$$\tilde{\mathcal{L}}_{TOT} = J_\mu A_\mu + \tilde{\mathcal{L}}^{(0)}_{EM} + \mathcal{L}^{(1)}_{EM}$$ (DENSITA' DI LAGRANGIANA)

DOVE $\tilde{\mathcal{L}}^{(0)}_{EM} = \frac{1}{16\pi}\, tr F^2 = a_0\, tr F^2$

$$\begin{cases} \hat{\mathcal{L}}^{(1)}_{EM} = a\,(tr F^2)^2 + b\,(tr F^4) \\ a = -\frac{5}{180}\frac{e^4}{m^4}\frac{1}{(4\pi)^3} \\ b = \frac{14}{180}\frac{e^4}{m^4}\frac{1}{(4\pi)^3} \end{cases}$$

EQUAZIONI DI CAMPO:

$$\frac{\partial \tilde{\mathcal{L}}_{TOT}}{\partial A_\beta} - \frac{\partial}{\partial x_\alpha}\left\{\frac{\partial \hat{\mathcal{L}}_{TOT}}{\partial(\partial_\alpha A_\beta)}\right\} = 0$$

CALCOLIAMO I VARI TERMINI: ⇒ RETRO

$$\frac{\partial\, tr F^2}{\partial(\partial_\alpha A_\beta)} = -4 F_{\alpha\beta} \quad ; \quad \frac{\partial (tr F^2)^2}{\partial(\partial_\alpha A_\beta)} = 2\, tr F^2 \frac{\partial\, tr F^2}{\partial(\partial_\alpha A_\beta)} = -8 (tr F^2) F_{\alpha\beta}$$

$$\frac{\partial\, tr F^4}{\partial(\partial_\alpha A_\beta)} = -8 F_{\alpha\gamma} F_{\gamma\sigma} F_{\sigma\beta} \quad ; \quad \frac{\partial \hat{\mathcal{L}}_{TOT}}{\partial A_\beta} = \frac{\partial J_\mu A_\mu}{\partial A_\beta} = J_\beta$$

1) $\frac{\partial\, \mathrm{tr} F^2}{\partial(\partial_\alpha A_\beta)} = \frac{\partial\, F_{\mu\nu} F_{\nu\mu}}{\partial(\partial_\alpha A_\beta)} = \frac{\partial}{\partial(\partial_\alpha A_\beta)} \left\{(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial_\nu A_\mu - \partial_\mu A_\nu)\right\} =$

$= (\delta_{\mu\alpha}\delta_{\nu\beta} - \delta_{\nu\alpha}\delta_{\mu\beta})(\partial_\nu A_\mu - \partial_\mu A_\nu) + (\partial_\mu A_\nu - \partial_\nu A_\mu)(\delta_{\nu\alpha}\delta_{\mu\beta} - \delta_{\mu\alpha}\delta_{\nu\beta})$

$= (F_{\beta\alpha} - F_{\alpha\beta}) + (F_{\beta\alpha} - F_{\alpha\beta}) = -4 F_{\alpha\beta}$

2) $\frac{\partial\, \mathrm{tr} F^4}{\partial(\partial_\alpha A_\beta)} = \frac{\partial}{\partial(\partial_\alpha A_\beta)} \left\{(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial_\nu A_\gamma - \partial_\gamma A_\nu)(\partial_\gamma A_\sigma - \partial_\sigma A_\gamma)(\partial_\sigma A_\mu - \partial_\mu A_\sigma)\right\}$

$= (\delta_{\mu\alpha}\delta_{\nu\beta} - \delta_{\nu\alpha}\delta_{\mu\beta})(\partial_\nu A_\gamma - \partial_\gamma A_\nu)(\partial_\gamma A_\sigma - \partial_\sigma A_\gamma)(\partial_\sigma A_\mu - \partial_\mu A_\sigma) +$

$+ (\partial_\mu A_\nu - \partial_\nu A_\mu)(\delta_{\nu\alpha}\delta_{\gamma\beta} - \delta_{\gamma\alpha}\delta_{\nu\beta})(\partial_\gamma A_\sigma - \partial_\sigma A_\gamma)(\partial_\sigma A_\mu - \partial_\mu A_\sigma) +$

$+ (\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial_\nu A_\gamma - \partial_\gamma A_\nu)(\delta_{\gamma\alpha}\delta_{\sigma\beta} - \delta_{\sigma\alpha}\delta_{\gamma\beta})(\partial_\sigma A_\mu - \partial_\mu A_\sigma) +$

