Morphism-driven design of graph colouring institutions*

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Abstract. Maximal planar graphs with vertex resp. edge colouring are naturally casted as (deceiptively similar) institutions. One then tries to embody Tait's equivalence algorithms into morphisms between them, and is lead to a partial redesign of those institutions. This paper aims at elucidating the pragmatic questions which arise in this case study, which also showcases the use of relational concepts and notations in the design of the subject institutions, that is driven by the design of an isomorphism between them.

Keywords: abstract model theory, institution, institution morphisms, relation algebra, four colour theorem, graph colouring

1 Introduction

Institution morphisms are a lively, albeit controversial subject of debate in the community of researchers who investigate abstract model-theoretic concepts and methods [7] in computing.

The original definition for these structure maps [10] was soon to compete with differently conceived, variously motivated proposals, such as the "maps", "simulations", "transformations", respectively found in [15,6,18], among (several) others. Recent work [11] aims at systematic investigation of properties and interrelations of these notions, that surely is a promising, useful effort.

So far, lesser attention seems to have been attracted by pragmatic questions relating to institution morphisms, whatever sensible kind thereof, such as the understanding of how do those maps affect the design of institutions, meant as formalizations of given logical frameworks. This question is not necessarily to be understood in a "comparative" sense; that is to say, our expectation is that even

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in straightforward cases where different notions of institution morphism have essentially equivalent instances, it may well happen that institutions designed without taking morphisms into account need to be (partially) redesigned when the problem of mapping (relating, translating, structuring) them comes into play. The present paper is aimed at presenting a little exercise of this kind. We start with introducing and motivating the exercise idea.

The Four Colour Theorem (4CT) is a paradigmatic case of potential applicability of methods and results that are offspring of research on translations between logical frameworks. Here is why the 4CT offers an interesting case study for translation concepts and methods relating to logical frameworks.

Our starting point is a view of the 4CT as a consistency theorem of finite, ad-hoc logics of graph colouring. The plural form *logics* here is purposeful, since a well-known result by Tait [20,21] proves the equivalence between the 4CT with *vertex colouring* and the 3CT with *edge colouring*. The latter means proper colouring of edges rather than vertices, where "proper" is spelled out as the condition that adjacent edges, i.e. the border of a same triangular face, must be assigned different colours, whereas adjacent vertices must be assigned different colours by a proper vertex colouring.

Now, Tait's equivalence comes equipped with a constructive proof, whereby algorithms are exhibited that turn any given proper 4-colouring of vertices of any given maximal planar graph into a proper 3-colouring of its edges, and vice versa—see e.g. [8] for an outline of Tait's algorithms. In this paper we use somewhat simpler algorithms for graph colouring conversion, that exploit the nice algebraic properties of the Klein 4-group, as presented in [1].

So, here's our basic idea for an exercise aimed at testing practical impact of institution morphisms, possibly in different flavours, into institution design in the case study in question: 1) formalize maximal planar graph colouring by two distinct institutions, respectively with vertex colourings and edge colourings as models, and 2) (try to) cast Tait's equivalence into a pair of converse morphisms between the two institutions.

The first part of the exercise already raises institution design questions, e.g. the choice of signature morphisms; on pragmatic grounds, one might like to have such morphisms formalize edge contraction, in view of the relevant role played by this operation in reducibility proofs [5], yet contraction doesn't preserve maximality of planar graphs in all cases, which entails that the Set-valued sentence functor ought to map those morphisms to partial functions. One may take the design decision to formulate just vertex, resp. edge permutations as signature morphisms, since these operations are of practical interest, too. This leads to a straightforward solution of the first part of the exercise, where we also showcase the use of relational concepts and notations, whereby one gets a pleasing conciseness and elegance in their presentation. Moreover, sentences in the institution with edge colouring have an amazing syntactic representation by Matiyasevitch's polynomials [13], whereby the *number* of proper colourings of any given maximal planar graph is readily found.

The second part of the exercise raises new design questions. Since the solution of the first part was determined without taking mutual interpretability of the two institutions into account, one shouldn't be surprised at finding out that Tait's algorithms prove hard to get embodied into structure-preserving maps between those institutions. Our redesign work in this respect seems quite instructive: we summarize the lessons of method drawn from this experience in the conclusions, where we also propose a few hints which may stimulate further work.

2 Graph colouring preliminaries

The first proof of the 4CT [2,3,4] raised controversial discussions due to its combinatorial complexity which, for the analysis of the nearly 2000 graphs involved, required the construction of a program—whose correctness was not proven though. A new proof was obtained by [16], keeping the structure of the previous proof but cutting down to 633 the number of graphs involved. The impression yet remains that a simpler reason for the truth of this theorem may exist. One dreams of a logical construction where combinatorics weigh no more than necessary, that is, inherent to the problem rather than to specifics of the proof. There are obvious reasons to be pessimistic about such a dream, if only for no evidence being available to-date that inherent combinatorics may lose weight any further (evidence of the contrary seems even harder to achieve, though). May a formulation of the 4CT in the abstract model-theoretic language of institutions help the dream become true? Our educated guess is that it won't. This pessimistic expectation doesn't affect our interest and motivation in carrying out the exercise, since they arise in the converse research direction—investigation into pragmatics of institution design rather than search of a new proof of the 4CT.

As customary, we only consider colourings of *maximal* planar graphs, viz. those where no new edge may be added between existing vertices without losing planarity. This restriction is harmless w.r.t. the 4CT, since every planar graph is a subgraph of a maximal one on the same vertices, and a proper colouring of a planar graph is a proper colouring of any subgraph on the same vertices.

Maximal planar graphs are also referred to as *triangulations* of the sphere, thanks to the well-known bijection established by stereographic projection between the plane and the surface of the sphere, where the projection point on this surface maps to the infinite in the plane. The spheric view proves somewhat more convenient, in that the graph divides the surphace into *bounded* triangular *faces* (viz. regions delimited by graph edges but containing no edges besides those at the border), whereas in the planar view one face, albeit triangular like all other faces, is unbounded.

In the next sections we shall adopt the following notational conventions.

Notation

$\underline{\mathbf{n}}$: the finite ordinal with n elements, viz. the natural numbers lesser than n.

$1_{\underline{\mathbf{n}}}$, $1'_{\underline{\mathbf{n}}}$, $0'_{\underline{\mathbf{n}}}$: resp. the *universal*, *identity* and *diversity* binary relations on $\underline{\mathbf{n}}$, thus $0'_{\underline{\mathbf{n}}} = 1_{\underline{\mathbf{n}}} \backslash 1'_{\underline{\mathbf{n}}}$ or, with standard relation-algebraic notation for the Boolean complement operation: $0'_{\underline{\mathbf{n}}} = 1'^{\,-1}_{\underline{\mathbf{n}}}$.

$\mathrm{r}^{\smile}$: relation-algebraic converse of r. Consistently, we also let $\mathrm{f}^{\smile}$ denote the inverse of an invertible function f.
r ; s : relation-algebraic composition of binary relations r, s. Consistently, we also let f ; g denote the function composition g∘f.
$\mathrm{T_V}(n)$: the set of $\underline{\mathbf{n+2}}$-labeled $(n+2)$-vertex triangulations of the sphere.
$\mathrm{T_E}(n)$: the set of $\underline{\mathbf{3n}}$-labeled $3n$-edge triangulations of the sphere.

3 Institution preliminaries

The classic definition of institution, already appearing in the paper introducing this concept [9], will suffice for our purposes. Generalizations of this definition were proposed later [10], base on twisted relation categories, that allow one to choose one of set-structure or category-structure for sentences as well as for models, the two choices being independent of each other. This gives rise to four variants of the institution concept, while other variants have been proposed too, referred to as "close variants" in [11].

Let's fix some basic notation about categories first.

Notation
$|\mathcal{C}|$ is the set of objects of category $\mathcal{C}$.
$\mathcal{C}(a,b)$ is the set of morphisms from a to b in category $\mathcal{C}$, for $a,b \in |\mathcal{C}|$.
Set is the category of (small[1]) sets with total functions as morphisms.
Cat is the category of (locally small[2]) categories with functors as morphisms.

