DEFINIZIONE PARAMETRICA DI UNA SUPERFICIE

SIA $\Delta \equiv [a,b] \times [c,d] \subseteq R^2$ (INTERNAMENTE CONNESSO DI R^2)

$\gamma : \Delta \to R^3$ DEFINISCE UNA SUPERFICIE REGOLARE Σ SE VALGONO LE 3 PROPRIETA' SOTTOSTANTI

1)
$$\gamma = \begin{cases} x^1 = x^1(u,v) \\ x^2 = x^2(u,v) \\ x^3 = x^3(u,v) \end{cases} \quad (1)$$

RAPPRESENTAZIONE PARAMETRICA DELLA SUPERFICIE, CON $(u,v) \in \Delta$

LE FUNZIONI x^1, x^2, x^3 CLASSE $C^1(\Delta)$

2) SE L'APPLICAZIONE E' TALE CHE PRESI $\forall$ DUE PUNTI $P_1, P_2 \in \Delta$ (DI CUI ALMENO UNO INTERNO) $\gamma(P_1) \neq \gamma(P_2)$. (2)

3) SE $\forall P \in \Delta$ LA MATRICE JACOBIANA

$$J = \frac{\partial(x^1, x^2, x^3)}{\partial(u,v)} = \begin{pmatrix} \frac{\partial x^1}{\partial u} & \frac{\partial x^1}{\partial v} \\ \frac{\partial x^2}{\partial u} & \frac{\partial x^2}{\partial v} \\ \frac{\partial x^3}{\partial u} & \frac{\partial x^3}{\partial v} \end{pmatrix}$$

HA RANGO 2. (3)

CIOE' SE CONSIDERIAMO I MINORI

$$A_\gamma(u,v) = \frac{\partial(x^2, x^3)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x^2}{\partial u} & \frac{\partial x^2}{\partial v} \\ \frac{\partial x^3}{\partial u} & \frac{\partial x^3}{\partial v} \end{vmatrix} = \left(\frac{\partial x^2}{\partial u}\frac{\partial x^3}{\partial v} - \frac{\partial x^2}{\partial v}\frac{\partial x^3}{\partial u} \right) \quad (4)$$

$$B_\gamma(u,v) = -\frac{\partial(x^1, x^3)}{\partial(u,v)} = -\begin{vmatrix} \frac{\partial x^1}{\partial u} & \frac{\partial x^1}{\partial v} \\ \frac{\partial x^3}{\partial u} & \frac{\partial x^3}{\partial v} \end{vmatrix} = \left(\frac{\partial x^1}{\partial v}\frac{\partial x^3}{\partial u} - \frac{\partial x^1}{\partial u}\frac{\partial x^3}{\partial v} \right) \quad (5)$$

$$C_\gamma(u,v) = \frac{\partial(x^1, x^2)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x^1}{\partial u} & \frac{\partial x^1}{\partial v} \\ \frac{\partial x^2}{\partial u} & \frac{\partial x^2}{\partial v} \end{vmatrix} = \left(\frac{\partial x^1}{\partial u}\frac{\partial x^2}{\partial v} - \frac{\partial x^1}{\partial v}\frac{\partial x^2}{\partial u} \right) \quad (6)$$

Abbiamo almeno un minore con determinante $\neq 0$.
Osserviamo che se consideriamo i due vettori

$$\frac{\partial x^1}{\partial u}\underline{e}_1 + \frac{\partial x^2}{\partial u}\underline{e}_2 + \frac{\partial x^3}{\partial u}\underline{e}_3 = \frac{\partial P}{\partial u} = \underline{e}_u = \underline{z}_u \quad (7) \qquad (P-O) = \underline{z}$$

$$\frac{\partial x^1}{\partial v}\underline{e}_1 + \frac{\partial x^2}{\partial v}\underline{e}_2 + \frac{\partial x^3}{\partial u}\underline{e}_3 = \frac{\partial P}{\partial v} = \underline{e}_v = \underline{z}_v \quad (8)$$