$+ (\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial_\nu A_\gamma - \partial_\gamma A_\nu)(\partial_\gamma A_\sigma - \partial_\sigma A_\gamma)(\delta_{\sigma\alpha}\delta_{\mu\beta} - \delta_{\mu\alpha}\delta_{\sigma\beta}) =$

$= (F_{\beta\gamma} F_{\gamma\sigma} F_{\sigma\alpha} - F_{\alpha\gamma} F_{\gamma\sigma} F_{\sigma\beta}) + (F_{\mu\alpha} F_{\beta\sigma} F_{\sigma\mu} - F_{\mu\beta} F_{\alpha\sigma} F_{\sigma\mu}) +$

$+ (F_{\mu\nu} F_{\nu\alpha} F_{\beta\mu} - F_{\mu\nu} F_{\nu\beta} F_{\alpha\mu}) + (F_{\beta\nu} F_{\nu\gamma} F_{\gamma\alpha} - F_{\alpha\nu} F_{\nu\gamma} F_{\gamma\beta}) =$

$= (-2 F_{\alpha\gamma} F_{\gamma\sigma} F_{\sigma\beta}) + (-2 F_{\alpha\mu} F_{\mu\sigma} F_{\sigma\beta}) + (-2 F_{\alpha\nu} F_{\nu\mu} F_{\mu\beta})$

$+ (-2 F_{\alpha\nu} F_{\nu\gamma} F_{\gamma\sigma}) = -8 F_{\alpha\gamma} F_{\gamma\sigma} F_{\sigma\beta}$

$$\frac{\partial \hat{\mathcal{L}}_{TOT}}{\partial(\partial_\alpha A_\beta)} = \frac{1}{16\pi}\frac{\partial\, \mathrm{tr}F^2}{\partial(\partial_\alpha A_\beta)} - \frac{5}{180}\frac{e^4}{m^4}\frac{1}{(4\pi)^3}\frac{\partial(\mathrm{tr}F^2)^2}{\partial(\partial_\alpha A_\beta)} + \frac{14}{180}\frac{e^4}{m^4}\frac{1}{(4\pi)^3}\frac{\partial\,\mathrm{tr}F^4}{\partial(\partial_\alpha A_\beta)} =$$

$$= -\frac{1}{4\pi}\left\{ F_{\alpha\beta} - \frac{5}{90}\frac{e^4}{m^4}\frac{1}{4\pi^2}(\mathrm{tr}F^2)F_{\alpha\beta} + \frac{14}{90}\frac{e^4}{m^4}\frac{1}{4\pi^2}F_{\alpha\gamma}F_{\gamma\sigma}F_{\sigma\beta}\right\}$$

DA CUI L'EQUAZIONE

$$\boxed{\frac{\partial}{\partial x_\alpha}\left\{ F_{\beta\alpha} - \frac{5}{90}\frac{e^4}{m^4}\frac{1}{4\pi^2}(\mathrm{tr}F^2)F_{\beta\alpha} + \frac{14}{90}\frac{e^4}{m^4}\frac{1}{4\pi^2}F_{\beta\gamma}F_{\gamma\sigma}F_{\sigma\alpha}\right\} = 4\pi J_\beta} \quad (*)$$

RICORDANDO CHE $\mathrm{tr}F^2 = 2(E^2-H^2)$ PER $\beta=4$ ESSENDO $J_4 = i\rho$

AVREMO:

$$\frac{\partial}{\partial x_1}\left\{ F_{41} - \frac{5}{45}\frac{e^4}{m^4}\frac{1}{4\pi^2}(E^2-H^2)F_{41} + \frac{7}{45}\frac{e^4}{m^4}\frac{1}{4\pi^2}F_{4\gamma}F_{\gamma\sigma}F_{\sigma 1}\right\}$$

$$+\frac{\partial}{\partial x_2}\left\{ F_{42} - \frac{5}{45}\frac{e^4}{m^4}\frac{1}{4\pi^2}(E^2-H^2)F_{42} + \frac{7}{45}\frac{e^4}{m^4}\frac{1}{4\pi^2}F_{4\gamma}F_{\gamma\sigma}F_{\sigma 2}\right\}$$

$$+\frac{\partial}{\partial x_3}\left\{ F_{43} - \frac{5}{45}\frac{e^4}{m^4}\frac{1}{4\pi^2}(E^2-H^2)F_{43} + \frac{7}{45}\frac{e^4}{m^4}\frac{1}{4\pi^2}F_{4\gamma}F_{\gamma\sigma}F_{\sigma 3}\right\}$$

$$+\frac{\partial}{\partial x_4}\left\{ \cancel{\frac{7}{45}\frac{e^4}{m^4}\frac{1}{4\pi^2}\underbrace{F_{4\gamma}F_{\gamma\sigma}F_{\sigma 4}}_{=0}}\right\} = 4\pi i \rho$$