An *institution* is a 4-tuple $\mathcal{I} = (\mathsf{Sig}, \mathsf{Sen}, \mathsf{Mod}, \models)$, with:

(i) Sig a category, whose objects are called *signatures*,
(ii) $\mathsf{Sen}{:}\mathsf{Sig}\to\mathsf{Set}$ a functor, sending each signature Σ to the set $\mathsf{Sen}(\Sigma)$ of Σ-sentences, and each signature morphism $\pi{:}\Sigma_1\to\Sigma_2$ to the mapping $\mathsf{Sen}(\pi){:}\mathsf{Sen}(\Sigma_1)\to\mathsf{Sen}(\Sigma_2)$ that translates Σ_1-sentences to Σ_2-sentences,
(iii) $\mathsf{Mod}{:}\mathsf{Sig}^{\mathrm{op}}\to\mathsf{Cat}$ a contravariant functor, sending each signature Σ to the category $\mathsf{Mod}(\Sigma)$ of Σ-models, and each signature morphism $\pi{:}\Sigma_1\to\Sigma_2$ to the π-reduction functor $\mathsf{Mod}(\pi){:}\mathsf{Mod}(\Sigma_2)\to\mathsf{Mod}(\Sigma_1)$,
(iv) $\models$: a $|\mathsf{Sig}|$-indexed relation $\{\models_\Sigma \subseteq |\mathsf{Mod}(\Sigma)|\times\mathsf{Sen}(\Sigma) \mid \Sigma\in|\mathsf{Sig}|\}$, viz. a satisfaction relation between Σ-models and Σ-sentences for each $\Sigma\in|\mathsf{Sig}|$, such that the following *satisfaction condition* holds for all $\Sigma_1,\Sigma_2\in|\mathsf{Sig}|$, signature morphisms $\pi\in\mathsf{Sig}(\Sigma_1,\Sigma_2)$, Σ_2-models M and Σ_1-sentences φ:

$$\mathsf{Mod}(\pi)(\mathrm{M}) \models_{\Sigma_1} \varphi \Leftrightarrow \mathrm{M} \models_{\Sigma_2} \mathsf{Sen}(\pi)(\varphi)$$

Notation
A few notational conventions will simplify the presentation. We shall henceforth adopt the abbreviations: $\pi\varphi$ for $\mathsf{Sen}(\pi)(\varphi)$, and $\mathrm{M}\pi$ for $\mathsf{Mod}(\pi)(\mathrm{M})$, where $\pi{:}\Sigma_1\to\Sigma_2$ is a signature morphism, φ is a Σ_1-sentence, and M is a Σ_2-model.

[1] i.e., excluding proper classes
[2] i.e., those with a small set of morphisms between any two objects

When considering different institutions, it proves convenient to decorate the name of each element of the 4-tuple which an institution consists of, by adding the institution name as first subscript.

The original definition of institution morphism proposed in [10] is as follows[3].

Let $\mathcal{I} = (\mathsf{Sig}_{\mathcal{I}}, \mathsf{Sen}_{\mathcal{I}}, \mathsf{Mod}_{\mathcal{I}}, \models_{\mathcal{I}})$, $\mathcal{I}' = (\mathsf{Sig}_{\mathcal{I}'}, \mathsf{Sen}_{\mathcal{I}'}, \mathsf{Mod}_{\mathcal{I}'}, \models_{\mathcal{I}'})$ be institutions. An *institution morphism* $\mathcal{T}\colon \mathcal{I}{\to}\mathcal{I}'$ is a 3-tuple $\mathcal{T} = (\Phi, \alpha, \beta)$, with:

(i) Φ: $\mathsf{Sig}_{\mathcal{I}}{\to}\mathsf{Sig}_{\mathcal{I}'}$ a functor,
(ii) α: Φ ; $\mathsf{Sen}_{\mathcal{I}'} \to \mathsf{Sen}_{\mathcal{I}}$ a natural transformation (*sentence transformation*),
(iii) β: $\mathsf{Mod}_{\mathcal{I}} \to \Phi^{\mathrm{op}}$; $\mathsf{Mod}_{\mathcal{I}'}$ a natural transformation (*model transformation*),

such that the following *satisfaction condition* holds, for all signatures $\Sigma{\in}|\mathsf{Sig}_{\mathcal{I}}|$, models $\mathrm{M}{\in}|\mathsf{Mod}_{\mathcal{I}}(\Sigma)|$ and sentences $\varphi'{\in}\mathsf{Sen}_{\mathcal{I}'}(\Phi(\Sigma))$:

$$\mathrm{M} \models_{\mathcal{I},\Sigma} \alpha_{\Sigma}(\varphi') \Leftrightarrow \beta_{\Sigma}(\mathrm{M}) \models_{\mathcal{I}',\Phi(\Sigma)} \varphi'$$

Note that model transformation and the signature functor go in the same direction, whereas sentence transformation goes in the opposite direction.

A somewhat dual concept was proposed under the name of "plain map" of institutions in [15], and renamed "institution comorphism" in [11], to emphasize the duality with the original concept. We welcome this change of terminology, and we further refer to an "institution (co)morphism" whenever the difference doesn't matter. The essential difference is in the directions of the natural transformations involved, both of which change, according to the following definition.

An *institution comorphism* $\mathcal{T}\colon \mathcal{I}{\to}\mathcal{I}'$ is a 3-tuple $\mathcal{T} = (\Phi, \alpha, \beta)$, with:

(i) Φ: $\mathsf{Sig}_{\mathcal{I}}{\to}\mathsf{Sig}_{\mathcal{I}'}$ a functor,
(ii) α: $\mathsf{Sen}_{\mathcal{I}} \to \Phi$; $\mathsf{Sen}_{\mathcal{I}'}$ a natural transformation (*sentence transformation*),
(iii) β: Φ^{op} ; $\mathsf{Mod}_{\mathcal{I}'} \to \mathsf{Mod}_{\mathcal{I}}$ a natural transformation (*model transformation*),

such that the following *satisfaction condition* holds, for all signatures $\Sigma{\in}|\mathsf{Sig}_{\mathcal{I}}|$, models $\mathrm{M}'{\in}|\mathsf{Mod}_{\mathcal{I}'}(\Phi(\Sigma))|$ and sentences $\varphi{\in}\mathsf{Sen}_{\mathcal{I}}(\Sigma)$:

$$\beta_{\Sigma}(\mathrm{M}') \models_{\mathcal{I},\Sigma} \varphi \Leftrightarrow \mathrm{M}' \models_{\mathcal{I}',\Phi(\Sigma)} \alpha_{\Sigma}(\varphi)$$

We recall that a functor is an *isomorphism of categories* when it is a bijection (both on objects and on arrows), and that a natural transformation is a *natural isomorphism* when every component is an invertible arrow. An *isomorphism of institutions* is then an institution (co)morphism such that its signature functor is an isomorphism of categories, and both the sentence transformation and the model transformation are natural isomorphisms. It doesn't matter whether "morphism" or "comorphism" is taken in this definition, since the inverse components of each transformation of an institution morphism define the components of the corresponding transformation of an institution comorphism, and vice versa.

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[3] modulo a minor notational detail: in the definition of institution presented here, the type of the model functor follows a traditional convention for contravariant functors [12]; this explains the occurrence of the *dual* functor Φ^{op} in the type of the model transformation as presented here, in the definition of institution morphism as well as comorphism. Recall that Φ^{op} and Φ coincide on objects.

4 A vertex colouring institution

Syntax will be abstract, exploiting the fact that institutions do not force one to deal with concrete syntax. Signatures are just positive numbers, ranking maximal planar graphs by their size, and we take the *bijective* relabelings of vertices as signature morphisms. This restriction is a design decision, motivated as follows.

Each $n{>}0$ is the rank of the maximal planar graphs, or triangulations of the sphere, that have $n{+}2$ vertices. Vertex colouring of such structures require that each vertex be given a unique identity. To this purpose we consider vertices to be uniquely labeled by the elements of finite ordinal $\underline{\mathbf{n+2}}$, for triangulations of rank n. Bijective relabelings are thus just label permutations. The pragmatic question arises as to what purpose could be served by non-bijective maps on finite ordinals. On the one hand, loss of surjectivity appears useless, insofar as it introduces labels in the morphism codomain that are not made use of to label any vertex, according to the morphism image. On the other hand, though, loss of injectivity would seem to be of some use, inasmuch it amounts to identify formerly distinct vertices, thus it could prove useful to formalize edge contraction—whenever an edge connects two such vertices. This operation, however, does not preserve maximality of planar graphs. This happens when a vertex of degree 3 is opposite to the contracted edge, see e.g. fig. 1, where the dashed edge is subject to contraction and its opposite vertices, both of degree 3, are circled.

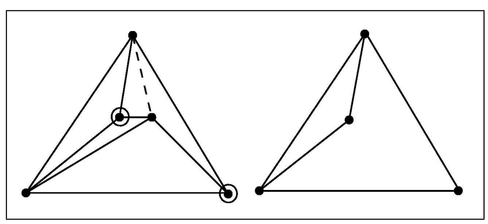

Fig. 1. Edge contraction not preserving maximality of planar graphs

Since edge contraction decreases by one the degree of those vertices which are opposite to the contracted edge, when one of them is of degree 3 the resulting graph is maximal only if it is the smallest triangulation, which consists of three vertices of degree 2 (no other triangulation has vertices of degree 2).

We must conclude that, if one admits non-injective relabelings as signature morphisms, then the Set-valued sentence functor, giving the set of vertex-labeled triangulations of rank n for each $n > 0$, ought to map those morphisms to *partial* functions, whereas only total functions are available as morphisms in Set.

