
$\mathrm{T}_{\mathrm{V}}(n)$**:** $(n{+}2)$-vertex-labeled triangulations
$\mathrm{T}_{\mathrm{E}}(n)$**:** $3n$-edge-labeled triangulations

from *relation algebra*:

$1_{\underline{\mathbf{n}}}$, $1'_{\underline{\mathbf{n}}}$, $0'_{\underline{\mathbf{n}}}$: *universal, identity, diversity*, binary relations on $\underline{\mathbf{n}}$
$\mathrm{r}\breve{}$: (anti-involutory) *converse* of binary relation r
r ; s : (associative) *composition* of binary relations r, s

categories

category = (typed) generalized monoid **A:** that is a *graph* $O \overset{\text{dom}}{\underset{\text{cod}}{\longleftarrow}} A$

with: O : objects, A : arrows, or morphisms

that is (writing $f : x{\rightarrow}y$ for dom$f$ = $x \wedge$ cod$f$ = y)

- closed under associative composition of (composable) arrows:

$$f : x{\rightarrow}y, g : y{\rightarrow}z \;\Rightarrow f;g : x{\rightarrow}z$$

- with local unit 1_x : $x{\rightarrow}x$ for each $x{\in}$O

$$f : x{\rightarrow}y \;\Rightarrow 1_x;f = f = f;1_y$$

functors

(categorial) structure-preserving maps between categories

notation: $|\mathcal{C}|$: the set of objects of category $\mathcal{C}$

$\mathcal{C}(a,b)$: the set of morphisms from a to b in $\mathcal{C}$, for $a,b{\in}|\mathcal{C}|$

then a functor F: $\mathcal{D}{\rightarrow}\mathcal{C}$ consists of maps

- on objects: F: $|\mathcal{D}|{\rightarrow}|\mathcal{C}|$
- on arrows : (F: $\mathcal{D}(a,b){\rightarrow}\mathcal{C}(\mathrm{F}a,\mathrm{F}b) \mid a,b{\in}|\mathcal{D}|$)

preserving units and composition: $\mathrm{F}1_x = 1_{\mathrm{F}x}$ $\;\mathrm{F}(f;g) = (\mathrm{F}f);(\mathrm{F}g)$

Set: category of (small) sets, morphisms: (total) functions

Cat: category of (locally small) categories, morphisms: functors

natural transformations

(functorial) structure-preserving maps between functor images

Let F, G: $\mathcal{D} \to \mathcal{C}$ (functors of the same type).

A natural transformation α: F$\dot{\to}$G is an assignment of an arrow $\alpha_x : \mathrm{F}x \to \mathrm{G}x$ in $\mathcal{C}$ to each object $x \in |\mathcal{D}|$, such that $\forall a, b \in |\mathcal{D}|$, $f \in \mathcal{D}(a, b)$

$$\mathrm{F}f; \alpha_b = \alpha_a; \mathrm{G}f$$

i.e. the following diagram in $\mathcal{C}$ commutes:

naturality

$$\begin{tikzcd}
\mathrm{F}a \arrow[r, "\alpha_a"] \arrow[d, "\mathrm{F}f"'] & \mathrm{G}a \arrow[d, "\mathrm{G}f"] \\
\mathrm{F}b \arrow[r, "\alpha_b"] & \mathrm{G}b
\end{tikzcd}$$

the arrows α_x are called the components of natural transformation α

institutions

abstract model-theoretic logics

A (classic) institution $\mathcal{I}$ = (Sig, Sen, Mod, $\models$) consists of:

