Graph colouring institutions*

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Abstract. