2.1

Si tratta di risolvere, nel campo complesso, il sistema $\begin{cases} z = w^3 \\ z^3 = w \end{cases}$.

Si ha

$$\begin{cases} z = w^3 \\ z^3 = w \end{cases} \iff \begin{cases} z = w^3 \\ w^9 = w \end{cases} \iff$$

$$\iff \begin{cases} z = 0 \\ w = 0 \end{cases} \cup (*) \begin{cases} z = w^3 \\ w^8 = 1 \end{cases}.$$

Risolviamo il sistema (*).

$$(*) \iff \begin{cases} z = w^3 \\ w = \cos \frac{2 k \pi}{8} + i \operatorname{sen} \frac{2 k\pi}{8} \\ \qquad k = 0, 1, \dots, 7 \end{cases} \iff$$

$$\iff \begin{cases} z = \cos \frac{3 k\pi}{4} + i \operatorname{sen} \frac{3k\pi}{4} \\ w = \cos \frac{k\pi}{4} + i \operatorname{sen} \frac{k\pi}{4} \end{cases} \quad k = 0, 1, \dots, 7 .$$

Pertanto le soluzioni del sistema (*) sono:

$(1,1)$, $\left(\frac{\sqrt{2}}{2}(-1+i), \frac{\sqrt{2}}{2}(1+i)\right)$,

$(-i,i)$, $\left(\frac{\sqrt{2}}{2}(1+i), \frac{\sqrt{2}}{2}(-1+i)\right)$,

$(-1,-1)$, $\left(\frac{\sqrt{2}}{2}(1-i), \frac{\sqrt{2}}{2}(-1-i)\right)$,

$(i,-i)$, $\left(\frac{\sqrt{2}}{2}(-1-i), \frac{\sqrt{2}}{2}(1-i)\right)$.

Dunque le coppie (z,w) ricercate sono le coppie $(0,0)$ e le otto coppie prima elencate.

2.2

(1) $2z^4 - (5-i)z^3 + (7-i)z^2 - (5-i)z + 2 = 0 \iff$

(si tratta di un'equazione reciproca di 1ª specie)

$$\iff 2\left(z^2 + \frac{1}{z^2}\right) - (5-i)\left(z + \frac{1}{z}\right) + 7 - i = 0$$

Ponendo $z + \frac{1}{z} = w$ $\left(\Rightarrow z^2 + \frac{1}{z^2} = w^2 - 2\right)$

si ottiene l'equazione di secondo grado

(2) $\quad 2(w^2-2) - (5-i)w + 7 - i = 0 \iff$

$$\iff 2w^2 - (5-i)w + 3 - i = 0 \quad ,$$

le cui soluzioni si ottengono dalle formule risolutive

$$w = \frac{5-i \pm \sqrt{(5-i)^2 - 8(3-i)}}{4} =$$

($\sqrt{}$ indica una delle due radici quadrate del numero considerato)

$$= \frac{5-i \pm \sqrt{25-10i-1-24+8i}}{4} =$$

$$= \frac{5-i \pm \sqrt{-2i}}{4} ;$$

$$-2i = 2(-i) = 2\left(\cos\frac{3\pi}{2} + i \operatorname{sen}\frac{3\pi}{2}\right) ,$$

quindi le radici quadrate di $-2i$ sono:

$$\pm\sqrt{2}\left(\cos\frac{3\pi}{4} + i \operatorname{sen}\frac{3\pi}{4}\right) = \pm\sqrt{2}\left(-\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right) =$$

$$= \pm(-1+i)$$

e le soluzioni delle (2) sono:

$$w_1 = \frac{5-i-1+i}{4} = 1 ,$$

$$w_2 = \frac{5-i+1-i}{4} = \frac{1}{2}(3-i) .$$

Le soluzioni delle (1) si trovano risolvendo le due equazioni:

(3) $z + \frac{1}{z} = 1$, (4) $z + \frac{1}{z} = \frac{1}{2}(3-i)$.