Signatures

$|\mathsf{Sig}_{\mathcal{V}}| = \mathbf{N}\backslash\{0\}$
$\mathsf{Sig}_{\mathcal{V}}(n,n) = \{\pi{:}\underline{\mathbf{n+2}}{\rightarrow}\underline{\mathbf{n+2}} \mid \pi \text{ is bijective}\}$
$\mathsf{Sig}_{\mathcal{V}}(m,n) = \emptyset$ if $m \neq n$

Sentences

Each $\theta \in \mathrm{T_V}(n)$ is represented by the symmetric quotient of a binary relation on vertices, $\epsilon_\theta \stackrel{def}{=} \eta_\theta/\mathrm{Sym}$, where η_θ is the irreflexive, symmetric edge relation of θ, thus satisfies the relation-algebraic laws $\eta_\theta \leq 0'_{\underline{\mathbf{n+2}}}$, $\eta_\theta = \eta_\theta{}^{\smile}$, while $|\eta_\theta| = 6n$, but the Sym quotient turns ordered pairs into unordered ones, thus $|\epsilon_\theta| = 3n$. As a matter of notation, we write $i\ \epsilon_\theta\ j$ or $\{i, j\} \in \epsilon_\theta$, rather than the more cumbersome $\{(i, j), (j, i)\} \in \epsilon_\theta$, whenever $\{(i, j), (j, i)\} \subseteq \eta_\theta$. We thus define:

$$\mathsf{Sen}_{\mathcal{V}}(n) = \{\epsilon_\theta \mid \theta \in \mathrm{T_V}(n)\}$$

Sentence translation

If $\pi \in \mathsf{Sig}_{\mathcal{V}}(n, n)$ and $\epsilon_\theta \in \mathsf{Sen}_{\mathcal{V}}(n)$, then $\pi\epsilon_\theta \in \mathsf{Sen}_{\mathcal{V}}(n)$, with

$$(\pi i)\ \pi\epsilon_\theta\ (\pi j) \Leftrightarrow i\ \epsilon_\theta\ j$$

Models

The model functor assigns to each signature $n > 0$ the category of 4-colourings of the $n + 2$ vertices, with colour permutations as model morphisms, thus:

$$|\mathsf{Mod}_{\mathcal{V}}(n)| = \underline{\mathbf{4}}^{\underline{\mathbf{n+2}}}$$
$$\forall \mu, \mu' \in |\mathsf{Mod}_{\mathcal{V}}(n)|.\mathsf{Mod}_{\mathcal{V}}(n)(\mu, \mu') = \{\rho \in \underline{\mathbf{4}}^{\underline{\mathbf{4}}} | \rho \text{ is bijective}, \mu' = \mu\, ;\, \rho\}$$

Model reduction

If $\pi{:}\underline{\mathbf{n+2}} \rightarrow \underline{\mathbf{n+2}} \in \mathsf{Sig}_{\mathcal{V}}(n, n)$ and $\mu{:}\underline{\mathbf{n+2}} \rightarrow \underline{\mathbf{4}} \in |\mathsf{Mod}_{\mathcal{V}}(n)|$, then $\mu\pi \in |\mathsf{Mod}_{\mathcal{V}}(n)|$, with $(\mu\pi)i \stackrel{def}{=} \mu(\pi i)$, and $\rho\pi \stackrel{def}{=} \rho$ for all colour permutations $\rho{:}\underline{\mathbf{4}} \leftrightarrow \underline{\mathbf{4}}$. This makes model reduction to be a functor, thus $\rho\pi \in \mathsf{Mod}_{\mathcal{V}}(n)(\mu\pi, \mu'\pi)$ if $\rho \in \mathsf{Mod}_{\mathcal{V}}(n)(\mu, \mu')$, since $(\mu\pi)\, ;\, \rho = (\mu\, ;\, \rho)\pi$, by an easy check.

Satisfaction

In $\mathcal{V}$, a $n-$model satisfies a $n-$sentence iff it is a proper vertex colouring of that triangulation, that is:

$$\mu \models_{\mathcal{V},n} \epsilon_\theta \text{ iff } \forall i, j \in \underline{\mathbf{n+2}}.\ i\epsilon_\theta j \Rightarrow \mu i \neq \mu j$$

A relation-algebraic formulation of this definition may exploit the "oriented" edge relation η_θ from which ϵ_θ is obtained as a quotient, and the view of the 4-colouring map as a binary relation $\mu \subseteq \underline{\mathbf{n+2}} \times \underline{\mathbf{4}}$. Then we get:

$$\mu \models_{\mathcal{V},n} \epsilon_\theta \text{ iff } \mu^{\smile}; \eta_\theta; \mu \leq 0'_{\underline{\mathbf{4}}}$$

It is easy to show that this definition complies with the *satisfaction condition*:

$$\mu\pi \models_{\mathcal{V},n} \epsilon_\theta \Leftrightarrow \mu \models_{\mathcal{V},n} \pi\epsilon_\theta$$

therefore $\mathcal{V}$ is an *institution*.

5 An edge colouring institution

Syntax will be somewhat more concrete, inspired by Matiyasevich's polynomial representation of triangulations of the sphere [13]. Signatures remain the same, but we now take the bijective relabelings of *edges* as signature morphisms.

Signatures

$|\mathsf{Sig}_{\mathcal{E}}| = |\mathsf{Sig}_{\mathcal{V}}| = \mathbf{N}\setminus\{0\}$
$\mathsf{Sig}_{\mathcal{E}}(n,n) = \{\pi{:}\underline{\mathbf{3n}}{\to}\underline{\mathbf{3n}} \mid \pi \text{ is bijective}\}$
$\mathsf{Sig}_{\mathcal{E}}(m,n) = \emptyset \text{ if } m \neq n$

Sentences

Sentences in $\mathsf{Sen}_{\mathcal{E}}(n)$, ranged over by ψ_ϑ, are represented by Matiyasevich's polynomials in product form:

$$\psi_\vartheta \;=\; \prod_{\mathrm{t}_{ijk}\in\vartheta} (x_i - x_j)(x_j - x_k)(x_k - x_i)$$

where $\vartheta{\in}\mathrm{T_E}(n)$ and t_{ijk} is a triangular face of ϑ having edges labeled i, j, k in clockwise order. We thus define:

$$\mathsf{Sen}_{\mathcal{E}}(n) = \{\psi_\vartheta \mid \vartheta{\in}\mathrm{T_E}(n)\}$$

Sentence translation

If $\pi{\in}\mathsf{Sig}_{\mathcal{E}}(n,n)$ and $\psi_\vartheta{\in}\mathsf{Sen}_{\mathcal{E}}(n)$ represented as above, then $\pi\psi_\vartheta{\in}\mathsf{Sen}_{\mathcal{E}}(n)$, with

$$\pi\psi_\vartheta \;=\; \prod_{\mathrm{t}_{ijk}\in\vartheta} (x_{\pi i} - x_{\pi j})(x_{\pi j} - x_{\pi k})(x_{\pi k} - x_{\pi i})$$

Models

The model functor assigns to each signature $n > 0$ the category of 3-colourings of the $3n$ edges, with colour permutations as model morphisms, thus:

$|\mathsf{Mod}_{\mathcal{E}}(n)| = \underline{\mathbf{3}}^{\underline{\mathbf{3n}}}$
$\forall \nu, \nu'{\in}|\mathsf{Mod}_{\mathcal{E}}(n)|.\mathsf{Mod}_{\mathcal{E}}(n)(\nu,\nu') = \{\rho{\in}\underline{\mathbf{3}}^{\underline{\mathbf{3}}} | \rho \text{ is bijective}, \nu' = \nu\,;\,\rho\}$

Model reduction

If $\pi{:}\underline{\mathbf{3n}}{\to}\underline{\mathbf{3n}} \in \mathsf{Sig}_{\mathcal{E}}(n,n)$ and $\nu{:}\underline{\mathbf{3n}}{\to}\underline{\mathbf{3}} \in |\mathsf{Mod}_{\mathcal{E}}(n)|$, then $\nu\pi \in |\mathsf{Mod}_{\mathcal{E}}(n)|$, with $(\nu\pi)i \stackrel{def}{=} \nu(\pi i)$, and $\rho\pi \stackrel{def}{=} \rho$ for all colour permutations $\rho{:}\underline{\mathbf{3}}{\leftrightarrow}\underline{\mathbf{3}}$. This makes model reduction to be a functor, thus $\rho\pi{\in}\mathsf{Mod}_{\mathcal{E}}(n)(\nu\pi,\nu'\pi)$ if $\rho{\in}\mathsf{Mod}_{\mathcal{E}}(n)(\nu,\nu')$, since $(\nu\pi)\,;\,\rho = (\nu\,;\,\rho)\pi$, by an easy check.