osserviamo che:

$$F_{4\gamma} F_{\gamma\sigma} F_{\sigma 3} = i\{(H_3 H_1) E_1 + (H_3 H_2) E_2 + (H_3 H_3) E_3 + (E^2 - H^2) E_3\}$$

$$F_{4\gamma} F_{\gamma\sigma} F_{\sigma 2} = i\{(H_2 H_1) E_1 + (H_2 H_2) E_2 + (H_2 H_3) E_3 + (E^2 - H^2) E_2\}$$

$$F_{4\gamma} F_{\gamma\sigma} F_{\sigma 1} = i\{(H_1 H_1) E_1 + (H_1 H_2) E_2 + (H_1 H_3) E_3 + (E^2 - H^2) E_1\}$$

quindi

$$\cancel{i}\frac{\partial}{\partial x_1}\left\{ E_1 - \frac{5}{45}\frac{e^4}{m^4}\frac{1}{4\pi^2}(E^2 - H^2) E_1 + \frac{7}{45}\frac{e^4}{m^4}\cdot\frac{1}{4\pi^2}\left[(H_1 H_1) E_1 + (H_1 H_2) E_2 + (H_1 H_3) E_3 + (E^2 - H^2) E_1\right]\right\} +$$

$$+\cancel{i}\frac{\partial}{\partial x_2}\left\{ E_2 - \frac{5}{45}\frac{e^4}{m^4}\frac{1}{4\pi^2}(E^2 - H^2) E_2 + \frac{7}{45}\frac{e^4}{m^4}\frac{1}{4\pi^2}\left[(H_2 H_1) E_1 + (H_2 H_2) E_2 + (H_2 H_3) E_3 + (E^2 - H^2) E_2\right]\right\} +$$

$$+\cancel{i}\frac{\partial}{\partial x_3}\left\{ E_3 - \frac{5}{45}\frac{e^4}{m^4}\frac{1}{4\pi^2}(E^2 - H^2) E_3 + \frac{7}{45}\frac{e^4}{m^4}\frac{1}{4\pi^2}\left[(H_3 H_1) E_1 + (H_3 H_2) E_2 + (H_3 H_3) E_3 + (E^2 - H^2) E_3\right]\right\} = \cancel{i}\, 4\pi\rho$$

$$= \frac{\partial}{\partial x_1}\left\{ E_1 + \frac{1}{45}\frac{e^4}{m^4}\frac{1}{4\pi^2}\left[2(E^2 - H^2) E_1 + 7\left[(H_1 H_1) E_1 + (H_1 H_2) E_2 + (H_1 H_3) E_3\right]\right]\right\} +$$

$$+ \frac{\partial}{\partial x_2}\left\{ E_2 + \frac{1}{45}\frac{e^4}{m^4}\frac{1}{4\pi^2}\left[2(E^2 - H^2) E_2 + 7\left[(H_2 H_1) E_1 + (H_2 H_2) E_2 + (H_2 H_3) E_3\right]\right]\right\} +$$

$$+ \frac{\partial}{\partial x_3}\left\{ E_3 + \frac{1}{45}\frac{e^4}{m^4}\frac{1}{4\pi^2}\left[2(E^2 - H^2) E_3 + 7\left[(H_3 H_1) E_1 + (H_3 H_2) E_2 + (H_3 H_3) E_3\right]\right]\right\} = 4\pi\rho$$

DA CUI DEFINENDO IL VETTORE

$$\boxed{D_i = \varepsilon_{ik} E_k} \quad \text{CON} \quad \boxed{\varepsilon_{ik} = \delta_{ik} + \frac{1}{45}\frac{e^4}{m^4}\frac{1}{4\pi^2}\left[2(E^2-H^2)\delta_{ik} + 7 H_i H_k\right]}$$

AVREMO

$$\frac{\partial D_1}{\partial x_1} + \frac{\partial D_2}{\partial x_2} + \frac{\partial D_3}{\partial x_3} = 4\pi\rho \Rightarrow \boxed{\vec{\nabla}\cdot\vec{D} = 4\pi\rho}$$

ANALOGAMENTE CALCOLANDO LA (*) PER $\beta = 1, 2, 3$ E DEFINENDO (VEDI NOTA SOTTOSTANTE) IL VETTORE

$$\boxed{B_i = \mu_{ik} H_k} \quad \text{CON} \quad \boxed{\mu_{ik} = \delta_{ik} + \frac{1}{45}\frac{e^4}{m^4}\frac{1}{4\pi^2}\left[2(E^2-H^2)\delta_{ik} - 7 E_i E_k\right]}$$

