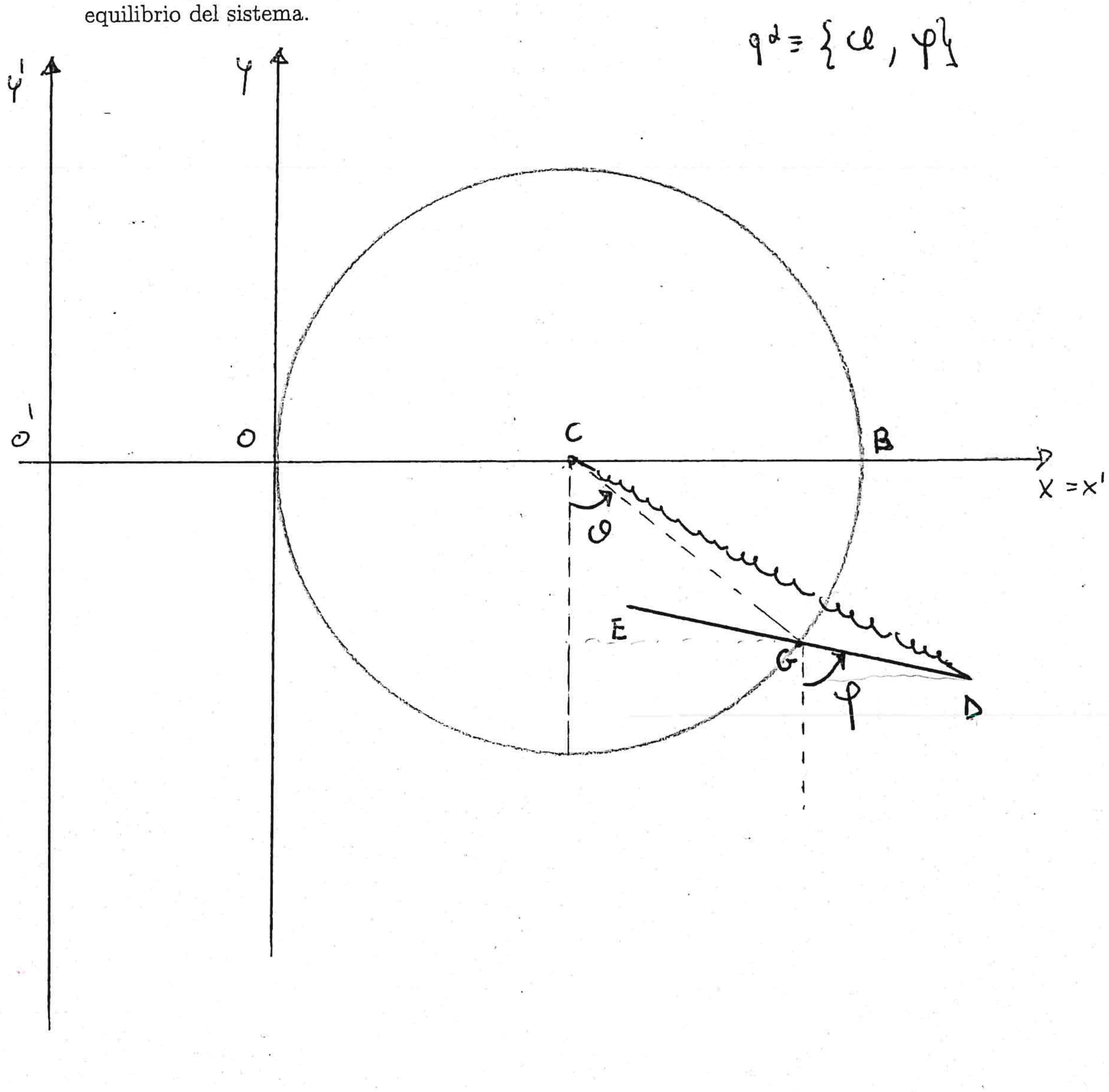
Università degli studi di Catania
Corso di laurea Triennale in Fisica
Prova scritta di Meccanica Analitica
Appello del 02.12.2016

In un piano verticale, si consideri un riferimento fisso $\{o', x', y'\}$ ed un riferimento mobile relativo $\{o, x, y\}$, con gli assi orizzontali x ed x' costantemente sovrapposti, che si muove di moto traslatorio lungo la direzione x' con accelerazione costante $a = g$ (essendo g l'accelerazione di gravitá). Solidale con il riferimento mobile relativo, sia dato un sistema S costituito da una circonferenza di raggio R, centro C e diametro OB sull'asse delle x (vedi figura), e da un'asta ED di lunghezza $2L$, massa m, avente densitá non omogenea $\varrho(s) = \alpha\,(L - s)^2$ (essendo s la distanza di un generico punto p dell'asta a partire da uno dei suoi estremi) con $0 \leq s \leq 2L$, ed α una costante positiva, il cui baricentro G puó scorrere lungo la guida [redacted]circolare. Sul sistema oltre alla forza peso (lungo la verticale) agisce la forza elastica

$$\{F = -k\,(D - C),\, D\} \tag{1}$$

essendo k una costante positiva. Scegliendo le coordinate θ e ϕ come in figura, e supponendo tutti i vincoli lisci, si chiede di determinare nel riferimento mobile relativo:

1. Le configurazioni di equilibrio del sistema, discutendone la stabilitá;
2. Le equazioni del moto e gli eventuali integrali primi del moto;
3. I moti in prima approssimazione attorno alle possibili configurazioni di equilibrio del sistema.