(3) $\iff z^2 - z + 1 = 0 \iff$

$$\iff z = \frac{1 \pm \sqrt{1-4}}{2} = \boxed{\frac{1}{2}\left(1 \pm \sqrt{3}\, i\right)} .$$

(4) $\iff 2z^2 - (3-i)z + 2 = 0 \iff$

$$\iff z = \frac{3-i \pm \sqrt{(3-i)^2 - 16}}{4} =$$

$$= \frac{3-i \pm \sqrt{9-6i-1-16}}{4} =$$

$$= \frac{3-i \pm \sqrt{-8-6i}}{4} ;$$

$$-8-6i = 10\left(-\frac{4}{5} - \frac{3}{5}i\right) =$$

$$= 10\left(\cos\varphi + i \operatorname{sen}\varphi\right) ,$$

dove $\varphi \in \left]-\pi, -\frac{\pi}{2}\right[$

(l'estremo dell'arco appartiene al terzo quadrante)

e quindi $\frac{\varphi}{2} \in \left]-\frac{\pi}{2}, -\frac{\pi}{4}\right[\quad \left(\subseteq \left]-\frac{\pi}{2}, 0\right[\right)$;

usando le formule di bisezione otteniamo:

$$\cos\frac{\varphi}{2} = \sqrt{\frac{1+\cos\varphi}{2}} = \sqrt{\frac{1-\frac{4}{5}}{2}} = \sqrt{\frac{1}{10}} ,$$

$$\operatorname{sen}\frac{\varphi}{2} = -\sqrt{\frac{1-\cos\varphi}{2}} = -\sqrt{\frac{1+\frac{4}{5}}{2}} = -\frac{3}{\sqrt{10}} ,$$

pertanto le radici quadrate di $-8-6i$

sono

$$\pm\sqrt{10}\left(\frac{1}{\sqrt{10}} - \frac{3}{\sqrt{10}}\,i\right) = \pm(1-3i)$$

e le soluzioni della (4) sono :

$$\frac{3-i \pm (1-3i)}{4} = \begin{cases} \boxed{1-i} \\ \boxed{\frac{1}{2}(1+i)} \end{cases} .$$

In conclusione le soluzioni dell'equazione assegnata sono:

$$\frac{1}{2}(1 \pm \sqrt{3}\, i) \quad , \quad 1-i \quad , \quad \frac{1}{2}(1+i) \quad .$$

2.3

Osserviamo che l' equazione assegnata

$$(1) \qquad z^4 \bar{z} = 1+i$$

può scriversi

$$z^3 |z|^2 = 1+i \quad ;$$

inoltre

$$(1) \Rightarrow |z^4 \bar{z}| = |1+i| \Leftrightarrow$$

$$\Leftrightarrow |z|^5 = \sqrt{2} \Leftrightarrow |z| = \sqrt[10]{2} \quad ;$$

pertanto

$$(1) \Leftrightarrow \begin{cases} z^3 |z|^2 = 1+i \\ |z| = \sqrt[10]{2} \end{cases} \Leftrightarrow \begin{cases} z^3 = \frac{1}{\sqrt[5]{2}}(1+i) \\ |z| = \sqrt[10]{2} \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \quad z^3 = \frac{1}{\sqrt[5]{2}}(1+i) \qquad (2)$$

l' implicazione $\Rightarrow$ è ovvia ; per provare la $\Leftarrow$ notiamo che dalla (2) segue $|z|^3 = \frac{1}{\sqrt[5]{2}}\sqrt{2}$ e quindi $|z| = 2^{\frac{1}{3}\left(\frac{1}{2}-\frac{1}{5}\right)} = 2^{\frac{1}{10}}$.

Risolviamo la (2) (si tratta di trovare le radici cubiche di $\frac{1}{\sqrt[5]{2}}(1+i)$).

$$\frac{1}{\sqrt[5]{2}}(1+i) = \frac{\sqrt{2}}{\sqrt[5]{2}}\left(\frac{\sqrt{2}}{2} + i\,\frac{\sqrt{2}}{2}\right) =$$

$$= 2^{\frac{1}{2}-\frac{1}{5}}\left(\cos\frac{\pi}{4} + i \operatorname{sen}\frac{\pi}{4}\right) =$$

$$= 2^{\frac{3}{10}}\left(\cos\frac{\pi}{4} + i \operatorname{sen}\frac{\pi}{4}\right) ;$$

pertanto le soluzioni dell'equazione assegnata sono :

$$z_k = 2^{\frac{1}{10}}\left(\cos\left(\frac{\pi}{12}+\frac{2k\pi}{3}\right) + i \operatorname{sen}\left(\frac{\pi}{12}+\frac{2k\pi}{3}\right)\right)$$

$$k = 0, 1, 2 .$$

$$z_0 = 2^{\frac{1}{10}}\left(\cos\frac{\pi}{12} + i \operatorname{sen}\frac{\pi}{12}\right) =$$

(per il calcolo di $\cos\frac{\pi}{12}$, $\operatorname{sen}\frac{\pi}{12}$ vedi pagg. 2.6-1 – 2.6-3)

$$= 2^{\frac{1}{10}} \left(\frac{1}{2\sqrt{2}} (\sqrt{3}+1) + i \frac{1}{2\sqrt{2}} (\sqrt{3}-1) \right) =$$

$$= 2^{\frac{1}{10}-1-\frac{1}{2}} \left(\sqrt{3}+1 + i (\sqrt{3}-1) \right) =$$

$$= 2^{-\frac{7}{5}} \left(\sqrt{3}+1 + i (\sqrt{3}-1) \right) =$$

$$= \boxed{\frac{1}{2\sqrt[5]{4}} \left((\sqrt{3}+1) + i (\sqrt{3}-1) \right)} \quad ;$$

$$z_1 = 2^{\frac{1}{10}} \left(\cos\left(\frac{\pi}{12} + \frac{2\pi}{3}\right) + i \,\mathrm{sen}\left(\frac{\pi}{12} + \frac{2\pi}{3}\right) \right) =$$

$$= 2^{\frac{1}{10}} \left(\cos \frac{3\pi}{4} + i \,\mathrm{sen} \frac{3\pi}{4} \right) =$$

$$= 2^{\frac{1}{10}} \left(- \frac{\sqrt{2}}{2} + i \frac{\sqrt{2}}{2} \right) =$$

$$= 2^{\frac{1}{10}-\frac{1}{2}} \left(-1 + i \right) = 2^{-\frac{2}{5}} (-1+i) =$$

$$= \boxed{\frac{1}{\sqrt[5]{4}} (-1+i)} \quad ;$$

2.3 - 4

$$z_2 = 2^{\frac{1}{10}} \left(\cos\left(\frac{\pi}{12} + \frac{4\pi}{3}\right) + i \operatorname{sen}\left(\frac{\pi}{12} + \frac{4\pi}{3}\right)\right) =$$

$$= 2^{\frac{1}{10}} \left(\cos \frac{17}{12}\pi + i \operatorname{sen} \frac{17}{12}\pi \right) =$$

$$\frac{17\pi}{12} = \frac{12 + 5}{12}\pi = \pi + \frac{5\pi}{12} = \pi + \left(\frac{\pi}{2} - \frac{\pi}{12}\right)$$

$$= 2^{\frac{1}{10}} \left(-\cos\left(\frac{\pi}{2} - \frac{\pi}{12}\right) - i \operatorname{sen}\left(\frac{\pi}{2} - \frac{\pi}{12}\right)\right) =$$

$$= -2^{\frac{1}{10}} \left(\operatorname{sen} \frac{\pi}{12} + i \cos \frac{\pi}{12} \right) =$$

$$= \boxed{-\frac{1}{2\sqrt[5]{4}} \left(\sqrt{3} - 1 + i\left(\sqrt{3}+1\right)\right)}$$

2.4

$$\frac{i+3}{3i-1} = \frac{(3+i)(-1-3i)}{(-1+3i)(-1-3i)} = \frac{-3+3-9i-i}{10} =$$

$$= -i = \cos\left(-\frac{\pi}{2}\right) + i\,\mathrm{sen}\left(\frac{\pi}{2}\right) ,$$

quindi le radici richieste sono

$$z_k = \cos\left(-\frac{\pi}{8} + k\frac{\pi}{2}\right) + i\,\mathrm{sen}\left(-\frac{\pi}{8} + k\frac{\pi}{2}\right)$$

$$k = 0, 1, 2, 3 \quad ;$$

per scriverle in forma algebrica calcoliamo

$$\cos\frac{\pi}{8} = \sqrt{\frac{1+\cos\frac{\pi}{4}}{2}} = \sqrt{\frac{1+\frac{\sqrt{2}}{2}}{2}} = \frac{1}{2}\sqrt{2+\sqrt{2}} ,$$

$$\mathrm{sen}\,\frac{\pi}{8} = \sqrt{\frac{1-\cos\frac{\pi}{4}}{2}} = \frac{1}{2}\sqrt{2-\sqrt{2}} \quad ;$$

pertanto

$$z_0 = \cos\left(-\frac{\pi}{8}\right) + i \operatorname{sen}\left(-\frac{\pi}{8}\right) =$$

$$= \frac{1}{2}\left(\sqrt{2+\sqrt{2}} - i\sqrt{2-\sqrt{2}}\right),$$

$$z_2 = \cos\left(-\frac{\pi}{8} + \pi\right) + i \operatorname{sen}\left(-\frac{\pi}{8} + \pi\right) =$$

$$= -z_0 = -\frac{1}{2}\left(\sqrt{2+\sqrt{2}} - i\sqrt{2-\sqrt{2}}\right),$$

$$z_1 = \cos\left(-\frac{\pi}{8} + \frac{\pi}{2}\right) + i \operatorname{sen}\left(-\frac{\pi}{8} + \frac{\pi}{2}\right) =$$

$$= -\operatorname{sen}\left(-\frac{\pi}{8}\right) + i\cos\left(-\frac{\pi}{8}\right) =$$

$$= \frac{1}{2}\left(\sqrt{2-\sqrt{2}} + i\sqrt{2+\sqrt{2}}\right),$$

$$z_3 = \cos\left(-\frac{\pi}{8} + \frac{\pi}{2} + \pi\right) + i \operatorname{sen}\left(-\frac{\pi}{8} + \frac{\pi}{2} + \pi\right) =$$

$$= -z_1 = -\frac{1}{2}\left(\sqrt{2-\sqrt{2}} + i\sqrt{2+\sqrt{2}}\right).$$

2.5

Osserviamo che il numero dato può scrivere come

$$-\frac{2}{1+i}\cdot\frac{(1+i)^6(-\sqrt{3}+i)^6}{2^6(\sqrt{3}+i)^3} =$$

$$= -\frac{2}{1+i}\left[\frac{(1+i)^2(-\sqrt{3}+i)^2}{2^2(\sqrt{3}+i)}\right]^3 ,$$

pertanto le radici cubiche cercate sono:

$$wz_0 \quad , \quad wz_1 \quad e \quad wz_2 \quad ,$$

essendo

$$w = \frac{(1+i)^2(-\sqrt{3}+i)^2}{4(\sqrt{3}+i)} = \frac{2i(2-2\sqrt{3}i)}{4(\sqrt{3}+i)} =$$

$$= \frac{i\,(1-\sqrt{3}\,i)}{\sqrt{3}+i} = \frac{\sqrt{3}+i}{\sqrt{3}+i} = 1$$

e z_0, z_1 e z_2 le radici cubiche di

$$-\frac{2}{1+i} = -\frac{2(1-i)}{(1+i)(1-i)} =$$

$$= -1+i = \sqrt{2}\left(-\frac{\sqrt{2}}{2} + i\,\frac{\sqrt{2}}{2}\right) =$$

$$= \sqrt{2}\left(\cos\frac{3\pi}{4} + i\,\operatorname{sen}\frac{3\pi}{4}\right) ;$$

quindi le radici cubiche del numero

dato sono

$$z_k = \sqrt[6]{2}\left(\cos\left(\frac{\pi}{4}+\frac{2k\pi}{3}\right) + i\,\operatorname{sen}\left(\frac{\pi}{4}+\frac{2k\pi}{3}\right)\right)$$

$$k = 0, 1, 2 .$$

Scriviamole in forma algebrica :

$$z_0 = \sqrt[6]{2}\left(\cos\frac{\pi}{4} + i \operatorname{sen}\frac{\pi}{4}\right) =$$

$$= \sqrt[6]{2}\left(\frac{\sqrt{2}}{2} + i\,\frac{\sqrt{2}}{2}\right) =$$

$$= 2^{\frac{1}{6}-\frac{1}{2}}\,(1+i) = \boxed{\frac{1}{\sqrt[3]{2}}\,(1+i)} \quad ;$$

$$z_1 = \sqrt[6]{2}\left(\cos\left(\frac{\pi}{4}+\frac{2\pi}{3}\right) + i \operatorname{sen}\left(\frac{\pi}{4}+\frac{2\pi}{3}\right)\right) =$$

$$= \sqrt[6]{2}\left[\cos\frac{\pi}{4}\cos\frac{2\pi}{3} - \operatorname{sen}\frac{\pi}{4}\operatorname{sen}\frac{2\pi}{3} +\right.$$

$$\left. + i\left(\operatorname{sen}\frac{\pi}{4}\cos\frac{2\pi}{3} + \cos\frac{\pi}{4}\cdot\operatorname{sen}\frac{2\pi}{3}\right)\right] =$$

$$= \sqrt[6]{2}\left[-\frac{\sqrt{2}}{2}\,\frac{1}{2} - \frac{\sqrt{2}}{2}\,\frac{\sqrt{3}}{2} + i\left(-\frac{\sqrt{2}}{2}\,\frac{1}{2} + \frac{\sqrt{2}}{2}\,\frac{\sqrt{3}}{2}\right)\right] =$$

$$= -2^{\frac{1}{6}+\frac{1}{2}-2}\left[1+\sqrt{3}+(1-\sqrt{3})i\right] =$$

$$= -2^{-\frac{8}{6}}\left[1+\sqrt{3}+(1-\sqrt{3})i\right] =$$

$$= \boxed{-\frac{1}{2\sqrt[3]{2}}\left[1+\sqrt{3}+(1-\sqrt{3})i\right]} \quad ;$$

$$z_2 = \sqrt[6]{2}\left(\cos\left(\frac{\pi}{4}+\frac{4\pi}{3}\right)+i\,\mathrm{sen}\left(\frac{\pi}{4}+\frac{4\pi}{3}\right)\right) =$$

$$= \sqrt[6]{2}\left[\frac{\sqrt{2}}{2}\left(-\frac{1}{2}\right)-\frac{\sqrt{2}}{2}\left(-\frac{\sqrt{3}}{2}\right)\right. +$$

$$\left. + i\left(\frac{\sqrt{2}}{2}\left(-\frac{1}{2}\right)+\frac{\sqrt{2}}{2}\left(-\frac{\sqrt{3}}{2}\right)\right)\right] =$$

$$= \boxed{\frac{1}{2\sqrt[3]{2}}\left[\sqrt{3}-1-(\sqrt{3}+1)i\right]} \quad .$$

2.6

$$z^6 + 2i z^3 - 2 = 0 \iff$$

$$\iff z^6 + 2i z^3 + i^2 - i^2 - 2 = 0 \iff$$

$$\iff (z^3 + i)^2 - 1 = 0 \iff$$

$$\iff (z^3 + i - 1)(z^3 + i + 1) = 0 \iff$$

$$\iff z^3 = 1 - i \quad \text{oppure} \quad z^3 = -1 - i .$$

$$1 - i = \sqrt{2}\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} i\right) =$$

$$= \sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right) + i \operatorname{sen}\left(-\frac{\pi}{4}\right)\right) ;$$

pertanto le radici cubiche di $1-i$ sono

$$z_k = \sqrt[6]{2}\left(\cos\left(-\frac{\pi}{12} + \frac{2k\pi}{3}\right) + i \operatorname{sen}\left(-\frac{\pi}{12} + \frac{2k\pi}{3}\right)\right)$$

$$k = 0, 1, 2 .$$

$$\cos\frac{\pi}{12} = \sqrt{\frac{1 + \cos\frac{\pi}{6}}{2}} = \sqrt{\frac{1 + \frac{\sqrt{3}}{2}}{2}} =$$

$$= \frac{1}{2}\sqrt{2+\sqrt{3}} =$$

(osserviamo che nel radicale doppio

$\sqrt{a+\sqrt{b}} = \sqrt{2+\sqrt{3}}$ si ha $a^2 - b = 1$,

quindi è conveniente applicare la

formula di riduzione dei radicali

doppi

$$\sqrt{a \pm \sqrt{b}} = \sqrt{\frac{a+\sqrt{a^2-b}}{2}} \pm \sqrt{\frac{a-\sqrt{a^2-b}}{2}} \;)$$

$$= \frac{1}{2}\left(\sqrt{\frac{3}{2}} + \sqrt{\frac{1}{2}}\right) = \frac{1}{2\sqrt{2}}(\sqrt{3}+1) \;;$$

$$\operatorname{sen}\frac{\pi}{12} = \sqrt{\frac{1-\cos\frac{\pi}{6}}{2}} = \sqrt{\frac{1-\frac{\sqrt{3}}{2}}{2}} =$$

$$= \frac{1}{2}\sqrt{2-\sqrt{3}} = \frac{1}{2}\left(\sqrt{\frac{3}{2}} - \sqrt{\frac{1}{2}}\right) =$$

$$= \frac{1}{2\sqrt{2}}\left(\sqrt{3} - 1\right) .$$

Allora :

$$z_0 = \sqrt[6]{2}\left[\frac{1}{2\sqrt{2}}\left(\sqrt{3}+1\right) - \frac{1}{2\sqrt{2}}\left(\sqrt{3}-1\right)i\right] =$$

$$= 2^{+\frac{1}{6}-1-\frac{1}{2}}\left[\left(\sqrt{3}+1\right) - \left(\sqrt{3}-1\right)i\right] =$$