e costruiamo il loro prodotto vettoriale euclideo cartesiano

$$\underline{e}_u \wedge \underline{e}_v = \begin{vmatrix} \underline{e}_1 & \underline{e}_2 & \underline{e}_3 \\ \frac{\partial x^1}{\partial u} & \frac{\partial x^2}{\partial u} & \frac{\partial x^3}{\partial u} \\ \frac{\partial x^1}{\partial v} & \frac{\partial x^2}{\partial v} & \frac{\partial x^3}{\partial v} \end{vmatrix} = \underline{z}_u \wedge \underline{z}_v =$$

$$= \left(\frac{\partial x^2}{\partial u}\frac{\partial x^3}{\partial v} - \frac{\partial x^2}{\partial v}\frac{\partial x^3}{\partial u}\right)\underline{e}_1 + \left(\frac{\partial x^1}{\partial v}\frac{\partial x^3}{\partial u} - \frac{\partial x^1}{\partial u}\frac{\partial x^3}{\partial v}\right)\underline{e}_2$$

$$+ \left(\frac{\partial x^1}{\partial u}\frac{\partial x^2}{\partial v} - \frac{\partial x^1}{\partial v}\frac{\partial x^2}{\partial u}\right)\underline{e}_3 = \underline{N} = N_1\underline{e}_1 + N_2\underline{e}_2 + N_3\underline{e}_3 \quad (9)$$

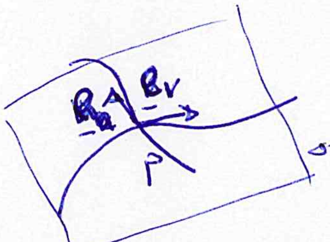
dove $N_1 = A_\gamma(u,v)$ $\quad N_2 = B_\gamma(u,v)$ $\quad N_3 = C_\gamma(u,v)$ $\quad$ (9B)

Dire quindi che J ha rango 2, significa dire

che il vettore $\boxed{\underline{N} = \underline{e}_u \wedge \underline{e}_v \text{ è diverso da zero}}$ (10)

Osserviamo inoltre che:

il vettore $\underline{z}_u = \underline{e}_u = \frac{\partial P}{\partial u}$ è tangente alla curva passante per $P \in \Sigma$ (ottenuta fissando v e facendo variare u)



← piano tangente in $P \in \Sigma$

Analogamente $\underline{z}_v = \underline{e}_v = \frac{\partial P}{\partial v}$ è tangente alla curva passante per $P \in \Sigma$ (ottenuta variando v ed avendo fissato u)

cioè $\underline{e}_u$ ed $\underline{e}_v$ appartengono al piano tangente in $P \in \Sigma$

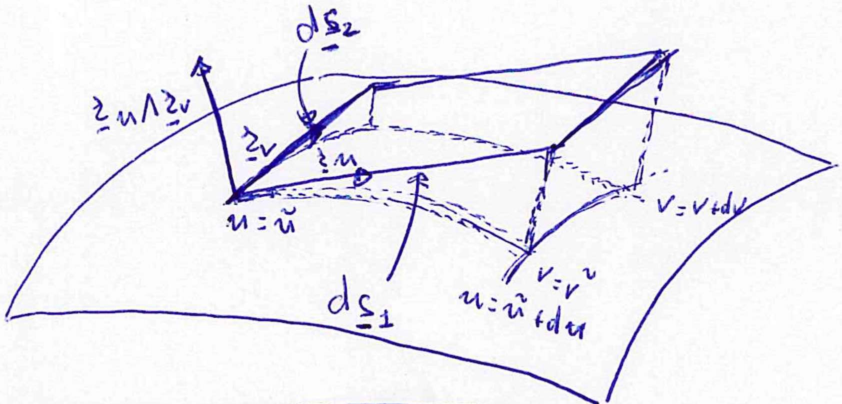
$\{\underline{e}_u, \underline{e}_v\}$ BASE DEL PIANO TANGENTE $\Sigma_P^{(t)}$