Sia $\rho = a(L-x)^2$ $\qquad 0 \le x \le 2L$

$$m = \int_0^{2L} a(L-x)^2\,dx = a\int_0^{2L}(L^2+x^2-2Lx)\,dx =$$

$$= a\left\{L^2(2L) + \frac{(2L)^3}{3} - 2L\frac{(2L)^2}{2}\right\} =$$

$$= a\left\{2L^3 + \frac{8}{3}L^3 - \frac{8L^3}{2}\right\} = aL^3\left\{2+\frac{8}{3}-4\right\} =$$

$$= aL^3\left\{\frac{8}{3}-2\right\} = a\frac{2}{3}L^3 = m \quad \Rightarrow \quad \boxed{a = \frac{3}{2}\frac{m}{L^3}}$$

calcolo del baricentro

$$G-O = \frac{1}{m}\int_0^{2L} x\,\rho(x)\,dx =$$

$$= \frac{a}{m}\int_0^{2L} x(L-x)^2\,dx =$$

$$= \frac{a}{m}\cdot\int_0^{2L}(L^2x + x^3 - 2Lx^2)\,dx$$

$$= \frac{a}{m}\cdot\left\{L^2\frac{(2L)^2}{2} + \frac{(2L)^4}{4} - 2L\frac{(2L)^3}{3}\right\} =$$

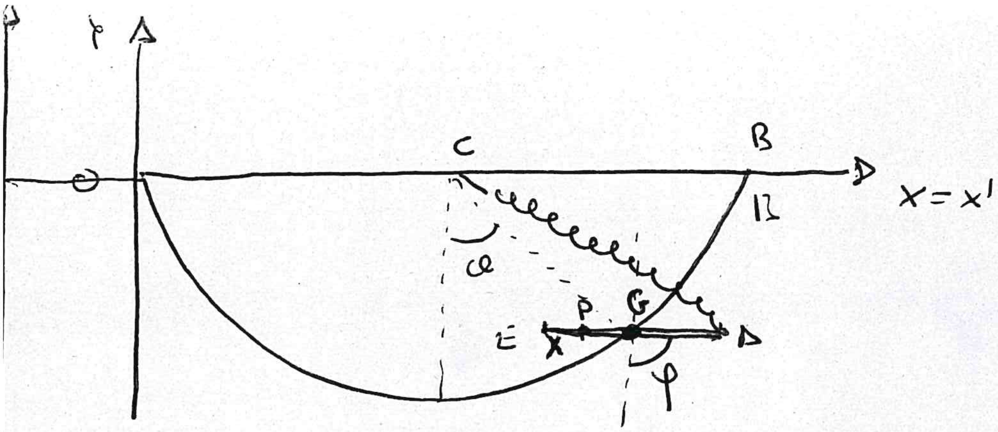
$$= \frac{a}{m}\left\{2L^4 + 4L^4 - \frac{16L^4}{3}\right\} = \frac{a}{m}L^4\left\{6-\frac{16}{3}\right\} =$$

$$= \frac{a}{m}L^4\cdot\frac{2}{3} = \frac{L^4}{m}\cdot\frac{2}{3}\cdot\left(\frac{3}{2}\frac{m}{L^3}\right) = L$$

Allora il baricentro si trova ancora nel punto medio (come ci aspettavamo, in quanto $\rho(x)$ è simmetrica rispetto al punto medio) quindi:

$$E \equiv \{x_G - L\sin\varphi,\ -[|y_G| - L\cos\varphi]\} = \{R\sin\alpha - L\sin\varphi,\ -R\cos\alpha + L\cos\varphi\}$$

$$D = \{x_G + L\sin\varphi,\ -(|y_G| + L\cos\varphi)\} = \{R\sin\alpha + L\sin\varphi,\ -R\cos\alpha - L\cos\varphi\}$$



$G \equiv \{R + R \operatorname{sen}\alpha, -R\cos\alpha\}$ $\qquad C = [R, 0]$

METODO: CALCOLO DELLA T UTILIZZANDO KÖNIG

$$I_{z,E} = \int_0^{2L} x^2 \rho(x)\, dx = a \int_0^{2L} x^2 (L-x)^2\, dx =$$

$$= a \int_0^{2L} (L^2 x^2 + x^4 - 2L x^3) dx =$$

$$= a \left\{ L^2 \cdot \frac{(2L)^3}{3} + \frac{(2L)^5}{5} - 2L \frac{(2L)^4}{4} \right\} =$$

$$= a \left\{ \frac{8}{3} L^5 + \frac{32}{5} L^5 - \frac{32 L^5}{4} \right\} = a L^5 \left\{ \frac{8}{3} + \frac{32}{5} - 8 \right\} =$$

$$= a L^5 \cdot \frac{40 + 96 - 120}{15} = \frac{16}{15} a L^5 = \frac{16}{15} \cdot \frac{3}{2} \frac{m}{L^3} L^5 = \frac{8}{5} m L^2$$

$I_{z,E} = \frac{8}{5} m L^2$ $\qquad I_{z,E} = I_{zG} + m L^2 \Rightarrow I_{zG} = I_{z,E} - m L^2$

$$= \frac{8}{5} m L^2 - m L^2 =$$

$$= \frac{3}{5} m L^2$$

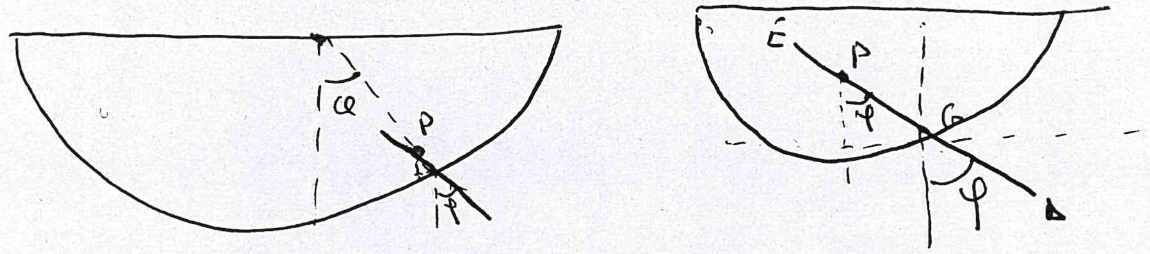
$$T = \frac{1}{2} m \dot{G}^2 + \frac{1}{2} I_{zG} \dot{\varphi}^2$$

$\dot{G} = (R\cos\alpha\, \dot{\alpha}, R \operatorname{sen}\alpha\, \dot{\alpha})$ $\qquad \dot{G}^2 = R^2 \dot{\alpha}^2$

$$\boxed{T = \frac{1}{2} m R^2 \dot{\alpha}^2 + \frac{3}{10} m L^2 \dot{\varphi}^2}$$

"UTILIZZIAMO IL METODO DELLA INTEGRAZIONE PER IL CALCOLO DELLA T"

(3)



$\overline{PG} = \overline{EG} - x = (L - x)$

$P \equiv \{ x_G - (L-x)\sin\varphi \,,\, -[\,|y_G| - (L-x)\cos\varphi\,] \} =$

$\{ R + R\sin\theta - (L-x)\sin\varphi \,,\, -R\cos\theta + (L-x)\cos\varphi \}$

$\dot{P} = \{ R\cos\theta\,\dot\theta - (L-x)\cos\varphi\,\dot\varphi \,,\, R\sin\theta\,\dot\theta - (L-x)\sin\varphi\,\dot\varphi \}$

$\dot{P}^2 = R^2\dot\theta^2 + (L-x)^2\dot\varphi^2 - 2R(L-x)\dot\theta\dot\varphi[\cos\theta\cos\varphi + \sin\theta\sin\varphi]$

DA CUI

$$T = \frac{1}{2} R^2 \dot\theta^2 \int dm + \frac{1}{2}\dot\varphi^2 \int_0^{2L} (L-x)^2 \rho(x)\,dx$$

$$- R[\cos\theta\cos\varphi + \sin\theta\sin\varphi]\dot\theta\dot\varphi \int_0^{2L} (L-x)\rho(x)\,dx$$

$$= \frac{1}{2} m R^2 \dot\theta^2 + \frac{1}{2}\dot\varphi^2 a \int_0^{2L} (L-x)^4\,dx$$

$$- R[\cos\theta\cos\varphi + \sin\theta\sin\varphi]\dot\theta\dot\varphi\, a \int_0^{2} (L-x)^3\,dx$$

PONIAMO $z = L - x$, $dz = -dx$

$$\int_0^{2L} (L-x)^4\,dx = -\int_{L}^{-L} z^4\,dz = \int_{-L}^{L} z^4\,dz = \frac{2}{5} L^5$$

$$\int_0^{2L} (L-x)^3\, dx = -\int_L^{-L} z^3\, dz = \int_{-L}^{L} z^3\, dz = 0$$

QUINDI

$$T = \frac{1}{2} m R^2 \dot{\alpha}^2 + \frac{1}{2} \dot{\varphi}^2 \cdot \left(\frac{3}{2} \frac{m}{L^3} \right) \cdot \frac{2}{5} L^{5}$$

$$\boxed{T = \frac{1}{2} m R^2 \dot{\alpha}^2 + \frac{3}{10} m L^2 \dot{\varphi}^2}$$

CHE È L'OPPOSTO RISULTATO OTTENUTO CON KONIG.

(CALCOLO DEL POTENZIALE)

OSSERVIAMO CHE SE CALCOLIAMO LE FORZE AGENTI AVREMO: FORZA PESO, FORZA ELASTICA, FORZA FITTIZIA DI TRASCINAMENTO

$\{ \underline{F}_P = mg(0,-1) \ , G \}$ $\qquad$ $\underline{a}_\tau = \ddot{O} = g \cdot (1,0)$

$\{ \underline{F}_E = -k(D-C) \ , D \}$ $\qquad$ $\underline{F}_\tau = -m\, \underline{a}_\tau = -mg(1,0)$

$\{ \underline{F}_\tau = mg(-1,0) \ , G \}$

$$U = U_P + U_E + U_\tau = mg(0,-1) \cdot (G-O) - \frac{1}{2} k (D-C)^2$$

$$+ mg(-1,0) \cdot (G-O) =$$

$$D - C = \{ R \sin\alpha + L \sin\varphi \ , -R\cos\alpha - L\cos\varphi \}$$

$$(D-C)^2 = R^2 + L^2 + 2RL \sin\alpha \sin\varphi$$

$$+ 2RL \cos\alpha \cos\varphi = R^2 + L^2 + 2RL [\cos\alpha \cos\varphi + \sin\alpha \sin\varphi]$$

$$= R^2 + L^2 + 2RL \cos(\alpha - \varphi)$$

DA CUI: $\boxed{U = mgR\cos\alpha - kRL\cos(\alpha-\varphi) - mgR\sin\alpha + C}$

$$Q_\alpha = \frac{\partial U}{\partial \alpha} = -mgR\,\text{sen}\,\alpha + KRL\,\text{sen}(\alpha-\varphi) - mgR\cos\alpha$$

$$= -mgR(\text{sen}\,\alpha + \cos\alpha) + KRL\,\text{sen}(\alpha-\varphi)$$

$$Q_\varphi = \frac{\partial U}{\partial \varphi} = -KRL\,\text{sen}(\alpha-\varphi)$$