- Sig, a category of signatures
- Sen: Sig$\rightarrow$Set, a sentence functor
- Mod: Sig$^{\mathrm{op}}\rightarrow$Cat, a contravariant model functor
- $\models$: $\{\models_\Sigma \subseteq |\mathrm{Mod}(\Sigma)|\times\mathrm{Sen}(\Sigma) \mid \Sigma\in|\mathrm{Sig}|\}$, a $|$Sig$|$-indexed satisfaction relation that complies with the satisfaction condition:

satisfaction invariance in institutions

$\forall\Sigma_1,\Sigma_2\in|\mathrm{Sig}|$, $\pi\in\mathrm{Sig}(\Sigma_1,\Sigma_2)$, $M\in|\mathrm{Mod}(\Sigma_2)|$, $\varphi\in\mathrm{Sen}(\Sigma_1)$:

$$\mathrm{Mod}(\pi)(M) \models_{\Sigma_1} \varphi \Leftrightarrow M \models_{\Sigma_2} \mathrm{Sen}(\pi)(\varphi)$$

notation: $\pi\varphi$ for Sen(π)(φ), $M\pi$ for Mod(π)(M)

the satisfaction condition is more simply rewritten as:

$$M\pi \models_{\Sigma_1} \varphi \Leftrightarrow M \models_{\Sigma_2} \pi\varphi$$

institution comorphisms

abstract model-theoretic interpretations

An institution comorphism $\mathcal{T}\colon \mathcal{I} \to \mathcal{I}' = (\Phi, \alpha, \beta)$ consists of

- Φ: $\mathsf{Sig}_{\mathcal{I}} \to \mathsf{Sig}_{\mathcal{I}'}$ a signature functor
- α: $\mathsf{Sen}_{\mathcal{I}} \dot{\to} \Phi\,;\mathsf{Sen}_{\mathcal{I}'}$ a natural (sentence transformation)
- β: $\Phi^{\mathrm{op}}\,;\mathsf{Mod}_{\mathcal{I}'} \dot{\to} \mathsf{Mod}_{\mathcal{I}}$ a natural (model transformation)

complying with the following satisfaction condition,
$\forall \Sigma \in |\mathsf{Sig}_{\mathcal{I}}|$, $\mathrm{M}' \in |\mathsf{Mod}_{\mathcal{I}'}(\Phi(\Sigma))|$, $\varphi \in \mathsf{Sen}_{\mathcal{I}}(\Sigma)$:

$$\beta_\Sigma(\mathrm{M}') \models_{\mathcal{I},\Sigma} \varphi \Leftrightarrow \mathrm{M}' \models_{\mathcal{I}',\Phi(\Sigma)} \alpha_\Sigma(\varphi)$$

institution morphisms

(just reverse the transformation arrows. . .)

An institution morphism $\mathcal{T}\colon \mathcal{I} \to \mathcal{I}' = (\Phi, \alpha, \beta)$ consists of

- Φ: $\mathsf{Sig}_{\mathcal{I}} \to \mathsf{Sig}_{\mathcal{I}'}$ a signature functor
- α: $\Phi\,;\mathsf{Sen}_{\mathcal{I}'} \to \mathsf{Sen}_{\mathcal{I}}$ a natural (sentence transformation)
- β: $\mathsf{Mod}_{\mathcal{I}} \to \Phi^{\mathrm{op}}\,;\mathsf{Mod}_{\mathcal{I}'}$ a natural (model transformation)

complying with the following satisfaction condition,
$\forall \Sigma \in |\mathsf{Sig}_{\mathcal{I}}|$, $\mathrm{M} \in |\mathsf{Mod}_{\mathcal{I}}(\Sigma)|$ $\varphi' \in \mathsf{Sen}_{\mathcal{I}'}(\Phi(\Sigma))$:

$$\mathrm{M} \models_{\mathcal{I},\Sigma} \alpha_\Sigma(\varphi') \Leftrightarrow \beta_\Sigma(\mathrm{M}) \models_{\mathcal{I}',\Phi(\Sigma)} \varphi'$$