$$= 2^{-\frac{4}{3}}\left[\left(\sqrt{3}+1\right) - \left(\sqrt{3}-1\right)i\right] =$$

$$= \frac{1}{2\sqrt[3]{2}}\left[\left(\sqrt{3}+1\right) - \left(\sqrt{3}-1\right)i\right] ;$$

$$z_1 = \sqrt[6]{2}\left[\cos\left(-\frac{\pi}{12}+\frac{2\pi}{3}\right) + i \operatorname{sen}\left(-\frac{\pi}{12}+\frac{2\pi}{3}\right)\right] =$$

$$= \sqrt[6]{2}\left[\cos\frac{7\pi}{12} + i \operatorname{sen}\frac{7\pi}{12}\right] =$$

$$= \sqrt[6]{2}\left[\cos\left(\frac{\pi}{2}+\frac{\pi}{12}\right) + i \operatorname{sen}\left(\frac{\pi}{2}+\frac{\pi}{12}\right)\right]$$

$$= \sqrt[6]{2}\left[-\operatorname{sen}\frac{\pi}{12} + i\cos\frac{\pi}{12}\right] =$$

$$= \sqrt[6]{2}\left[-\frac{1}{2\sqrt{2}}\left(\sqrt{3}-1\right) + \frac{1}{2\sqrt{2}}\left(\sqrt{3}+1\right)\right] =$$

$$= \frac{1}{2\sqrt[3]{2}}\left[-\left(\sqrt{3}-1\right) + \left(\sqrt{3}+1\right)i\right] \quad ;$$

$$z_2 = \frac{1}{\sqrt[6]{2}}\left[\cos\left(-\frac{\pi}{12}+\frac{4\pi}{3}\right) + i \operatorname{sen}\left(-\frac{\pi}{12}+\frac{4\pi}{3}\right)\right] =$$

$$= \sqrt[6]{2}\left[\cos\frac{15\pi}{12} + i\,\text{sen}\,\frac{15\pi}{12}\right] =$$

$$= \sqrt[6]{2}\cdot\left[\cos\left(\pi+\frac{\pi}{4}\right) + i\,\text{sen}\left(\pi+\frac{\pi}{4}\right)\right] =$$

$$= \sqrt[6]{2}\left[-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\,i\right] =$$

$$= -\frac{1}{2^{-\frac{1}{6}+\frac{1}{2}}}(1+i) = -\frac{1}{\sqrt[3]{2}}(1+i).$$

$$-1-i = \sqrt{2}\left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\,i\right) =$$

$$= \sqrt{2}\left(\cos\frac{5\pi}{4} + i\,\text{sen}\,\frac{5\pi}{4}\right);$$

pertanto le radici cubiche di $-1-i$

sono

$$w_k = \sqrt[6]{2}\left(\cos\left(\frac{5\pi}{12}+\frac{2k\pi}{3}\right)+i\,\text{sen}\left(\frac{5\pi}{12}+\frac{2k\pi}{3}\right)\right)$$

$k = 0, 1, 2.$

$$w_0 = \sqrt[6]{2}\left[\cos\frac{5\pi}{12} + i\,\text{sen}\,\frac{5\pi}{12}\right] =$$

$$= \sqrt[6]{2}\left[\cos\left(\frac{\pi}{2}-\frac{\pi}{12}\right)+i\,\text{sen}\left(\frac{\pi}{2}-\frac{\pi}{12}\right)\right] =$$

$$= \sqrt[6]{2}\left[\text{sen}\,\frac{\pi}{12} + i\cos\frac{\pi}{12}\right] =$$

$$= \sqrt[6]{2}\left[\frac{1}{2\sqrt{2}}(\sqrt{3}-1) + \frac{1}{2\sqrt{2}}(\sqrt{3}+1)i\right] =$$

$$= 2^{\frac{1}{6}-1-\frac{1}{2}}\left[(\sqrt{3}-1) + (\sqrt{3}+1)i\right] =$$

$$= 2^{-\frac{4}{3}}\left[(\sqrt{3}-1)+(\sqrt{3}+1)i\right] ;$$

$$w_1 = \sqrt[6]{2}\left[\cos\left(\frac{5\pi}{12}+\frac{2\pi}{3}\right)+i\,\text{sen}\left(\frac{5\pi}{12}+\frac{2\pi}{3}\right)\right] =$$

$$= \sqrt[6]{2}\left[\cos\frac{13\pi}{12} + i\,\text{sen}\,\frac{13\pi}{12}\right] =$$