$$Q_\alpha = KRL\,\text{sen}(\alpha-\varphi) - mgR(\text{sen}\,\alpha + \cos\alpha)$$

$$Q_\varphi = -KRL\,\text{sen}(\alpha-\varphi)$$

CALCOLIAMO DIRETTAMENTE LE SOLLECITAZIONI ⇒ DA CUI IL POTENZIALE

NOTA: OSSERVIAMO CHE:

$$dF_z = -g(1,0)\,dm \qquad \text{DA CUI}$$

$$dQ_\alpha^{z} = g(-1,0)\,dm \cdot \frac{\partial P}{\partial \alpha}$$

$$dQ_\varphi^{z} = g(-1,0)\,dm \cdot \frac{\partial P}{\partial \varphi}$$

$$P = \{R + R\,\text{sen}\,\alpha - (L-x)\,\text{sen}\,\varphi,\ -R\cos\alpha + (L-x)\cos\varphi]$$

$$\frac{\partial P}{\partial \alpha} = [\mp R\cos\alpha,\ R\,\text{sen}\,\alpha] \quad ; \quad \frac{\partial P}{\partial \varphi} = [-(L-x)\cos\varphi,\ -(L-x)\,\text{sen}\,\varphi]$$

$$= -(L-x)[\cos\varphi,\ \text{sen}\,\varphi]$$

$$dQ_\alpha^{z} = -gR\cos\alpha\,dm = -gR\cos\alpha\,\rho(x)\,dx$$

$$dQ_\varphi^{z} = (L-x)\cos\varphi\,dm = (L-x)\cos\varphi\,\rho(x)\,dx$$

DA CUI INTEGRANDO:

$$Q_\alpha^{z} = -mgR\cos\alpha \qquad Q_\varphi^{z} = \cos\varphi \cdot a\int_0^{2}(L-x)^3\,dx =$$

$$= \cos\varphi \cdot a\int^{L} z^3\,dz = 0$$

DA CUI IL POTENZIALE ASSOCIATO ALLE FORZE DI TRASCINAMENTO

$$U = \int Q_\alpha^{\tau} \, d\alpha + c_1(\varphi) = -mgR \int \cos\alpha \, d\alpha + c_1(\varphi) = -mgR\,\text{sen}\,\alpha + c_1(\varphi)$$

$$U = \int \underbrace{Q_\varphi^{\tau}}_{0} \, d\alpha + c_2(\alpha) \Rightarrow c_2(\alpha) = -mgR\,\text{sen}\,\alpha$$

AVREMO QUINDI $c_1(\varphi) = 0$ $U = -mgR\,\text{sen}\,\alpha$.

NOTA: CALCOLO DEL POTENZIALE: PROVIAMO CHE LA FORZA DI TRASCINAMENTO PUÒ ESSERE CALCOLATA SUL G

[struck-through] POTEVAMO CALCOLARE DIRETTAMENTE IL POTENZIALE DELLE FORZE DI TRASCINAMENTO

$$U_P^{\tau} = g(-1,0)\,dm \cdot (P-O) = -g\,[R\,\text{sen}\,\alpha - (L-x)\,\text{sen}\,\varphi]\,dm$$

DA CUI

$$U_P^{\tau} = -mgR\,\text{sen}\,\alpha + g\,\text{sen}\,\varphi \underbrace{\int_0^{2L} (L-x)\,\rho(x)\,dx}_{0} = -mgR\,\text{sen}\,\alpha.$$

CIOÈ COME SE LA FORZA DI TRASCINAMENTO FOSSE DIRETTAMENTE APPLICATA AL BARICENTRO, IL CHE È VERO PERCHÉ È UNA FORZA COSTANTE INFATTI

$$U_P^{\tau} = g(-1,0)\,dm \cdot (P-O) \Rightarrow U_\tau = g(-1,0) \cdot \underbrace{\int (P-O)\,dm}_{m(G-O)} =$$

$$= mg(-1,0) \cdot (G-O).$$

(COME ABBIAMO FATTO ALL'INIZIO.

ABBIAMO COSÌ CHE:

$$\begin{cases} T = \frac{1}{2} mR^2\dot\alpha^2 + \frac{3}{10} mL^2\dot\varphi^2 \\ U = mgR(\cos\alpha - \text{sen}\,\alpha) - KRL\cos(\alpha-\varphi) \end{cases}$$

$$\begin{cases} Q_\alpha = KRL\,\text{sen}(\alpha-\varphi) - mgR(\text{sen}\,\alpha + \cos\alpha) \\ Q_\varphi = -KRL\,\text{sen}(\alpha-\varphi) \end{cases}$$

"EQUILIBRIO E STABILITÀ"

$Q_\varphi = -KR\,\text{sen}(\alpha-\varphi) = 0 \qquad \alpha - \varphi = 0, \pi \Rightarrow \varphi = \alpha \qquad \boxed{\alpha = \varphi + \pi}$

$Q_\alpha = 0 \Rightarrow \text{sen}\,\alpha + \cos\alpha = 0 \Rightarrow \text{sen}\,\alpha = -\cos\alpha \Leftarrow$

$\Rightarrow \alpha = \frac{3}{4}\pi, -\frac{\pi}{4}$

Quindi $\quad \alpha = -\frac{\pi}{4} \qquad \varphi = -\frac{\pi}{4} \quad$ e $\varphi = -\frac{\pi}{4} - \pi = -\frac{5}{4}\pi = \frac{3}{4}\pi$

$\alpha = \frac{3}{4}\pi \qquad \varphi = \frac{3}{4}\pi \quad$ e $\varphi = \frac{3}{4}\pi - \pi = -\frac{\pi}{4}$