the 4-colouring institution $\mathcal{V}$

$$|\mathsf{Sig}_{\mathcal{V}}| = \mathbf{N}\backslash\{0\}, \mathsf{Sig}_{\mathcal{V}}(m,n) \;= \emptyset \text{ if } m \neq n$$
$$\mathsf{Sig}_{\mathcal{V}}(n,n) \;= \{\pi : \underline{\mathbf{n+2}} \to \underline{\mathbf{n+2}} \mid \pi \text{ is bijective}\}$$
$$\mathsf{Sen}_{\mathcal{V}}(n) = \{\epsilon_\theta \mid \theta \in \mathrm{T_V}(n)\}, (\pi i)\ \pi\epsilon_\theta\ (\pi j) \Leftrightarrow i\ \epsilon_\theta\ j$$
$$|\mathsf{Mod}_{\mathcal{V}}(n)| = \underline{\mathbf{4}}^{\underline{\mathbf{n+2}}}$$
$$\mathsf{Mod}_{\mathcal{V}}(n)(\mu,\mu') = \{\rho \in \underline{\mathbf{4}}^{\underline{\mathbf{4}}} | \rho \text{ is bijective}, \mu' = \mu\,;\,\rho\}$$
$$(\mu\pi)i \stackrel{def}{=} \mu(\pi i), \rho\pi \stackrel{def}{=} \rho\ \forall\rho : \underline{\mathbf{4}} \leftrightarrow \underline{\mathbf{4}}$$
$$\mu \models_{\mathcal{V},n} \epsilon_\theta \text{ iff } \mu^{\smile}; \eta_\theta; \mu \leq 0'_{\underline{\mathbf{4}}}$$

design decisions in $\mathcal{V}$

why just signature isomorphisms (vertex label permutations)?

- non-surjective morphisms are useless
- non-injective morphisms could support edge contraction (a useful concept in reducibility proofs), but this operation does not preserve maximality of planar graphs, thus the sentence functor should map such morphisms to partial functions, unavailable in Set…

the 3-colouring institution $\mathcal{E}$

syntax: Matiyasevitch's polynomials

$$|\mathsf{Sig}_{\mathcal{E}}| = \mathbf{N}\backslash\{0\}, \mathsf{Sig}_{\mathcal{E}}(m,n) \;= \emptyset \text{ if } m \neq n$$

$$\mathsf{Sig}_{\mathcal{E}}(n,n) \;= \{\pi : \underline{\mathbf{3n}}\to\underline{\mathbf{3n}} \mid \pi \text{ is bijective}\}$$

$$\psi_\vartheta \;=\; \prod_{\mathrm{t}_{ijk}\in\vartheta} (x_i - x_j)(x_j - x_k)(x_k - x_i)$$

$$\mathsf{Sen}_{\mathcal{E}}(n) = \{\psi_\vartheta \mid \vartheta\in\mathrm{T}_{\mathrm{E}}(n)\}$$

$$\pi\psi_\vartheta \;=\; \prod_{\mathrm{t}_{ijk}\in\vartheta} (x_{\pi i} - x_{\pi j})(x_{\pi j} - x_{\pi k})(x_{\pi k} - x_{\pi i})$$

models and satisfaction in $\mathcal{E}$

$$|\mathsf{Mod}_{\mathcal{E}}(n)| = \underline{\mathbf{3}^{\mathbf{3n}}}$$

$$\mathsf{Mod}_{\mathcal{E}}(n)(\nu,\nu') = \{\rho\in\underline{\mathbf{3}^{\mathbf{3}}} | \rho \text{ is bijective}, \nu' = \nu\,;\,\rho\}$$

$$(\nu\pi)i \stackrel{def}{=} \nu(\pi i), \rho\pi \stackrel{def}{=} \rho\;\forall\rho : \underline{\mathbf{3}}\leftrightarrow\underline{\mathbf{3}}$$

$$\xi_{\psi_\vartheta} \leq 1_{\underline{\mathbf{3n}}} : i\;\xi_{\psi_\vartheta}\;j \text{ iff } (x_i - x_j) \text{ occurs in } \psi_\vartheta$$

$$\nu \models_{\mathcal{E},n} \psi_\vartheta \text{ iff } \nu^{\smile}; \xi_{\psi_\vartheta}; \nu \leq 0'_{\underline{\mathbf{3}}}$$

the Klein 4-group

enables an elegant simplification of Tait's algorithms

$$\text{carrier} : \underline{\mathbf{4}}$$

$$x +_K x = 0, x +_K 0 = 0 +_K x = x$$

$$x +_K y = z \text{ whenever } \{x, y, z\} = \underline{\mathbf{4}} \setminus \underline{\mathbf{1}}$$