$$= \sqrt[6]{2} \left[\cos\left(\pi + \frac{\pi}{12}\right) + i \operatorname{sen}\left(\pi + \frac{\pi}{12}\right) \right] =$$

$$= \sqrt[6]{2} \left[- \frac{1}{2\sqrt{2}} (\sqrt{3} + 1) - \frac{1}{2\sqrt{2}} (\sqrt{3} - 1) i \right] =$$

$$= - \frac{1}{2\sqrt[3]{2}} \left[\sqrt{3} + 1 + (\sqrt{3} - 1) i \right] \quad ;$$

$$w_2 = \sqrt[6]{2} \left[\cos\left(\frac{5\pi}{12} + \frac{4\pi}{3}\right) + i \operatorname{sen}\left(\frac{5\pi}{12} + \frac{4\pi}{3}\right) \right] =$$

$$= \sqrt[6]{2} \left[\cos \frac{21\pi}{12} + i \operatorname{sen} \frac{21\pi}{12} \right] =$$

$$= \sqrt[6]{2} \left[\cos \frac{7\pi}{4} + i \operatorname{sen} \frac{7\pi}{4} \right] =$$

$$= \sqrt[6]{2} \left[\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} i \right] =$$

$$= 2^{\frac{1}{6} - \frac{1}{2}} (1 - i) = \frac{1}{\sqrt[3]{2}} (1 - i)$$

In conclusione le soluzioni dell'equazione proposte sono:

$$\frac{1}{2\sqrt[3]{2}}\left[\pm(\sqrt{3}+1)-(\sqrt{3}-1)i\right] \qquad \{z_0, w_1\}$$

$$\frac{1}{2\sqrt[3]{2}}\left[\pm(\sqrt{3}-1)+(\sqrt{3}+1)i\right] \qquad \{z_1, w_0\}$$

$$\frac{1}{\sqrt[3]{2}}\left(\pm 1 - i\right) \qquad \{z_2, w_2\}.$$

2.7

Sappiamo che, se $z \in \mathbb{C}$ è soluzione dell'equazione algebrica

$$(1) \quad z^8 - (3+i)z^7 - 3i\,z^5 - (1-4i)z^2 + (3+5i)z - 3i = 0,$$

allora il numero complesso coniugato $\bar{z}$ è soluzione dell'equazione coniugata:

$$(2) \quad z^8 - (3-i)z^7 + 3i\,z^5 - (1+4i)z^2 + (3-5i)z + 3i = 0,$$

pertanto le eventuali radici reali della (1) sono anche radici della (2) e quindi anche radici dell'equazione ottenuta sommando membro a membro (1) e (2) (la quale è a coefficienti reali):

$$2\left[z^8-3z^7-z^2+3z\right]=0 \iff$$

$$\iff z\left[z^7-3z^6-z+3\right]=0 \iff$$

$$\iff z\left[z^6(z-3)-(z-3)\right]=0 \iff$$

$$\iff z(z-3)(z^6-1)=0 \qquad (3).$$

Le radici reali della (3) sono:

$$0, \quad 3, \quad \pm 1 .$$

Vediamo quali di queste sono soluzioni della (1). Indichiamo con $P(z)$ il polinomio che figura al primo membro dalla (1):

$$P(0) = -3i$$

$$P(1) = \cancel{1} - (\cancel{3}+i) - 3i - (\cancel{1}-4i) + \cancel{3}+5i - 3i =$$

$$= 2i$$

$$P(-1) = \cancel{1} + (\cancel{3}+i) + 3i - (\cancel{1}-4i) - (\cancel{3}+5i) - 3i =$$

$$= 0$$

$$P(3) = 3^8 - (3+i)3^7 - 3i \cdot 3^5 - (1-4i)3^2 + 3(3+5i) - 3i =$$

$$= 3^8 - 3^8 - 3^2 + 3^2 +$$

$$+ i\left(-3^7 - 3^6 + 36 + 15 - 3\right) =$$

$$= \underbrace{(-3^7 \underbrace{- 3^6 + 48}_{<0}}_{<0})\, i \neq 0 .$$

Dunque l'unica radice reale della (1)

è $z = -1$

2.8

Come nell'esercizio 2.5 cerchiamo di esprimere il numero dato

$$u = -\frac{5(1+2i)^7}{(1-2i)^4} \cdot \frac{3+i}{1+i}$$

nella forma $v \cdot w^3$ $(v, w \in \mathbb{C})$ in modo che le radici cubiche del numero v siano facilmente calcolabili.