$S = (\alpha, \varphi) = \left\{ s_1 \equiv \left(-\frac{\pi}{4}, -\frac{\pi}{4}\right), s_2 \equiv \left(-\frac{\pi}{4}, \frac{3}{4}\pi\right), s_3 \equiv \left(\frac{3}{4}\pi, \frac{3}{4}\pi\right) \quad s_4 = \left(\frac{3}{4}\pi, -\frac{\pi}{4}\right) \right\}$

$$\begin{cases} Q_{\varphi\varphi} = KRL\cos(\alpha-\varphi) \\ U_{\alpha\alpha} = KRL\cos(\alpha-\varphi) - mgR(\cos\alpha - \text{sen}\,\alpha) \\ U_{\alpha\varphi} = U_{\varphi\alpha} = -KRL\cos(\alpha-\varphi) \end{cases}$$

<u>Da qui</u>: $\quad U_{\varphi\varphi}\big|_{s_1, s_3} = KRL > 0$

$$\begin{cases} U_{\alpha\alpha}\big|_{s_3} = KRL + mgR\sqrt{2} \\ U_{\alpha\alpha}\big|_{s_1} = KRL - mgR\sqrt{2} \end{cases}$$

$U_{\alpha\varphi}\big|_{s_1; s_3} = -KRL < 0$

$$H_{s_1} = \begin{vmatrix} KRL - mgR\sqrt{2} & -KRL \\ -KRL & KRL \end{vmatrix} = -\sqrt{2}\, mgKLR^2 < 0$$

s_1 INSTABILE

$$H_{s_2} = \begin{vmatrix} KRL + \sqrt{2}\, mgR & -KRL \\ -KRL & KRL \end{vmatrix} = \sqrt{2}\, mgKLR^2 > 0$$

$U_{\varphi\varphi} > 0 \quad H > 0$ NO MAX **s_3 INSTABILE**

$$\begin{cases} U_{\varphi\varphi}\big|_{s_2} = -KRL \\ U_{\alpha\alpha}\big|_{s_2} = -KRL + \sqrt{2}\, mgR \\ U_{\alpha\varphi}\big|_{s_2} = KRL \end{cases}$$

$$H = \begin{vmatrix} -\sqrt{2}\, mgR - KRL & KRL \\ KRL & -KRL \end{vmatrix}$$

$$= (KRL)^2 + \sqrt{2}\, mgKR^2L - (KRL)^2$$
$$= +\sqrt{2}\, mgKLR^2 > 0$$

$U_{\varphi\varphi}\big|_{s_2} < 0 \quad H > 0$

MAX **s_2 STABILE**

$$\begin{cases} U_{\alpha\alpha}\big|_{s_4} = -KRL + \sqrt{2}\, mgR \\ U_{\varphi\varphi}\big|_{s_4} = -KRL \\ U_{\alpha\varphi}\big|_{s_4} = KRL \end{cases}$$

$$H = \begin{vmatrix} \sqrt{2}\, mgR - KRL & KRL \\ KRL & -KRL \end{vmatrix}$$

$$= -\sqrt{2}\, mgKLR^2 < 0$$

NO MAX $\Rightarrow$ **s_4 INSTABILE**

EQUAZIONI DEL MOTO

$$T = \frac{1}{2} m R^2 \dot{\alpha}^2 + \frac{3}{10} m L^2 \dot{\varphi}^2$$

$$\frac{\partial T}{\partial \dot{\alpha}} = m R^2 \dot{\alpha}, \quad \frac{d}{dt}\frac{\partial T}{\partial \dot{\alpha}} = m R^2 \ddot{\alpha}, \quad \frac{\partial T}{\partial \alpha} = 0$$

$$\frac{\partial T}{\partial \dot{\varphi}} = \frac{3}{5} m L^2 \dot{\varphi}, \quad \frac{d}{dt}\frac{\partial T}{\partial \dot{\varphi}} = \frac{3}{5} m L^2 \ddot{\varphi}, \quad \frac{\partial T}{\partial \varphi} = 0$$

da cui:

1) $$m R^2 \ddot{\alpha} = KRL \sin(\varphi - \alpha) - mgR(\sin\alpha + \cos\alpha) = Q_\alpha$$

2) $$\frac{3}{5} m L^2 \ddot{\varphi} = -KRL \sin(\alpha - \varphi) = Q_\varphi$$

LE FORZE SONO CONSERVATIVE QUINDI COME INTEGRALE PRIMO AVREMO SOLTANTO L'ENERGIA $E = T - U$

"MOTI IN PRIMA APPROSSIMAZIONE"

CONSIDERIAMO I MOTI APPROSSIMATI ATTORNO LA $S_2 \equiv \left(-\frac{\pi}{4}, \frac{3}{4}\pi\right)$

LINEARIZZIAMO AVREMO:

$$m R^2 \ddot{\alpha} = \cancel{Q_\alpha\big|_{S_2}} + \frac{\partial Q_\alpha}{\partial \alpha}\bigg|_{S_2}\left(\alpha + \frac{\pi}{4}\right) + \frac{\partial Q_\alpha}{\partial \varphi}\bigg|_{S_2}\left(\varphi - \frac{3}{4}\pi\right)$$

$$\frac{\partial Q_\alpha}{\partial \alpha}\bigg|_{S_2} = U_{\alpha\alpha}\big|_{S_2} = -KRL - \sqrt{2}\, mgR$$

$$\frac{\partial Q_\alpha}{\partial \varphi}\bigg|_{S_2} = U_{\alpha\varphi}\big|_{S_2} = KRL$$

da cui:

$$mR^2\ddot{\alpha} = -(KRL+\sqrt{2}\,mgR)\left(\alpha+\frac{\pi}{4}\right) + KRL\left(\varphi-\frac{3}{4}\pi\right)$$