Satisfaction

In $\mathcal{E}$, a n-model satisfies a n-sentence iff it is a proper edge colouring of that triangulation, that is:

$$\nu \models_{\mathcal{E},n} \psi_\vartheta \text{ iff } \forall i,j{\in}\underline{\mathbf{3n}}.(x_i - x_j) \text{ occurs in } \psi_\vartheta \Rightarrow \nu i{\neq}\nu j$$

A relation-algebraic formulation of this definition may use the binary relation of "occurrence in ψ_ϑ", $\xi_{\psi_\vartheta}{\leq}1_{\underline{\mathbf{3n}}}$: $i\ \xi_{\psi_\vartheta}\ j$ iff $(x_i - x_j)$ occurs in ψ_ϑ. Then, by using the view of a 3-colouring map as a binary relation $\nu \subseteq \underline{\mathbf{3n}}{\times}\underline{\mathbf{3}}$, we get:

$$\nu \models_{\mathcal{E},n} \psi_\vartheta \text{ iff } \nu^{\smile};\xi_{\psi_\vartheta};\nu \leq 0'_{\underline{\mathbf{3}}}$$

It is easy to show that this definition complies with the *satisfaction condition*:

$$\nu\pi \models_{\mathcal{E},n} \psi_\vartheta \Leftrightarrow \nu \models_{\mathcal{E},n} \pi\psi_\vartheta$$

therefore $\mathcal{E}$, too, is an *institution*.

6 Tait's equivalence

A triangulation admits a proper 4-colouring of its vertices if, and only if, it admits a proper 3-colouring of its edges. This is Tait's classical result [20,21], albeit here stated in graph-theoretic terms rather than, as in its original formulation, in terms of cubic map colourings. The equivalence is shown by exhibiting two algorithms, which we are going to recast in graph-theoretic terms, that for any given triangulation respectively turn any proper 4-colouring of its vertices into a proper 3-colouring of its edges, and vice versa.

We take $\underline{\mathbf{4}}$ as the set of colours for vertex-colouring and $\underline{\mathbf{4}}\backslash\underline{\mathbf{1}}$ that for edge-colouring. Taking the latter rather than $\underline{\mathbf{3}}$ somewhat simplifies the presentation of Tait's algorithms, thanks to the properties of an elegant, algebraic construction which uses the *Klein 4-group*, as provided in [1]. We take $\underline{\mathbf{4}}$ as the group carrier, with 0 as its neutral element. Every element is self-inverse, and the binary group operation $\oplus$ further satisfies $x \oplus y = z$ whenever $\{x, y, z\} = \underline{\mathbf{4}}\backslash\underline{\mathbf{1}}$. This defines $\oplus$, since $0 \oplus x = x \oplus 0 = x \oplus x = 0$ for all $x{\in}\underline{\mathbf{4}}$, by the previous conditions.

4CT $\Rightarrow$ 3CT

Let $\mu{:}\underline{\mathbf{n{+}2}}{\to}\underline{\mathbf{4}}$ be a proper 4-colouring of given triangulation $\theta{\in}\mathrm{T_V}(n)$. For each edge x in θ, let $\mu i{\neq}\mu j$ be the colours assigned by μ to the vertices connected by x. Then their Klein sum $\mu i \oplus \mu j$ is the colour assigned to edge x.

By the properties of the Klein 4-group, this colour is never 0 insofar as $\mu i{\neq}\mu j$ (by assumption, μ is a proper colouring of the vertices of θ). Furthermore, any two edges sharing a face get different colours since they share one vertex (thus one addend of the Klein sums yielding their respective colours), whereas the other two vertices they resp. join are coloured differently by μ, as they are the ends of the third edge sharing the same face. We thus have a proper 3-colouring of the edges of θ, with colours out of $\underline{\mathbf{4}}\backslash\underline{\mathbf{1}}$.

3CT ⇒ 4CT

The construction in the converse direction is a bit more complex. Let ν:$\underline{\mathbf{3n}}\rightarrow\underline{\mathbf{4}}\backslash\underline{\mathbf{1}}$ be a proper 3-colouring of given triangulation $\vartheta\in\mathrm{T_E}(n)$. Choose a vertex in the triangulation as start-vertex, and assign it colour 0. Every other vertex is then coloured by the Klein sum of the colours assigned by ν to the edges of any path from the start-vertex to that vertex.

Of course, the specified construction is only sound if Klein summation of the colours assigned by ν proves invariant for all paths joining any given pair of vertices. This holds because (i) every element is self-inverse in the Klein group and (ii) Klein summation of the colours assigned by ν along every circuit turns out to be 0. A proof of this fact is worked out in [1] (pp. 22–23), for the colouring of cubic maps, but it is readily interpreted in our present setting as follows.

Let $\mathrm{S}=\sum \nu x_i$ be the Klein sum of the colours assigned by ν to the edges of a given circuit K, and consider those triangular faces which belong to one of the two regions of the sphere having K as border (no matter which one). Since ν is a proper 3-colouring with colours out of $\underline{\mathbf{4}}\backslash\underline{\mathbf{1}}$, the Klein sum of the colours of the edges of any given triangular face is always $1 \oplus 2 \oplus 3{=}0$. Summation over all faces considered above must obviously yield 0 as well. Now, each edge in K is counted only once in this summation, whereas each other edge is counted twice, giving thus a null contribution to the summation since $x \oplus x{=}0$ for all $x\in\underline{\mathbf{4}}$. We thus conclude that summation over the edges in K alone must yield 0, viz. S=0.

Finally, it is immediately seen that the 4-colouring of vertices specified above is proper, since adjacent vertices have paths from the start-vertex that differ by one edge only, and ν assigns a non-zero colour to this edge, whence the two vertices get different colours.

One may wonder whether the present construction has *exactly* the previous one as its inverse, for every given triangulation. That is to say, if one starts with a proper 3-colouring ν of edges, gets a proper 4-colouring of vertices out of it as specified here above, and then gives this as input to the previous algorithm, does then this yield back the 3-colouring ν one started with? The answer is positive, and an *almost* similar exactness holds in the converse direction, where one starts with a proper 4-colouring μ of vertices, and gets it back *if* in the second, 3CT→4CT stage the start-vertex is chosen among those which are assigned colour 0 by μ. The proof of these facts exploits the algebraic properties of the Klein 4-group, and is left to the reader as an exercise.

7 Morphism-driven redesign of institutions

A basic obstacle makes it impossible to embody Tait's algorithms into (whatever kind of) morphism between the $\mathcal{V}$ and $\mathcal{E}$ institutions presented above, and that is: the lack of a non-trivial functorial mapping between their categories of signatures. Although those categories share their objects, their signature morphisms differ, and these prove hard to map. It's seems worthwhile to review the implicit reason for the choice of different signature morphisms in the design of the aforementioned institutions.

The choice of signature morphisms for $\mathcal{V}$ was just the obvious one, as far as abstract syntax for vertex colourings is concerned. Similarly, that for $\mathcal{E}$ was inspired by Matiyasevich's polynomial representation of triangulations, where only the naming of edges matter, thus it seemed fairly natural to take edge renamings as the edge colouring counterpart of vertex renamings for vertex colouring, as far as abstract syntax for edge colourings is concerned. This choice is actually sentence independent, in that it only depends on the rank of the triangulation (since every triangulation of given rank n has the same number of edges, that is $3n$), therefore it was appropriate as a design choice for signature morphisms.

Our "local" design choices of signature morphisms prove no longer appropriate when a wider perspective is taken, that is to say, as soon as one needs to know which vertices are connected by which edges—as it happens to be the case with Tait's algorithms, for example.

As a side remark, we note that the situation whereby a structure is first designed according to a local view of its purpose, and only later a wider context of its operation comes into play, seems to be a fairly general trait of human design activities. Since the main vehicle of context interaction on abstract algebraic structures is the concept of structure-preserving map, or morphism, it is hardly surprising to find out that context-driven redesign becomes morphism-driven redesign in the case under study. Let's now turn our attention from general remarks into a search for a solution to our institution redesign problem.

The unordered edge relation ϵ_θ on vertices shows up in $\mathcal{V}$ as sentence representation. We need to keep this information when moving to Matiyasevich's representation used in $\mathcal{E}$. Here any enumeration of the edges does the job, thus one may choose a particular enumeration that be determined by the vertex labeling only. To this purpose we define the *lexicographic edge-labeling map* $\lambda_\theta{:}\epsilon_\theta{\to}\underline{\mathbf{3n}}$ for every triangulation $\theta{\in}\mathrm{T_V}(n)$ represented in $\mathcal{V}$ by ϵ_θ, as the unique enumeration of edges which satisfies:

$$\begin{aligned}&\forall\ i,j,i',j'.i\ \epsilon_\theta\ j, i'\ \epsilon_\theta\ j', i<j, i'<j' \Rightarrow\\ &(\lambda_\theta\{i,j\}<\lambda_\theta\{i',j'\} \Leftrightarrow (i<i' \vee (i=i' \wedge j<j')))\end{aligned}$$

Now, since the λ_θ map is sentence-dependent, it cannot be used to recover permutations of $\underline{\mathbf{3n}}$ as signature morphisms in the edge-colouring institution. However, by this map, for each θ every permutation π of $\underline{\mathbf{n+2}}$ will induce a unique permutation $\pi^\#_\theta$ of $\underline{\mathbf{3n}}$ that commutes with the lexicographic edge-labeling map. More precisely, let π^2 denote the straightforward extension of π to (unordered) pairs of elements of $\underline{\mathbf{n+2}}$, $\pi^2\{i,j\}\stackrel{def}{=}\{\pi i,\pi j\}$, and let then, for any given $\theta{\in}\mathrm{T_V}(n)$, π^2_θ denote the restriction of π^2 to ϵ_θ; finally, let $\pi\theta{\in}\mathrm{T_V}(n)$ be the triangulation which is obtained from θ by relabeling its vertices with π—then $\pi\theta$ is represented by $\pi\epsilon_\theta$ in $\mathsf{Sen}_{\mathcal{V}}(n)$, $\epsilon_{\pi\theta}=\pi\epsilon_\theta$ thus. Since λ_θ is a bijection, we can define the permutation $\pi^\#_\theta{:}\underline{\mathbf{3n}}{\to}\underline{\mathbf{3n}}$ as follows:

$$\pi^\#_\theta \stackrel{def}{=} \lambda_\theta{}^{\smile};\pi^2_\theta\ ;\lambda_{\pi\theta}\ .$$

Although, for any given triangulation in $\mathrm{T_V}(n)$ with $n>1$, one does not get all permutations of $\underline{\mathbf{3n}}$ in this way, one does not need all of them either.