e che il vettore $\underline{N} = \underline{e}_u \wedge \underline{e}_v$ è ortogonale al piano tangente $\Sigma_P^{(t)}$ in $P \in \Sigma$. (NOTA: potremmo considerare $-N = \underline{e}_v \wedge \underline{e}_u$ come normale)

inoltre $\forall\, \underline{u}, \underline{v} \in D$ $\underline{e}_u \wedge \underline{e}_v \neq 0$ quindi $\underline{e}_v$ ed $\underline{e}_u$ sono vettori linearmente indipendenti (base di $\Sigma_P^{(t)}$) $\forall P \in \Sigma$

Consideriamo l'integrale di superficie

$$\mathrm{AREA}(\Sigma) = \int_{\Sigma} d\sigma = \iint_D \underbrace{|\overbrace{\underline{e}_u \wedge \underline{e}_v}^{\underline{\xi}_u \wedge \underline{\xi}_v}|\, du\, dv}_{d\sigma} \qquad (11)$$



$\boxed{\text{PROVIAMO CHE } d\sigma = |\underline{e}_u \wedge \underline{e}_v|\, du\, dv}$ (12)

Data la superficie $(\Sigma, \underline{\xi})$, i lati del parallelogramma curvilineo sono approssimati, a meno di infinitesimi di ordine superiore a du e a dv, da quelli del parallelogramma rettilineo individuati dai vettori $d\underline{S}_1 = \underline{\xi}_u\, dv$ e $d\underline{S}_2 = \underline{\xi}_v\, du$

(13) $d\underline{S}_1 = dx^1 \underline{e}_1 + dx^2 \underline{e}_2 + dx^3 \underline{e}_3 = \frac{\partial x^1}{\partial u} du\, \underline{e}_1 + \frac{\partial x^2}{\partial u} du\, \underline{e}_2 + \frac{\partial x^3}{\partial u} du$

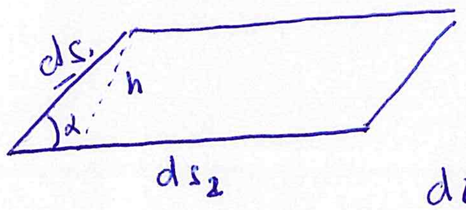
$= \underline{\xi}_u\, du$ (dove x^1, x^2, x^2 hanno variazioni solo nella variabile u (v è costante essendo $\tilde{v} = v$))

(14) $d\underline{S}_2 = dx^1 \underline{e}_2 + dx^2 \underline{e}_2 + dx^3 \underline{e}_3 = \frac{\partial x^1}{\partial v} dv\, \underline{e}_1 + \frac{\partial x^2}{\partial v} dv\, \underline{e}_2 + \frac{\partial x^3}{\partial v} dv$

$= \underline{\xi}_v\, dv$ (in questo caso x^1, x^2, x^3 hanno variazioni solo nella

$|d\underline{s}_1| = |\underline{\xi}_u|\,du \qquad |d\underline{s}_2| = |\underline{\xi}_v|\,dv$

MA L'AREA DI UN PARALLELOGRAMMA RETTILINEO AVENTE COME LATI I DUE VETTORI $d\underline{s}_1$ e $d\underline{s}_2$ È DATA DAL MODULO DEL LORO PRODOTTO VETTORIALE (AREA = BASE × h)



$h = |ds_1| \operatorname{sen}\alpha$

DA CUI $d\sigma = |ds_1| \operatorname{sen}\alpha\, |ds_2| =$

$$d\sigma = |\underline{e}_u \wedge \underline{e}_v|\,du\,dv \qquad (15)$$

ESSENDO $\underline{e}_u = \underline{\xi}_u \quad \underline{e}_v = \underline{\xi}_v$