Da 2ª equazione

$$\Rightarrow \frac{3}{5}mL^2\ddot{\varphi} = \cancel{Q_\varphi} + \left.\frac{\partial Q_\varphi}{\partial \alpha}\right|_{S_2}\left(\alpha+\frac{\pi}{4}\right) + \left.\frac{\partial Q_\varphi}{\partial \varphi}\right|_{S_2}\left(\varphi-\frac{3}{4}\pi\right)$$

$$\frac{\partial Q_\varphi}{\partial \alpha} = U_{\varphi\alpha}\Big|_{S_2} = KRL \qquad \left.\frac{\partial Q_\varphi}{\partial \varphi}\right|_{S_2} = U_{\varphi\varphi}\Big|_{S_2} = -KRL$$

$$\frac{3}{5}mL^2\ddot{\varphi} = KRL\left(\alpha+\frac{\pi}{4}\right) - KRL\left(\varphi-\frac{3}{4}\pi\right)$$

da cui

$$\begin{cases} mR^2\ddot{\alpha} + (KRL+\sqrt{2}\,mgR)\left(\alpha+\frac{\pi}{4}\right) - KRL\left(\varphi-\frac{3}{4}\pi\right) = 0 \\ \frac{3}{5}mL^2\ddot{\varphi} - KRL\left(\alpha+\frac{\pi}{4}\right) + KRL\left(\varphi-\frac{3}{4}\pi\right) = 0 \end{cases}$$

Cerco soluzioni $\alpha+\frac{\pi}{4} = \alpha_0 e^{\lambda t}$ $\quad \varphi-\frac{3}{4}\pi = \varphi_0 e^{\lambda t}$

$$\left[mR^2\lambda^2 + (KRL+\sqrt{2}\,mgR)\right]\alpha_0 - KRL\,\varphi_0 = 0$$

$$\left[\frac{3}{5}mL^2\lambda^2 + KRL\right]\varphi_0 - KRL\,\alpha_0 = 0$$

~~$KRL\left[mR^2\lambda^2 + (KRL+\sqrt{2}mgR)\right] + KRL\left[\frac{3}{5}mL^2\lambda^2 + KRL\right] = 0$~~

~~$mKRL\left[R^2 + \frac{3}{5}L^2\right]\lambda$~~

$$\left(\frac{3}{5}m^2R^2L^2\right)\lambda^4 + \left[mKR^3L + \frac{3}{5}mL^2(KRL+\sqrt{2}\,mgR)\right]\lambda^2$$

$$+ \cancel{(KRL)^2} + \sqrt{2}\,mgKLR^2 - \cancel{(KRL)^2} = 0$$

AVREMO QUINDI L'EQUAZIONE (PONENDO $\lambda^2 = z$)

$$a z^2 + b z + c = 0 \quad \text{CON} \quad \begin{cases} a = \frac{3}{5} m^2 R^2 L^2 > 0 \\ b = mKR^3 L + \frac{3}{5} m L^2 (KRL + \sqrt{2}\, mgR) > 0 \\ c = \sqrt{2}\, mgKLR^2 > 0 \end{cases}$$

SE VERIFICHIAMO CHE (VEDI RETRO) $\Delta = b^2 - 4ac \geq 0$ AVREMO RADICI REALI, E UTILIZZEREMO LA REGOLA DI CARTESIO

DA CUI ESSENDO LE DUE RADICI

z_1, z_2 TALI CHE $z_1 + z_2 = -\frac{b}{a} < 0$ $\quad z_1 \cdot z_2 = \frac{c}{a} > 0$

$z_1, z_2 < 0$ DA CUI $\lambda_{1,2} = \pm i \sqrt{|z_1|}$ $\quad \lambda_{3,4} = \pm i \sqrt{|z_2|}$

QUINDI COMPOSIZIONE DI MOTI ARMONICI.

CONSIDERIAMO I MOTI APPROSSIMATI ATTORNO AD $S_1 = \left(-\frac{\pi}{4}, -\frac{\pi}{4}\right)$

DA CUI LINEARIZZANDO AVREMO:

$$m R^2 \ddot{\vartheta} = \cancel{Q_\vartheta}\Big|_{S_1} + \frac{\partial Q_\vartheta}{\partial \vartheta}\Big|_{S_1} \left(\vartheta + \frac{\pi}{4}\right) + \frac{\partial Q_\vartheta}{\partial \varphi}\Big|_{S_1} \left(\varphi + \frac{\pi}{4}\right)$$

$$\frac{\partial Q_\vartheta}{\partial \vartheta}\Big|_{S_1} = KRL - mgR\sqrt{2}$$

$$\frac{\partial Q_\vartheta}{\partial \varphi}\Big|_{S_1} = -KRL$$

DA CUI

$$m R^2 \ddot{\vartheta} = (KRL - mgR\sqrt{2})\left(\vartheta + \frac{\pi}{4}\right) - KRL\left(\varphi + \frac{\pi}{4}\right)$$

DALLA 2ª EQUAZIONE

$$\frac{3}{5} m L^2 \ddot{\varphi} = \cancel{Q_\varphi}\Big|_{S_1} + \frac{\partial Q_\varphi}{\partial \vartheta}\Big|_{S_1} \left(\vartheta + \frac{\pi}{4}\right) + \frac{\partial Q_\varphi}{\partial \varphi}\Big|_{S_1} \left(\varphi + \frac{\pi}{4}\right)$$

$$\frac{\partial Q_\varphi}{\partial \vartheta}\Big|_{S_1} = -KRL \qquad \frac{\partial Q_\varphi}{\partial \varphi}\Big|_{S_1} = KRL$$