On pragmatic grounds, in order to get all isomorphic triangulations of a given one, only vertex label permutations are needed, regardless of whether vertices or edges are coloured. It makes thus sense to try to redesign the edge-colouring institution only. The new version of it, let it be $\mathcal{E}'$, has the same signature morphisms as $\mathcal{V}$, the same category of signatures thus: $\mathsf{Sig}_{\mathcal{E}'} = \mathsf{Sig}_{\mathcal{V}}$.

For sentences in $\mathcal{E}'$, we keep Matiyasevich's polynomial representation, but enumerating edges by the λ_θ map, thus for $\theta \in \mathrm{T_V}(n)$ we define

$$\psi_\theta = \prod_{\mathrm{t}_{ijk} \in \theta} (x_{\lambda_\theta\{i,j\}} - x_{\lambda_\theta\{j,k\}})(x_{\lambda_\theta\{j,k\}} - x_{\lambda_\theta\{k,i\}})(x_{\lambda_\theta\{k,i\}} - x_{\lambda_\theta\{i,j\}})$$

where t_{ijk} is a triangular face having *vertices* labeled i, j, k, in clockwise order. Let $\mathsf{Sen}_{\mathcal{E}'}(n)$ be the set of all such sentences.

The change of signature morphisms just made, requires a straightforward adaptation of sentence translation along them. If $\pi \in \mathsf{Sig}_{\mathcal{E}'}(n, n)$ and $\psi_\theta \in \mathsf{Sen}_{\mathcal{E}'}(n)$ is a Matiyasevich's polynomial as above, then $\pi\psi_\theta \in \mathsf{Sen}_{\mathcal{E}'}(n)$ is defined by

$$\pi\psi_\theta \stackrel{def}{=} \pi_\theta^\# \psi_\theta \stackrel{def}{=}$$
$$\prod_{\mathrm{t}_{ijk} \in \theta} (x_{\pi_\theta^\# \lambda_\theta\{i,j\}} - x_{\pi_\theta^\# \lambda_\theta\{j,k\}})(x_{\pi_\theta^\# \lambda_\theta\{j,k\}} - x_{\pi_\theta^\# \lambda_\theta\{k,i\}})(x_{\pi_\theta^\# \lambda_\theta\{k,i\}} - x_{\pi_\theta^\# \lambda_\theta\{i,j\}}).$$

The check that $\mathsf{Sen}_{\mathcal{E}'}$ so defined is indeed a functor is straightforward.

The change with the signature morphisms affects the model category more deeply, in that model reduction is to be defined uniquely, for each vertex-label permutation $\pi{:}\underline{\mathbf{n+2}} \rightarrow \underline{\mathbf{n+2}}$, in a sentence-independent way, whereas the edge labeling defined by the λ_θ map clearly depends on the sentence. We've thus got to define edge colourings using (unordered) pairs of vertex labels as edge labels, and we shall do so for all such pairs (but those in the identity relation), to let model reduction not depend on any sentence.

This is not the only problem with model design, though. Our present aim is to get such edge colouring models that the vertex colouring models found in $\mathcal{V}$ could be naturally mapped onto them, embodying Tait's algorithms. However, these algorithms only specify mapping between *proper* vertex colourings and *proper* edge colourings. This raises two design problems: i) a colouring is proper with respect to a given triangulation, thus we are still faced with sentence dependence, and ii) models in $\mathcal{V}$ comprise *all* 4-colourings of vertices, including those which satisfy no sentence at all, so our mapping will have to account for them too.

We may deal with both problems at once if we consider what happens if we drop the assumption, made in the 4CT⇒3CT part of Tait's equivalence proof, that the vertex colouring $\mu{:}\underline{\mathbf{n+2}} \rightarrow \underline{\mathbf{4}}$ be a proper one. Then, by Klein sum, we may get edges "coloured 0", meaning that the two vertices have the same colour. One may use this within an edge colouring model to mean that it will not satisfy any sentence having some edge which links a pair of vertices whose label pair is "coloured 0" in that model. In the converse direction, we may only well-define vertex colourings as images of edge colourings in terms of Klein summation of edge colours along paths if the soundness condition is met that enabled us to do so in Tait's equivalence proof. We thus require edge-colouring models to have Klein sum 0 along every triangle. Summing up, we are led to the following definition.

Let $\mathrm{E}_{\underline{\mathbf{n+2}}}=0'_{\underline{\mathbf{n+2}}}/\mathrm{Sym}$ be the universe of (unordered) pairs of different vertex labels available for all triangulations in $\mathrm{T_V}(n)$, thus $\epsilon_\theta \subseteq \mathrm{E}_{\underline{\mathbf{n+2}}}$ for all $\theta \in \mathrm{T_V}(n)$, and $|\mathrm{E}_{\underline{\mathbf{n+2}}}|=(n+1)(n+2)/2$. For each $n>0$ we take as n-models the maps in the following set:

$$|\mathsf{Mod}_{\mathcal{E}'}(n)| = \{\nu : E_{\underline{\mathbf{n+2}}} \rightarrow \underline{\mathbf{4}} \mid i \neq j \neq k \neq i \Rightarrow \nu\{i,j\} \oplus \nu\{j,k\} \oplus \nu\{k,i\} = 0\}$$

and the n-model morphisms should be the 0-preserving colour permutations $\rho{:}\underline{\mathbf{4}} \leftrightarrow \underline{\mathbf{4}}$, thus for all $\nu, \nu' \in |\mathsf{Mod}_{\mathcal{E}'}(n)|$:

$$\mathsf{Mod}_{\mathcal{E}'}(n)(\nu,\nu') = \{\rho \in \underline{\mathbf{4}}^{\underline{\mathbf{4}}} | \rho \text{ is bijective}, \rho 0 = 0, \nu' = \nu \,;\, \rho\}$$

Edge colour permutations are required to preserve 0 because of its distinguished role in the Klein 4-group, that is consistent with our interpretation of that role as an edge "colour". This will be apparent in the definition of satisfaction below.

Model reduction is defined as expected: if $\nu \in |\mathsf{Mod}_{\mathcal{E}'}(n)|$ and $\pi{:}\underline{\mathbf{n+2}} \rightarrow \underline{\mathbf{n+2}}$ is a signature morphism, then $\nu\pi\{i,j\} = \nu\{\pi i, \pi j\}$, while $\rho\pi = \rho$ for all model morphisms ρ. The check that $\mathsf{Mod}_{\mathcal{E}'}$ so defined is indeed a functor is straightforward, since $(\nu\pi)\,;\rho = (\nu\,;\rho)\pi$, as it is easily seen.

Finally, according to the chosen sentence representation in $\mathcal{E}'$ we have

$$\begin{aligned}&\nu \models_{\mathcal{E}',n} \psi_\theta \text{ iff}\\ &\forall i,j,k \in \underline{\mathbf{n+2}}.(x_{\lambda_\theta\{i,j\}} - x_{\lambda_\theta\{j,k\}}) \text{ occurs in } \psi_\theta \Rightarrow 0 \neq \nu\{i,j\} \neq \nu\{j,k\} \neq 0.\end{aligned}$$

To get a relation-algebraic formulation of satisfaction in $\mathcal{E}'$, we use the view of a colouring map as a binary relation $\nu \subseteq \mathrm{E}_{\underline{\mathbf{n+2}}} \times \underline{\mathbf{4}}$, and we let ξ_{ψ_θ} denote the binary relation on $\underline{\mathbf{3n}}$ that is defined, for each $\psi_\theta \in \mathsf{Sen}_{\mathcal{E}'}(n)$, just as ξ_{ψ_ϑ} is, for each $\psi_\vartheta \in \mathsf{Sen}_{\mathcal{E}}(n)$ (cf. end of section 5). Let further $\iota_\theta{:}\epsilon_\theta \rightarrow \mathrm{E}_{\underline{\mathbf{n+2}}}$ be the inclusion map, viewed as a binary relation $\iota_\theta \subseteq \epsilon_\theta \times \mathrm{E}_{\underline{\mathbf{n+2}}}$. Then satisfaction in $\mathcal{E}'$ is characterized by

$$\nu \models_{\mathcal{E}',n} \psi_\theta \text{ iff } \nu\breve{}\,; \iota_\theta\breve{}\,; \lambda_\theta ; \xi_{\psi_\theta} ; \lambda_\theta\breve{}\,; \iota_\theta ; \nu \leq 0'_{\underline{\mathbf{4}}\backslash\underline{\mathbf{1}}}$$

This definition complies with the *satisfaction condition*:

$$\nu\pi \models_{\mathcal{E}',n} \psi_\theta \Leftrightarrow \nu \models_{\mathcal{E}',n} \pi\psi_\theta$$

as it is easily seen using the definitions of sentence translation and model reduction given above, therefore $\mathcal{E}'$, too, is an institution. Perhaps surprisingly, this turns out not to be apt to our redesign purpose yet. The story of our ultimate design problem leads us straight into the subject of the next section.