Tait's equivalence

4CT $\Rightarrow$ 3CT

if μ:**n+2**$\to$**4** is a proper 4-colouring of $\theta \in \mathrm{T_V}(n)$, then a proper 3-colouring of its edges ν:**3n**$\to$**3** is given by ν: $x \mapsto \mu i +_K \mu j$ whenever edge x connects vertices i, j in θ.

3CT $\Rightarrow$ 4CT

Let ν:**3n**$\to$**4**$\setminus$**1** be a proper 3-colouring of $\vartheta \in \mathrm{T_E}(n)$:

- choose a start-vertex s$\in$**n+2**, assign it colour 0;
- assign to every other vertex the Klein sum of the edge colours along any path from the start-vertex to it (well-defined? Yes!).

morphism = context $\rightsquigarrow$ redesign $\mathcal{E}$

basic trouble with relating $\mathcal{V}$ and $\mathcal{E}$:

vertex labels and edge labels are unrelated!

So, how to map signature morphisms functorially? The trivial functor is useless (why? try this as an exercise. . .)

Another challenge: Tait's equivalence only maps proper colourings, yet model transformation must take all models from the source institution to the target one

Moreover: a colouring is proper w.r.t. a given triangulation, but model transformation is to be defined in a sentence-independent way

the edge colouring institution $\mathcal{E}'$

solution ideas $\rightsquigarrow$ $\mathcal{E}'$

- keep the same category of signatures $\mathsf{Sig}_{\mathcal{V}}$ (vertex label permutations yield graph isomorphisms)
- enumerate edges of a sentence depending on their vertex labels (e.g.: lexicographic ordering of vertex label pairs)
- extend edge colouring models to all edges, by "colouring" with 0 those edges that must be absent in every sentence that satisfies the given model (why?)
- extend model morphisms to Klein 4-group isomorphisms

natural fine tuning

Is $\mathcal{E}'$ isomorphic to $\mathcal{V}$?

)-8= **NO!** =8-(

Why? Well, what about the start-vertex in Tait's algorithm? What place should get it, in a natural model transformation?

Our last refinement solves this problem: a slight extension of the model functor, in both institutions, making room for the start-vertex.

There are obvious forgetful morphisms from the so extended institutions $\mathcal{V}'$, $\mathcal{E}''$, to $\mathcal{V}$, $\mathcal{E}'$, respectively.

$\mathcal{V}'$ and $\mathcal{E}''$ are isomorphic.

a combinatorial remark

Our ultimate design refinement does actually solve another nasty problem as well, that is:

a natural isomorphism between two institutions requires each component of the model transformation to be an invertible functor, thus a bijection on each set of model morphisms between any two models.

Now, our model morphisms are colour permutations: there are **4!** of them on 4-colourings, but only **3!** (Klein 4-group isomorphisms) of them on 3-colourings. The task would seem barely impossible, yet. . . (never trust appearances !-)

abstract finite model theory?

looks like a kind of oxymoron, yet

this is more or less what this institution design exercise is about: the use of abstract model-theoretic concepts in finite model-theoretic questions.

One might object: What's the question in this exercise? The Four Colour Theorem has been proven (oh, well. . .), Tait's equivalence is all but a brand new discovery. . .

a few research questions

An interesting, perhaps surprising side-effect of our redesign exercise is precisely that, although we started from a very ad-hoc view of logical design, viz. problem logic rather than problem formulation in a general-purpose logic, the model structures we end-up with prove easily re-usable for more general questions in graph colouring, such as:

- sufficient conditions for 4-colorability, weaker than planarity
- syntactic characterization of k-colorability