AVREMO L'EQUAZIONE

$$\boxed{\vec{\nabla}\wedge\vec{B} = \frac{1}{c}\frac{\partial\vec{D}}{\partial t} + \frac{4\pi}{c}\vec{J}}$$

NOTA: PER $\beta = 1$

$$\frac{\partial}{\partial x^\alpha}\left\{F_{1\alpha} - \frac{5}{90}\frac{e^4}{m^4}\frac{1}{4\pi^2}2(E^2-H^2)F_{1\alpha} + \frac{14}{90}\frac{e^4}{m^4}\frac{1}{4\pi^2}F_{1\gamma}F_{\gamma\kappa}F_{\kappa\alpha}\right] = 4\pi\frac{\rho}{c}v_1$$

ESSENDO

$$\begin{cases} F_{1\gamma}F_{\gamma\kappa}F_{\kappa 1} = 0 \\ F_{1\gamma}F_{\gamma\kappa}F_{\kappa 2} = (E^2-H^2)H_3 - (E_1H_1 + E_2H_2 + E_3H_3)E_3 \\ F_{1\gamma}F_{\gamma\kappa}F_{\kappa 3} = -\left\{(E^2-H^2)H_2 - (E_1H_1 + E_2H_2 + E_3H_3)E_2\right] \\ F_{1\gamma}F_{\gamma\kappa}F_{\kappa 4} = -i\left\{(E^2-H^2)E_1 + (E_1H_1 + E_2H_2 + E_3H_3)H_1\right] \end{cases}$$

DA cui:

$$\frac{\partial}{\partial x^2}\left\{H_3 + \frac{5}{45}\frac{e^4}{m^4}\frac{1}{4\pi^2}(E^2-H^2)H_3 + \frac{7}{45}\frac{e^4}{m^4}\frac{1}{4\pi^2}\left[(E^2-H^2)H_3 - (E_1H_1+E_2H_2+E_3H_3)E_3\right]\right\}$$

$$-\frac{\partial}{\partial x^3}\left\{H_2 - \frac{5}{45}\frac{e^4}{m^4}\frac{1}{4\pi^2}(E^2-H^2)H_2 + \frac{7}{45}\frac{e^4}{m^4}\frac{1}{4\pi^2}\left[(E^2-H^2)H_2 - (E_1H_1+E_2H_2+E_3H_3)E_2\right]\right\}$$

$$-\frac{\cancel{x}}{\cancel{x}c}\frac{\partial}{\partial t}\left\{E_1 - \frac{5}{45}\frac{e^4}{m^4}\frac{1}{4\pi^2}(E^2-H^2)E_1 + \frac{7}{45}\frac{e^4}{m^4}\frac{1}{4\pi^2}\left[(E^2-H^2)E_1 + (E_1H_1+E_2H_2+E_3H_3)H_1\right]\right\} = 4\pi\frac{\rho}{c}V_1$$

DA cui

$$\frac{\partial}{\partial x^2}\left\{H_3 + \frac{1}{45}\frac{e^4}{m^4}\frac{1}{4\pi^2}\left[2(E^2-H^2)H_3 - 7(E_1H_1+E_2H_2+E_3H_3)E_3\right]\right\}$$

$$-\frac{\partial}{\partial x^3}\left\{H_2 + \frac{1}{45}\frac{e^4}{m^4}\frac{1}{4\pi^2}\left[2(E^2-H^2)H_2 - 7(E_1H_1+E_2H_2+E_3H_3)E_2\right]\right\} =$$

$$= 4\pi\frac{\rho}{c}V_1 + \frac{1}{c}\frac{\partial}{\partial t}\left\{E_1 + \frac{1}{45}\frac{e^4}{m^4}\frac{1}{4\pi^2}\left[2(E^2-H^2)E_1 + 7(E_1H_1+E_2H_2+E_3H_3)H_1\right]\right\}$$

cioè

$$\left(\frac{\partial B_3}{\partial x^2} - \frac{\partial B_2}{\partial x^3}\right) = 4\pi\frac{\rho V_1}{c} + \frac{1}{c}\frac{\partial}{\partial t}D_1 \Rightarrow (\nabla\wedge B)_1 = 4\pi\frac{\rho V_1}{c} + \frac{1}{c}\frac{\partial D_1}{\partial t}$$

considerando i casi $\beta = 2, 3$ avremo l'equazione vettoriale

$$\boxed{(\underline{\nabla}\wedge\underline{B}) = \frac{4\pi}{c}\rho\underline{v} + \frac{1}{c}\frac{\partial\underline{D}}{\partial t}}$$

con $B_i = \mu_{ij}H_j$

$D_i = \varepsilon_{ij}E_j$