8 Natural fine tuning of institution design

The construction of a 4-colouring of vertices out of a given 3-colouring of edges presented in section 6 begins with the choice of a start-vertex, which is assigned colour 0. At a first glance, it may seem natural to make that choice appear as a constituent of the model transformation part of an institution (co)morphism between the institutions $\mathcal{V}$ and $\mathcal{E}'$, resp. introduced in sections 4 and 7. At a closer look, however, such an idea proves troublesome precisely with respect to naturality of the transformation! This is shown by the following analysis.

Suppose one defines, for $n > 0$, the components β_n of the model transformation $\beta{:}\mathsf{Mod}_{\mathcal{E}'}{\to}\mathsf{Mod}_{\mathcal{V}}$ as pairs $\beta_n{=}\langle \mathrm{b}_n, \mathrm{s}_n\rangle$, where $\mathrm{s}_n{\in}\underline{\mathbf{n{+}2}}$ be the start vertex and the functor $\mathrm{b}_n{:}\mathsf{Mod}_{\mathcal{E}'}(n){\to}\mathsf{Mod}_{\mathcal{V}}(n)$ be defined on objects according to the construction recalled above: for all $\nu{\in}|\mathsf{Mod}_{\mathcal{E}'}(n)|$, let $(\mathrm{b}_n\nu)\mathrm{s}_n = 0$ and $i{\neq}\mathrm{s}_n \Rightarrow (\mathrm{b}_n\nu)i{=}\nu\{\mathrm{s}_n, i\}$. So, s_n is fixed for each β_n. This destroys naturality, for the choice of the start-vertex is insensitive to vertex relabeling, so it doesn't commute with model reduction; if $\pi{:}\underline{\mathbf{n{+}2}}{\to}\underline{\mathbf{n{+}2}}$ is a signature morphism, then $(\mathrm{b}_n(\nu\pi))\mathrm{s}_n{=}0$ whilst $((\mathrm{b}_n\nu)\pi)\mathrm{s}_n{=}(\mathrm{b}_n\nu)(\pi\mathrm{s}_n)$ depends on π and is not always 0.

Now, the components of a natural transformation may only depend on the *objects* of the underlying category (the common domain category of the source and target functors), not on its arrows, so one may only make the choice of the start-vertex prove sensitive to signature morphisms if that choice shows up as an *institution* constituent, where those morphisms live.

The result of our analysis still leaves room to design decisions, such as: should the choice of the start-vertex be a constituent of signatures, or of models? In both institutions or just in one of them? To answer the first question, we take adherence to meaning as a pragmatic design criterion; since that choice is only relevant to model transformation, and is only needed in the construction of a vertex colouring model from an edge colouring model, we choose to extend the model structure to accommodate for the choice of the start-vertex. Our answer to the second question is driven by our wish to build an isomorphism between the subject institutions. This requires that every component of the model transformation be invertible. Now, this is only possible if one keeps track of the choice of the start-vertex when mapping models from either institution to the other (one of the two mappings would not be injective otherwise), so model structure is to be extended in both institutions.

Summing up, we introduce yet new versions $\mathcal{V}'$ and $\mathcal{E}''$ of both institutions, that respectively inherit both the category of signatures and the sentence functor from $\mathcal{V}$ and $\mathcal{E}'$, but have slightly modified model functors, defined as follows.

Extended model functor for vertex colouring

$$|\mathsf{Mod}_{\mathcal{V}'}(n)| = \{\upsilon = \langle \mu_\upsilon, \mathrm{s}_\upsilon\rangle \mid \mu_\upsilon{\in}\underline{\mathbf{4}}^{\underline{\mathbf{n+2}}}, \mathrm{s}_\upsilon{\in}\underline{\mathbf{n+2}}, \mu_\upsilon\mathrm{s}_\upsilon = 0\}$$
$$\forall\upsilon, \upsilon'{\in}|\mathsf{Mod}_{\mathcal{V}'}(n)|.\mathsf{Mod}_{\mathcal{V}}(n)(\upsilon, \upsilon') = \{\sigma{\in}\underline{\mathbf{4}}^{\underline{\mathbf{4}}} \mid \sigma \text{ is bijective}, \mu_{\upsilon'} = \mu_\upsilon;\sigma\}$$

So, the n-model morphisms still are the colour permutations, but constrained (by the target model) to keep 0 as the colour of the start-vertex (in the target model, where the start-vertex need not be the same as in the source model). Then the condition $\mu_{\upsilon'} = \mu_\upsilon ; \sigma$ entails $\sigma(\mu_\upsilon \mathrm{s}_{\upsilon'}) = 0$ by the constraint on models that fixes the colour of the start vertex to be 0. This cuts down by a factor of 4 the number of those colour permutations which induce morphisms between any two given models, thus from 4! down to 3!, whence a natural isomorphism construction between the model functors of the two institutions under design will prove possible (take it into account that colour permutations in the edge colouring institution, even though extended to "colour 0", are required to be Klein 4-group isomorphisms, thus 0-preserving permutations, hence there are just 3! of them).

Model reduction, for a given n-model $\upsilon{=}\langle \mu_\upsilon, \mathrm{s}_\upsilon \rangle$ and a signature morphism $\pi{:}\underline{\mathbf{n{+}2}}{\to}\underline{\mathbf{n{+}2}}$, is defined by $\upsilon\pi{=}\ \langle \mu_\upsilon \pi, \mathrm{s}_{\upsilon\pi} \rangle$, where $(\mu_\upsilon\pi)i = \mu_\upsilon(\pi i)$ like in $\mathcal{V}$, and moreover $\mathrm{s}_{\upsilon\pi} \stackrel{def}{=} \pi^{\smile} \mathrm{s}_\upsilon$. Letting $\sigma\pi = \sigma$ for all n-model morphisms σ and n-signature morphisms π, makes model reduction to be a functor, since $(\mu_\upsilon\pi) \,;\, \sigma = (\mu_\upsilon \,;\, \sigma)\pi$, as it is straightforward to check.

Extended model functor for edge colouring

$$|\mathsf{Mod}_{\mathcal{E}''}(n)| = \{\varepsilon = \langle \nu_\varepsilon : E_{\underline{\mathbf{n+2}}} {\to} \underline{\mathbf{4}}, \mathrm{s}_\varepsilon \rangle \mid \\ \mathrm{s}_\varepsilon {\in} \underline{\mathbf{n+2}} \wedge (i{\neq}j{\neq}k{\neq}i \Rightarrow \nu_\varepsilon\{i,j\} \oplus \nu_\varepsilon\{j,k\} \oplus \nu_\varepsilon\{k,i\} = 0)\}$$

$$\forall \varepsilon, \varepsilon' {\in} |\mathsf{Mod}_{\mathcal{E}''}(n)|.\ \ \mathsf{Mod}_{\mathcal{E}''}(n)(\varepsilon, \varepsilon') = \mathsf{Mod}_{\mathcal{E}'}(n)(\nu_\varepsilon, \nu_{\varepsilon'}) = \\ \{\rho {\in} \underline{\mathbf{4}}^{\underline{\mathbf{4}}}, \mid \rho \text{ is bijective}, \rho\, 0 = 0, \nu_{\varepsilon'} = \nu_\varepsilon ; \rho\}$$

So, the n-model morphisms still are the 0-preserving colour permutations. Model reduction, for a given n-model $\varepsilon{=}\langle \nu_\varepsilon, \mathrm{s}_\varepsilon \rangle$ and a signature morphism $\pi{:}\underline{\mathbf{n{+}2}}{\to}\underline{\mathbf{n{+}2}}$, is defined by $\varepsilon\pi{=}\ \langle \nu_\varepsilon\pi, \mathrm{s}_{\varepsilon\pi} \rangle$, where $(\nu_\varepsilon\pi)\{i,j\} = \nu_\varepsilon\{\pi i, \pi j\}$ like in $\mathcal{E}'$, and moreover $\mathrm{s}_{\varepsilon\pi} \stackrel{def}{=} \pi^{\smile} \mathrm{s}_\varepsilon$. Letting $\rho\pi = \rho$ for all n-model morphisms ρ and n-signature morphisms π, makes model reduction to be a functor, since $(\nu_\varepsilon\pi) \,;\, \rho = (\nu_\varepsilon \,;\, \rho)\pi$, as it is straightforward to check.