Possiamo scrivere:

$$u = -5\,\frac{(1+2i)^6}{(1-2i)^3} \cdot \frac{1+2i}{1-2i} \cdot \frac{3+i}{1+i} =$$

$$= -5\left[\frac{(1+2i)^2}{1-2i}\right]^3 \cdot \frac{(1+2i)^2}{5} \cdot \frac{(3+i)(1-i)}{2} =$$

$$= -\frac{1}{2}\left[\frac{(1+2i)^2}{1-2i}\right]^3 (-3+4i)(4-2i) =$$

$$= -\left[\frac{(1+2i)^2}{1-2i}\right]^3 (-3+4i)(2-i) =$$

$$= -\left[\frac{(1+2i)^2}{1-2i}\right]^3 (-2+11i) ,$$

ma non è facile calcolare le radici cubiche di

$$2-11i = \sqrt{125}\left(\frac{2}{\sqrt{125}} - \frac{11}{\sqrt{125}}\, i\right) .$$

Proviamo allora a scrivere

$$u = -5\,\frac{(1+2i)^6}{(1-2i)^6}\cdot (1+2i)(1-2i)^2\,\frac{3+i}{1+i} =$$

$$= -5\,\frac{(1+2i)^6}{(1-2i)^6}\cdot (1+4)(1-2i)\,\frac{3+i}{1+i} =$$

$$= -25\left[\frac{(1+2i)^2}{(1-2i)^2}\right]^3 (1-2i)\ \frac{(3+i)(1-i)}{2} =$$

$$= -\frac{25}{2}\left[\frac{(1+2i)^2}{(1-2i)^2}\right]^3 (1-2i)(4-2i) =$$

$$= -25\left[\frac{(1+2i)^2}{(1-2i)^2}\right]^3 (1-2i)(2-i) =$$

$$= -25\left[\frac{(1+2i)^2}{(1-2i)^2}\right]^3 (-5i) =$$

$$= \left[5\ \frac{(1+2i)^2}{(1-2i)^2}\right]^3 i \quad ;$$

ne segue, essendo $i = \cos\frac{\pi}{2} + i \operatorname{sen}\frac{\pi}{2}$, che

le radici cubiche cercate sono

$w z_0$, $w z_1$ e $w z_2$,

dove

$$w = 5\,\frac{(1+2i)^2}{(1-2i)^2} = 5\left[\frac{(1+2i)(1+2i)}{(1-2i)(1+2i)}\right]^2 =$$

$$= 5\,\frac{1}{5}\,(-3+4i)^2 = 5-24i\ ,$$

e

$$z_k = \cos\left(\frac{\pi}{6}+\frac{2k\pi}{3}\right) + i\,\mathrm{sen}\left(\frac{\pi}{6}+\frac{2k\pi}{3}\right)$$

$$k = 0, 1, 2\ ,$$

cioè

$$z_0 = \frac{\sqrt{3}}{2} + \frac{1}{2}\,i\ ,$$

$$z_1 = \cos\Big(\underbrace{\frac{\pi}{6}+\frac{2\pi}{3}}_{=\frac{1+4}{6}\pi}\Big) + i\,\mathrm{sen}\left(\frac{\pi}{6}+\frac{2\pi}{3}\right) =$$

$$= \cos\left(\pi - \frac{\pi}{6}\right) + i \operatorname{sen}\left(\pi - \frac{\pi}{6}\right) =$$

$$= -\frac{\sqrt{3}}{2} + \frac{1}{2} i \quad ,$$

$$z_2 = \cos\left(\frac{\pi}{6} + \frac{4\pi}{3}\right) + i \operatorname{sen}\left(\frac{\pi}{6} + \frac{4\pi}{3}\right) =$$

$$\left[\frac{\pi}{6} + \frac{4\pi}{3} = \frac{1+8}{6}\pi = \frac{3}{2}\pi \right]$$

$$= \cos\frac{3}{2}\pi + i \operatorname{sen}\frac{3}{2}\pi = -i \quad ,$$

quindi le soluzioni del quesito sono:

$$(5-24i)\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right) = \boxed{\frac{1}{2}\left[5\sqrt{3}+24 + (5-24\sqrt{3})i\right]} ,$$

$$(5-24i)\left(-\frac{\sqrt{3}}{2} + \frac{1}{2}i\right) = \boxed{\frac{1}{2}\left[-5\sqrt{3}+24 + (5+24\sqrt{3})i\right]} ,$$

$$(5-24i)(-i) = \boxed{-24-5i} .$$