QUINDI IN GENERALE (SIA CHE CONSIDERI IL VETTORE $\underline{N}$ O ANCHE $-\underline{N}$)

$$d\sigma = |\underline{\xi}_u \wedge \underline{\xi}_v|\,du\,dv = |N|\,du\,dv \qquad (16) \qquad \text{C.V.D.}$$

INFINE L'INTEGRALE DI SUPERFICIE DI UNA FUNZIONE $F(x,y,z)$ È DATO DA:

$$\int_{\partial\Sigma} F(x,y,z)\,d\sigma = \iint F[x^1(u,v), x^2(u,v), x^2(u,v)]\,|N|\,du\,dv$$

$$= \iint_D F(u,v)\,|N|\,du\,dv \qquad (17)$$

NEL CASO SPECIFICO POSSIAMO ANCHE SCRIVERE $|N|$ NELLA FORMA

$$|N| = \sqrt{A_\gamma^2 + B_\gamma^2 + C_\gamma^2} \Rightarrow |N|^2 = |\xi_u|^2 |\xi_v|^2 \operatorname{sen}^2\varphi =$$

$$= |\xi_v|^2 |\xi_u|^2 (1 - \cos^2\varphi) = |\xi_u|^2 |\xi_v|^2 - (\underline{\xi}_u \cdot \underline{\xi}_v)^2 =$$

$$= \underbrace{\left[\left(\frac{\partial x^1}{\partial u}\right)^2 + \left(\frac{\partial x^2}{\partial u}\right)^2 + \left(\frac{\partial x^3}{\partial u}\right)^2\right]}_{E(u,v) = |\xi_u|^2} \underbrace{\left[\left(\frac{\partial x^1}{\partial v}\right)^2 + \left(\frac{\partial x^2}{\partial v}\right)^2 + \left(\frac{\partial x^3}{\partial v}\right)^2\right]}_{G(u,v) = |\xi_v|^2}$$

$$- \Big[\overbrace{\frac{\partial x^1}{\partial u}\frac{\partial x^1}{\partial v} + \frac{\partial x^2}{\partial u}\frac{\partial x^2}{\partial v} + \frac{\partial x^3}{\partial u}\frac{\partial x^3}{\partial v}}^{F(u,v) = [\xi_u \cdot \xi_v]}\Big]^2 = EG - F^2$$

DA CUI $|N| = \sqrt{EG - F^2}$

$$\iint_{\Sigma} f(x^1, x^2, x^3)\, d\sigma = \iint_{D} f(u,v)\, |N|\, du\, dv = \iint_{D} f(u,v) \sqrt{EG - F^2}\, du\, dv$$

$$\iint_{\Sigma} d\sigma = \iint_{D} |N|\, du\, dv = \iint_{D} \sqrt{EG - F^2}\, du\, dv$$

SE INFINE VALUTIAMO UNA SUPERFICIE TRAMITE LA SUA RAPPRESENTAZIONE CARTESIANA, ~~ED~~ IL SUO GRAFICO:

$$\begin{cases} x = x \\ y = y \\ z = f(x,y) \end{cases} \qquad (\text{i.e. } x = u \text{ ed } y = v,\ z = f(u,v))$$

ALLORA AVREMO

$$\varphi_x = \left(\frac{\partial x}{\partial x}, \frac{\partial y}{\partial x}\ \frac{\partial z}{\partial x} \right) = \left(1, 0, \frac{\partial f}{\partial x}\right)$$

$$\varphi_y = \left(\frac{\partial x}{\partial y}, \frac{\partial y}{\partial y}, \frac{\partial z}{\partial y} \right) = \left(0, 1, \frac{\partial f}{\partial y}\right)$$

$$\varphi_x \wedge \varphi_y = \begin{vmatrix} e_1 & e_2 & e_3 \\ 1 & 0 & \frac{\partial f}{\partial x} \\ 0 & 1 & \frac{\partial f}{\partial y} \end{vmatrix} = -\frac{\partial f}{\partial x} \underline{e_1} - \frac{\partial f}{\partial y} \underline{e_2} + \underline{e_3}$$

$$|\varphi_x \wedge \varphi_y| = \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}$$