Satisfaction

Satisfaction in $\mathcal{V}'$ and $\mathcal{E}''$ is defined as in $\mathcal{V}$ and $\mathcal{E}'$, respectively, but only using the colouring map part of the model, whence the satisfaction condition is met just as it is in the respective, aforementioned previous versions of those institutions.

A final check would be in place, viz. that the fine tuning of the model functors just worked out does indeed solve the problem which motivated it. That turns out to be a natural task to be dealt with in the next section.

9 Graph colouring institution morphisms

Enough material is now available to build an isomorphism between the graph colouring institutions $\mathcal{V}'$ and $\mathcal{E}''$.

As a matter of notational convenience, for each of the three constituents of the isomorphism $\mathcal{T} : \mathcal{V}' \leftrightarrow \mathcal{E}''$, we use the same symbol in either direction, so we need not specify whether $\mathcal{T}$ is an institution morphism or comorphism. Furthermore, if γ is a bijection, we write $x \overset{\gamma}{\leftrightarrow} y$ to mean $y = \gamma x$ and $x = \gamma^{\smile} y$.

Let $\Phi \overset{def}{=} \mathbf{1}_{\mathsf{Sig}_{\mathcal{V}'}}$ be the identity functor on the category of signatures (the same in both institutions), then we are going to show the existence of natural isomorphisms α: $\mathsf{Sen}_{\mathcal{V}'} \leftrightarrow \mathsf{Sen}_{\mathcal{E}''}$ and β: $\mathsf{Mod}_{\mathcal{V}'} \leftrightarrow \mathsf{Mod}_{\mathcal{E}''}$, so that $\mathcal{T} = \langle \Phi, \alpha, \beta \rangle$ is an institution isomorphism $\mathcal{T} : \mathcal{V}' \leftrightarrow \mathcal{E}''$.
For each $n > 0$, the sentence transformation component $\alpha_n : \mathsf{Sen}_{\mathcal{V}'}(n) \leftrightarrow \mathsf{Sen}_{\mathcal{E}''}(n)$ is the bijection whereby $\epsilon_\theta \overset{\alpha_n}{\longleftrightarrow} \psi_\theta$ for each $\theta \in \mathrm{T_V}(n)$, with ϵ_θ as defined in section 4 and ψ_θ as defined in section 7.

The model transformation component $\beta_n : \mathsf{Mod}_{\mathcal{V}'}(n) \leftrightarrow \mathsf{Mod}_{\mathcal{E}''}(n)$ is the invertible functor defined by the following conditions.

Objects: for all $\upsilon = \langle \mu_\upsilon, \mathrm{s}_\upsilon \rangle \in |\mathsf{Mod}_{\mathcal{V}'}(n)|$ and $\varepsilon = \langle \nu_\varepsilon, \mathrm{s}_\varepsilon \rangle \in |\mathsf{Mod}_{\mathcal{E}''}(n)|$,

$$\upsilon \overset{\beta_n}{\longleftrightarrow} \varepsilon \text{ iff } \mathrm{s}_\upsilon = \mathrm{s}_\varepsilon \wedge \forall i, j \in \underline{\mathbf{n+2}}. i \neq j \Rightarrow \nu_\varepsilon\{i,j\} = \mu_\upsilon(i) \oplus \mu_\upsilon(j).$$

Arrows: for all $\upsilon, \upsilon' \in |\mathsf{Mod}_{\mathcal{V}'}(n)|$, $\varepsilon, \varepsilon' \in |\mathsf{Mod}_{\mathcal{E}''}(n)|$, $\sigma \in \mathsf{Mod}_{\mathcal{V}'}(n)(\upsilon, \upsilon')$, and $\rho \in \mathsf{Mod}_{\mathcal{E}''}(n)(\varepsilon, \varepsilon')$,

$$\sigma \overset{\beta_n}{\longleftrightarrow} \rho \text{ iff } \forall x \in \underline{\mathbf{4}}. \rho x = \sigma 0 \oplus \sigma x.$$

A little calculation shows that the clauses given above do indeed define a functor in either direction. For arrows, the map in the $\mathcal{E}'' \rightarrow \mathcal{V}'$ direction is also determined by the condition $\sigma(\mu_\upsilon \mathrm{s}_{\upsilon'}) = 0$, as previously pointed out. This, together with the conditions given above (on arrows as well as on objects), uniquely determines the σ permutation for a given ρ, thanks to the algebraic properties of the Klein sum. The verification of these facts is left to the reader.

As customary, the next checks address naturality of the transformations. These present no difficulties, but we include them to show that the fine tuning carried out in the previous section solves the motivating problem.

Naturality of sentence transformation

For all $n > 0$, π:$\underline{\mathbf{n{+}2}} \leftrightarrow \underline{\mathbf{n{+}2}}$ and $\theta \in \mathrm{T_V}(n)$, we have $\epsilon_{\pi\theta} \overset{\alpha_n}{\longleftrightarrow} \psi_{\pi\theta}$ by definition of α_n, and $\epsilon_{\pi\theta} = \pi\epsilon_\theta$ by definition of $\pi\theta$. So, in order to show naturality of the sentence transformation, that is $\pi\epsilon_\theta \overset{\alpha_n}{\longleftrightarrow} \pi\psi_\theta$, it is enough to prove $\pi\psi_\theta = \psi_{\pi\theta}$. Now, since π is a bijection, we can write the $\psi_{\pi\theta}$ polynomial as follows:

$$\prod_{\mathrm{t}_{ijk} \in \theta} (x_{\lambda_{\pi\theta}\{\pi i, \pi j\}} - x_{\lambda_{\pi\theta}\{\pi j, \pi k\}})(x_{\lambda_{\pi\theta}\{\pi j, \pi k\}} - x_{\lambda_{\pi\theta}\{\pi k, \pi i\}})(x_{\lambda_{\pi\theta}\{\pi k, \pi i\}} - x_{\lambda_{\pi\theta}\{\pi i, \pi j\}})$$

and since $\pi^2_\theta\, ; \lambda_{\pi\theta} = \lambda_\theta ; \pi^\#_\theta$ by definition of $\pi^\#_\theta$, the subscript replacement which this identity enables on the previous polynomial makes it identical to the polynomial which defines $\pi\psi_\theta$.

Naturality of model transformation

For all $n > 0$, $\pi{:}\underline{\mathbf{n+2}}\leftrightarrow\underline{\mathbf{n+2}}$, $\upsilon{=}\langle\mu_\upsilon, \mathrm{s}_\upsilon\rangle{\in}|\mathsf{Mod}_{\mathcal{V}'}(n)|$ and $\varepsilon{=}\langle\nu_\varepsilon, \mathrm{s}_\varepsilon\rangle{\in}|\mathsf{Mod}_{\mathcal{E}''}(n)|$,

$$\begin{aligned}
\upsilon \xleftrightarrow{\beta_n} \varepsilon \Leftrightarrow\ & \mathrm{s}_\upsilon = \mathrm{s}_\varepsilon \wedge (i \neq j \Rightarrow \nu_\varepsilon\{i,j\} = \mu_\upsilon(i) \oplus \mu_\upsilon(j)) \\
& \text{[by definition of } \beta_n\text{]} \\
\Leftrightarrow\ & \pi^{\smile}\mathrm{s}_\upsilon = \pi^{\smile}\mathrm{s}_\varepsilon \wedge (i \neq j \Rightarrow \nu_\varepsilon\{\pi i, \pi j\} = \mu_\upsilon(\pi i) \oplus \mu_\upsilon(\pi j)) \\
& \text{[since } \pi \text{ is a bijection]} \\
\Leftrightarrow\ & \mathrm{s}_{\upsilon\pi} = \mathrm{s}_{\varepsilon\pi} \wedge (i \neq j \Rightarrow \nu_{\varepsilon\pi}\{i,j\} = \mu_{\upsilon\pi}(i) \oplus \mu_{\upsilon\pi}(j)) \\
& \text{[by definition of model reduction]} \\
\Leftrightarrow\ & \upsilon\pi \xleftrightarrow{\beta_n} \varepsilon\pi \\
& \text{[by definition of } \beta_n\text{]}
\end{aligned}$$

Satisfaction condition

Let $n > 0$, $\epsilon_\theta{\in}\mathsf{Sen}_{\mathcal{V}'}(n)$, $\psi_\theta{\in}\mathsf{Sen}_{\mathcal{E}''}(n)$, thus $\epsilon_\theta{\xleftrightarrow{\alpha_n}}\psi_\theta$; let $\langle\mu, s\rangle{\in}\mathsf{Mod}_{\mathcal{V}'}(n)$ and $\langle\nu, s\rangle{\in}\mathsf{Mod}_{\mathcal{E}''}(n)$, with $\langle\mu, s\rangle{\xleftrightarrow{\beta_n}}\langle\nu, s\rangle$. Then we have got to show that

$$\langle\mu, s\rangle{\models_{\mathcal{V}',n}}\epsilon_\theta \Leftrightarrow \langle\nu, s\rangle{\models_{\mathcal{E}'',n}}\psi_\theta.$$