SEMPLIFICHIAMO I COEFF. DEL TRINOMIO

$$\bar a z^2 + \bar b z + \bar c = 0 \qquad \begin{cases} \bar a = \frac{3}{5} m R L > 0 \\ \bar b = K R^2 + \frac{3}{5} L \left(K L + \sqrt{2}\, m g \right) > 0 \\ \bar c = \sqrt{2}\, g K R \end{cases}$$

$$\bar\Delta = \bar b^2 - 4 \bar a \bar c = \left[K R^2 + \frac{3}{5} L \left(K L + \sqrt{2}\, m g \right) \right]^2 - \frac{12}{5} \sqrt{2}\, m g K L R^2$$

$$= (K R^2)^2 + \left(\frac{3}{5} L \right)^2 \underbrace{\left(K L + \sqrt{2}\, m g \right)^2}_{(KL - \sqrt{2} mg)^2 + 4\sqrt{2} mg KL} + \frac{6}{5} K R^2 L \left(K L - \sqrt{2}\, m g \right)$$

$$= \underbrace{\left[K R^2 + \frac{3}{5} L \left(K L - \sqrt{2}\, m g \right) \right]^2}_{\geq 0} + \underbrace{\frac{36}{25} \sqrt{2}\, m g K L^3}_{> 0} > 0$$

DA CUI IL Δ DEL TRINOMIO $a z^2 + b z + c = 0$ È TALE

CHE $\Delta > 0 \Rightarrow$ RADICI REALI.

da cui:

$$\frac{3}{5} m L^2 \ddot{\varphi} = -KRL\left(\alpha+\frac{\pi}{4}\right) + KRL\left(\varphi+\frac{\pi}{4}\right)$$

da cui le due equaz. linearizzate

$$m R^2 \ddot{\alpha} + (mgR\sqrt{2} - KRL)\left(\alpha+\frac{\pi}{4}\right) + KRL\left(\varphi+\frac{\pi}{4}\right) = 0$$

$$\frac{3}{5} m L^2 \ddot{\varphi} + KRL\left(\alpha+\frac{\pi}{4}\right) - KRL\left(\varphi+\frac{\pi}{4}\right) = 0$$

cerco soluzioni del tipo $\alpha+\frac{\pi}{4} = \alpha_0 e^{\lambda t}$ $\quad \varphi+\frac{\pi}{4} = \varphi_0 e^{\lambda t}$

da cui il sistema:

$$\left[m R^2 \lambda^2 + (mgR\sqrt{2} - KRL)\right]\alpha_0 + (KRL)\varphi_0 = 0$$

$$(KRL)\alpha_0 + \left[\frac{3}{5} m L^2 \lambda^2 - KRL\right]\varphi_0 = 0$$

da cui l'equazione:

$$\left(\frac{3}{5} m^2 R^2 L^2\right)\lambda^4 + \left[-mKR^3L + \frac{3}{5} m L^2 (mgR\sqrt{2} - KRL)\right]\lambda^2$$

$$-KRL(mgR\sqrt{2} - KRL) - (KRL)^2 = 0$$

$$\left(\frac{3}{5} m^2 R^2 L^2\right)\lambda^4 + mL\left[\frac{3}{5} L (mgR\sqrt{2} - KRL) - KR^3\right]\lambda^2$$

$$- mgKLR^2\sqrt{2} = 0$$ ⇐ (in questo caso è banale verificare che $\Delta > 0$)

da cui l'equazione $a z^2 + b z + c = 0$ con $a > 0$, $c < 0$

da cui essendo $z_1 \cdot z_2 = \frac{c}{a} < 0$ avremo certamente

una radice positiva ed una negativa da cui ad esempio

$z_1 > 0$ e $z_2 < 0 \Rightarrow \lambda_{1,2} = \pm\sqrt{z_1}$ moti iperbolici

c) MOTI APPROSSIMATI ATTORNO AA $S_3 = \left(\frac{3}{4}\pi, \frac{3}{4}\pi\right)$

$$mR^2\ddot{\theta} = \cancel{Q_\theta\big|_{S_3}} + \frac{\partial Q_\theta}{\partial\theta}\bigg|_{S_3}\left(\theta - \frac{3}{4}\pi\right) + \frac{\partial Q_\theta}{\partial\varphi}\bigg|_{S_3}\left(\varphi - \frac{3}{4}\pi\right)$$

$$= \left(KRL + mgR\sqrt{2}\right)\left(\theta - \frac{3}{4}\pi\right) - KRL\left(\varphi - \frac{3}{4}\pi\right)$$

$$\frac{3}{5}mL^2\ddot{\varphi} = \cancel{Q_\varphi\big|_{S_3}} + \frac{\partial Q_\varphi}{\partial\theta}\bigg|_{S_3}\left(\theta - \frac{3}{4}\pi\right) + \frac{\partial Q_\varphi}{\partial\varphi}\bigg|_{S_3}\left(\varphi - \frac{3}{4}\pi\right) =$$

$$= -KRL\left(\theta - \frac{3}{4}\pi\right) + KRL\left(\varphi - \frac{3}{4}\pi\right)$$

DA CUI LE DUE EQUAZIONI:

$$\begin{cases} mR^2\ddot{\theta} - \left(KRL + mgR\sqrt{2}\right)\left(\theta - \frac{3}{4}\pi\right) + KRL\left(\varphi - \frac{3}{4}\pi\right) = 0 \\ \frac{3}{5}mL^2\ddot{\varphi} + (KRL)\left(\theta - \frac{3}{4}\pi\right) - KRL\left(\varphi - \frac{3}{4}\pi\right) = 0 \end{cases}$$