DA CUI AVREMO IN COORDINATE CARTESIANE

$$\begin{cases} \text{AREA}(\Sigma) = \iint_{D} \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}\, dx\, dy \\[2ex] \iint F\, d\sigma = \iint_{D} F(x,y) \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}\, dx\, dy \end{cases}$$

ESEMPI:

1) SIA DATA LA SUPERFICIE ESPRESSA TRAMITE IL SUO GRAFICO IN COORDINATE POLARI:

$$\begin{cases} x = \rho\cos\alpha \\ y = \rho\,\mathrm{sen}\,\alpha \\ z = f(\rho\cos\alpha, \rho\,\mathrm{sen}\,\alpha) \end{cases}$$

$$\underline{r}_\rho = \left(\cos\alpha, \mathrm{sen}\,\alpha, \frac{\partial f}{\partial \rho}\right)$$

$$\underline{r}_\alpha = \left(-\rho\,\mathrm{sen}\,\alpha, \rho\cos\alpha, \frac{\partial f}{\partial \alpha}\right)$$

$$\underline{r}_\rho \wedge \underline{r}_\alpha = \begin{vmatrix} e_1 & e_2 & e_3 \\ \cos\alpha & \mathrm{sen}\,\alpha & \frac{\partial f}{\partial \rho} \\ -\rho\,\mathrm{sen}\,\alpha & \rho\cos\alpha & \frac{\partial f}{\partial \alpha} \end{vmatrix} =$$

$$= \left(\mathrm{sen}\,\alpha \frac{\partial f}{\partial \alpha} - \rho\cos\alpha\frac{\partial f}{\partial \rho}\right)\underline{e}_1 + \left(-\rho\,\mathrm{sen}\,\alpha\frac{\partial f}{\partial \rho} - \cos\alpha\frac{\partial f}{\partial \alpha}\right)\underline{e}_2 + \rho\,\underline{e}_3$$

DA CUI

$$(r_\rho \wedge r_\alpha)^2 = \left(\frac{\partial f}{\partial \alpha}\right)^2 + \rho^2\left(\frac{\partial f}{\partial \rho}\right)^2 + \rho^2$$

DA CUI $\quad \mathrm{AREA}(\Sigma) = \iint_D \sqrt{\rho^2 + \rho^2\left(\frac{\partial f}{\partial \rho}\right)^2 + \left(\frac{\partial f}{\partial \alpha}\right)^2}\, d\rho\, d\alpha$

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2) SIA DATA LA SFERA DI RAGGIO R DI RAPPRESENTAZIONE PARAMETRICA

$$\begin{cases} x = R\,\mathrm{sen}\,\alpha\cos\varphi & 0 \le \alpha \le \pi \\ y = R\,\mathrm{sen}\,\alpha\,\mathrm{sen}\,\varphi & 0 \le \varphi \le 2\pi \\ z = R\cos\alpha \end{cases}$$

$$\begin{cases} r_\alpha = (R\cos\alpha\cos\varphi, R\cos\alpha\,\mathrm{sen}\,\varphi, -R\,\mathrm{sen}\,\alpha) \\ r_\varphi = (-R\,\mathrm{sen}\,\alpha\,\mathrm{sen}\,\varphi, R\,\mathrm{sen}\,\alpha\cos\varphi, 0) \end{cases}$$