Let's first reduce our goal by the following sequence of implications:

$$\begin{aligned}
& \langle\mu, s\rangle{\models_{\mathcal{V}',n}}\epsilon_\theta \Leftrightarrow \langle\nu, s\rangle{\models_{\mathcal{E}'',n}}\psi_\theta \\
\text{if}\ & (\forall i,j.i\epsilon_\theta j \Rightarrow \mu i \neq \mu j) \Leftrightarrow (\forall i,j,k.\lambda_\theta\{i,j\}\xi_{\psi_\theta}\lambda_\theta\{j,k\} \Rightarrow 0 \neq \nu\{i,j\} \neq \nu\{j,k\} \neq 0) \\
& \text{[by definition of satisfaction in } \mathcal{V}' \text{ and } \mathcal{E}''\text{]} \\
\text{if}\ & (\forall i,j.i\epsilon_\theta j \Rightarrow \mu i \neq \mu j) \Leftrightarrow (\forall i,j,k.\lambda_\theta\{i,j\}\xi_{\psi_\theta}\lambda_\theta\{j,k\} \Rightarrow 0 \neq \mu i \oplus \mu j \neq \mu j \oplus \mu k \neq 0) \\
& \text{[by definition of } \beta_{\mathrm{n}}\text{].}
\end{aligned}$$

We now prove the two directions of this double implication separately.

(⇒)

$$\begin{aligned}
\lambda_\theta\{i,j\}\xi_{\psi_\theta}\lambda_\theta\{j,k\} & \Rightarrow (i\epsilon_\theta j \wedge j\epsilon_\theta k) && \text{[since } \epsilon_\theta \xleftrightarrow{\alpha_n} \psi_\theta\text{]} \\
& \Rightarrow \mu i \neq \mu j \neq \mu k && \text{[by assumption]} \\
& \Rightarrow (\mu i \oplus \mu j \neq 0) \wedge (\mu j \oplus \mu k \neq 0) && \text{[by properties of } \oplus\text{].}
\end{aligned}$$

Furthermore,

$$\begin{aligned}
\lambda_\theta\{i,j\}\xi_{\psi_\theta}\lambda_\theta\{j,k\} & \Rightarrow \lambda_\theta\{j,k\}\xi_{\psi_\theta}\lambda_\theta\{k,i\} && \text{[by triangular construction of } \psi_\theta\text{]} \\
& \Rightarrow k\epsilon_\theta i && \text{[since } \epsilon_\theta \xleftrightarrow{\alpha_n} \psi_\theta\text{]} \\
& \Rightarrow \mu k \neq \mu i && \text{[by assumption]} \\
& \Rightarrow \mu k \oplus \mu j \neq \mu i \oplus \mu j && \text{[by properties of } \oplus\text{],}
\end{aligned}$$

which, together with the outcome of the previous deduction sequence, concludes this part of the proof.

(⇐)

$$\begin{aligned}
i\epsilon_\theta j & \Rightarrow \exists k.\lambda_\theta\{i,j\}\xi_{\psi_\theta}\lambda_\theta\{j,k\} && \text{[since } \epsilon_\theta \xleftrightarrow{\alpha_n} \psi_\theta, \text{ and} \\
& && \text{by triangular construction of } \psi_\theta\text{]} \\
& \Rightarrow 0 \neq \mu i \oplus \mu j && \text{[by assumption]} \\
& \Rightarrow \mu i \neq \mu j && \text{[by properties of } \oplus\text{].}
\end{aligned}$$

Remark. The forgetful morphisms $\mathcal{E}''{\to}\mathcal{E}'$ and $\mathcal{V}'{\to}\mathcal{V}$, whereby the start-vertex disappears from the model structure, are also comorphisms $\mathcal{E}'{\to}\mathcal{E}''$ and $\mathcal{V}{\to}\mathcal{V}'$. Our case study seems to suggest a general technique to relate institutions, when some structure translation algorithm between their models is known, but that introduces additional, specific structure for the sole purpose of translation.

10 Conclusions

What general lessons of method can be drawn from the institution design exercises carried out in this work? We propose the following statements.

The first design decision to be taken is: Which variant of the institution concept should one use? Criteria to orientate this decision should be drawn from the intended purpose and context of the design, e.g. whether meant for model-theoretic analysis or proof-theoretic reasoning. Our approach to the case study elaborated in this paper belongs to the first mainstream; undoubtedly, similarly interesting design questions can be expected to arise in the syntactic mainstream.

A strong influence on the design of an institution is further expected to come from its intended, or desired, structural relationships (such as: embedding, equivalence, isomorphism) with other institutions. As a matter of fact, institution design turns out to be a byproduct of the design of appropriate institution (co)morphisms, when enough information about those relationships is available. The work here presented in sections 7 and 8 gives some support to consideration of the following as potentially relevant factors of influence in this respect:
1. A purposeful choice of the abstract syntactic map: not every functor between the given categories of signatures does the job, for example, the trivial one hardly ever does;
2. Totality of the sentence and model transformations, whereby *all* sentences of the source institution should get an image in the target one, not just a "meaningful" subset thereof; *idem* for the models;
3. Naturality of the sentence and model transformations, which may require fine-tuning of an institution design in order to make room, somewhere inside its structure, for support to information which is needed by a transformation algorithm but can't be embodied in the transformation itself without losing naturality.

The methodological conclusions proposed above are not necessarily the most useful outcome of this work. We rather expect, or at least hope, that outcome to show up as a rise of interest in carrying out further institution design exercises—in the fascinating area of finite model theory in particular. We are well aware that a term like *abstract finite model theory* is of a somewhat oxymoronic flavour, in that *finite* implies *set*, hence *concrete*, yet it might prove apt to designate an area of investigation where the use of abstract model-theoretic concepts in finite model-theoretic questions is taken as characteristic trait.

Further exercise ideas on the theme considered in this work, may be inspired by the vast literature on the subject, that over the years has resulted in several equivalent reformulations of the 4CT, see e.g. [17,14].

More closely relating to the problem considered in this paper, perhaps the following concluding remarks, while providing some kind of evaluation of what has been achieved, will also prove useful to stimulate further work.

We managed to find *one* solution to the problem proposed in [19]; there may be more... For instance, one could seek for design alternatives by exploring the feasibility of some "transfer of structure" from sentences to signatures. The careful reader may have noted sentence-dependence as a recurring problem in the analysis carried out in section 7. Clearly, if those sentences are taken as signatures, the problem disappears. This idea has not been pursued in this paper since it clashes against an original intuition of triangulations as abstract sentences, that we just wished to stick to, throughout the redesign work. We do not know whether it's a good or bad idea, we just point it out as an example of potential source of design alternatives.

Perhaps more interesting inspiration for further work also arises along the complementary evaluation path, viz. that of the design alternatives which *have* been undertaken here. This includes, for example, a critique of the design decisions which have shaped the models in $\mathcal{E}'$ and the model structure extension in $\mathcal{E}''$ and, $\mathcal{V}'$. One may argue that our solution to the problem has been obtained at the expense of a remarkable loss of simplicity, in comparison with the model structures in $\mathcal{V}$ and, especially in $\mathcal{E}$. While the objection is surely well-founded, two facts should not be missed, if a fair judgement is to be drawn. First, if Tait's equivalence had been taken as the core design topic from the outset, there would have been little reason to choose an abstract syntax based on self-standing edge labels, that is, unrelated to vertex labels. Without this choice, the $\mathcal{E}'$ model structure would have emerged rightaway, but we would have missed the chance of illustrating an instructive design problem. Second, which seems more relevant to a future work perspective, other combinatorial problems naturally arise from the model construction as in $\mathcal{E}'$, e.g. relating to characterization of 4-colourability of graphs, not necessarily planar ones. This view is justified by the following considerations.

1. All possible edges between the given vertices are available in models, viz. the set $\mathrm{E}_{\underline{\mathbf{n+2}}}$, so there's no obstacle to extend attention to nonplanar graphs.

2. The detachment of models from sentences that is enforced by the institution structure, has led us to build models where not only the colour of (some) edges is given, but also the admissibility of the presence of (other) edges, in sentences that are satisfied by a given model.

3. It is well known that some nonplanar graphs are 4-colourable (some even 2-colourable!), thus the 4CT gives an only *sufficient* condition for 4-colourability, that is planarity, which has the virtue of being *syntactic* precisely in the sense which this term has taken in this paper—a property of *sentences*, as it were. One may wonder whether a *syntactic characterization* of 4-colourability may exist, and of k-colourability more generally, besides the obvious one. This problem seems to be widely open.

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