CERCO SOLUZIONI $\theta - \frac{3}{4}\pi = \theta_0 e^{\lambda t}$ $\quad \varphi - \frac{3}{4}\pi = \varphi_0 e^{\lambda t}$

DA CUI IL SISTEMA:

$$\left[mR^2\lambda^2 - \left(KRL + mgR\sqrt{2}\right)\right]\theta_0 + KRL\,\varphi_0 = 0$$

$$KRL\,(\theta_0) + \left[\frac{3}{5}mL^2\lambda^2 - KRL\right]\varphi_0 = 0$$

DA CUI L'EQUAZIONE AGLI AUTOVAL.

$$\left(\frac{3}{5}m^2R^2L^2\right)\lambda^4 + \left[-mKR^3L - \frac{3}{5}mL^2\left(KRL + mgR\sqrt{2}\right)\right]\lambda^2$$

$$+ KRL\left(\cancel{KRL} + mgR\sqrt{2}\right) - \cancel{(KRL)^2} = 0$$

NOTA: IN QUESTO CASO CHE $\Delta > 0$ SI PROVA COME NEL CASO a)

DA CUI L'EQUAZIONE DEL TIPO $az^2 + bz + c = 0$

CON $a = \frac{3}{5}m^2R^2L^3 > 0 \quad b = -\left\{\frac{3}{5}mL^2\left(KRL + mgR\sqrt{2}\right) + mKR^3L\right\} < 0$

$c = mgKLR^3\sqrt{2} > 0$

da cui:

$$z_1 + z_2 = -\frac{b}{a} > 0 \quad ; \quad z_1 \cdot z_2 = \frac{c}{a} > 0$$

avremo quindi due soluzioni $z_1 > 0$ $z_2 > 0$

quindi **moti iperbolici**

$$\lambda_{1,2} = \pm\sqrt{z_1} \qquad \lambda_{3,4} = \pm\sqrt{z_2}$$

d) Moti approssimati attorno a $S_4 \equiv \left(\frac{3}{4}\pi, -\frac{\pi}{4}\right)$ (ϑ, φ)

$$mR^2\ddot{\vartheta} = \cancel{Q_\vartheta}\Big|_{S_4} + \frac{\partial Q_\vartheta}{\partial \vartheta}\Big|_{S_4}\left(\vartheta - \frac{3}{4}\pi\right) + \frac{\partial Q_\vartheta}{\partial \varphi}\Big|_{S_4}\left(\varphi + \frac{\pi}{4}\right)$$

$$= \left(-KRL + \sqrt{2}\, mgR\right)\left(\vartheta - \frac{3}{4}\pi\right) + KRL\left(\varphi + \frac{\pi}{4}\right)$$

$$\frac{3}{5} mL^2\ddot{\varphi} = \cancel{Q_\varphi}\Big|_{S_4} + \frac{\partial Q_\varphi}{\partial \vartheta}\Big|_{S_4}\left(\vartheta - \frac{3}{4}\pi\right) + \frac{\partial Q_\varphi}{\partial \varphi}\Big|_{S_4}\left(\varphi + \frac{\pi}{4}\right) =$$

$$= KRL\left(\vartheta - \frac{3}{4}\pi\right) - KRL\left(\varphi + \frac{\pi}{4}\right)$$

$$\begin{cases} mR^2\ddot{\vartheta} + \left(-\sqrt{2}\, mgR + KRL\right)\left(\vartheta - \frac{3}{4}\pi\right) - KRL\left(\varphi + \frac{\pi}{4}\right) = 0 \\ \frac{3}{5} mL^2\ddot{\varphi} - KRL\left(\vartheta - \frac{3}{4}\pi\right) + KRL\left(\varphi + \frac{\pi}{4}\right) = 0 \end{cases}$$

cerco soluzioni $\vartheta - \frac{3}{4}\pi = \vartheta_0 e^{\lambda t}$ $\quad \varphi + \frac{\pi}{4} = \varphi_0 e^{\lambda t}$

da cui:

$$\left[mR^2\lambda^2 + \left(KRL - \sqrt{2}\, mgR\right)\right]\vartheta_0 - KRL\,\varphi_0 = 0$$

$$-KRL\,\vartheta_0 + \left[\frac{3}{5} mL^2\lambda^2 + KRL\right]\varphi_0 = 0$$

$$\Rightarrow \left[\frac{3}{5} m^2R^2L^2\right]\lambda^4 + \left[mKR^3L + \frac{3}{5} mL^2\left(KRL - \sqrt{2}\, mgR\right)\right]\lambda^2$$

$$+KRL\,(\cancel{KRL}-\sqrt{2}\,mgR)\;-\cancel{(KRL)^2}=0$$

da cui

$$a\,z^2+b\,z+c=0 \quad \text{con}$$

$$a=\frac{3}{5}m^2R^2L^2>0$$

$$b=mL\left\{KR^3+\frac{3}{5}L\,(KRL-\sqrt{2}\,mgR)\right\}$$

$$c=-\sqrt{2}\,mgKR^2L<0$$

(in questo caso che $\Delta>0$ si prova banalmente)

da cui:

$$z_1+z_2=-\frac{b}{a} \qquad \text{ed} \qquad z_1\cdot z_2=+\frac{c}{a}<0$$

quindi avremo una soluzione positiva ed una negativa.

Ad esempio sia $z_1>0$ e $z_2<0$ da cui

$\lambda_1,\lambda_2=\pm\sqrt{z_1}$ moti iperbolici

$\lambda_3,\lambda_4=\pm i\sqrt{|z_2|}$ moti armonici

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