$$\underline{r}_\alpha \wedge \underline{r}_\varphi = \begin{vmatrix} e_1 & e_2 & e_3 \\ R\cos\alpha\cos\varphi & R\cos\alpha\,\mathrm{sen}\,\varphi & -R\,\mathrm{sen}\,\alpha \\ -R\,\mathrm{sen}\,\alpha\,\mathrm{sen}\,\varphi & R\,\mathrm{sen}\,\alpha\cos\varphi & 0 \end{vmatrix} = (+R^2\mathrm{sen}^2\alpha\cos\varphi)\,e_1 + (-R^2\,\mathrm{sen}^2\alpha\,\mathrm{sen}\,\varphi)\,\underline{e}_2 + (R^2\,\mathrm{sen}\,\alpha\cos\alpha)\,e_3$$

$$|r_\alpha \wedge r_\varphi|^2 = R^4\mathrm{sen}^4\alpha\cos^2\varphi + R^4\mathrm{sen}^4\alpha\,\mathrm{sen}^2\varphi + R^4\mathrm{sen}^2\alpha\cos^2\alpha = R^4\mathrm{sen}^4\alpha + R^4\mathrm{sen}^2\alpha\cos^2\alpha = R^4\mathrm{sen}^2\alpha$$

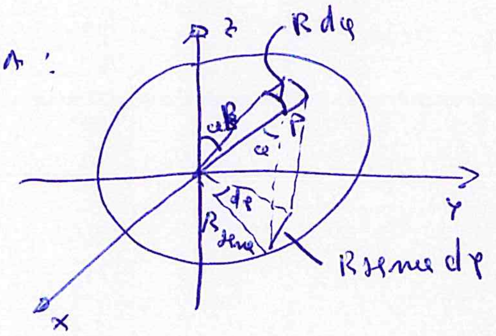
Da cui

$$|z_\alpha \wedge z_\varphi| = R^2 \operatorname{sen}\alpha$$

$$\text{AREA}(\Sigma) = \iint_D R^2 \operatorname{sen}\alpha \, d\alpha \, d\varphi = R^2 \int_0^{\pi} \operatorname{sen}\alpha \, d\alpha \int_0^{2\pi} d\varphi = 2\pi R^2 \left[-\cos\alpha\right]_0^{\pi} =$$

$$= 4\pi R^2$$

NOTA:



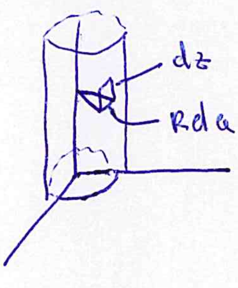
da cui $d\sigma = \underbrace{R^2 \operatorname{sen}\alpha \, d\alpha \, d\varphi}_{|z_\alpha \wedge z_\varphi|}$

Sia dato un cilindro (di base $2\pi R$ ed altezza h.) la cui rappresentazione parametrica si dà

$$\begin{cases} x = R\cos\alpha \\ y = R\operatorname{sen}\alpha \\ z = z \end{cases} \qquad \begin{array}{l} 0 \le z \le h \\ 0 \le \alpha \le 2\pi \end{array} \qquad \begin{array}{l} z_\alpha = (-R\operatorname{sen}\alpha, \; R\cos\alpha, \; 0) \\ z_z = (0, \; 0, \; 1) \end{array}$$

$$z_\alpha \wedge z_z = \begin{vmatrix} e_1 & e_2 & e_3 \\ -R\operatorname{sen}\alpha & R\cos\alpha & 0 \\ 0 & 0 & 1 \end{vmatrix} = R\cos\alpha \, e_1 + R\operatorname{sen}\alpha \, e_2$$

$|z_\alpha \wedge z_z|^2 = R^2$ da cui $\text{AREA}(\Sigma) = \iint_D R \, d\alpha \, dz = R \int_0^{2\pi} d\alpha \int_0^h dz = 2\pi R \cdot h$



$$d\sigma = dz \underset{\substack{\uparrow \\ |z_\alpha \wedge z_z|}}{R\, d\alpha} = |z_\alpha \wedge z_z| \, d\alpha \, dz.$$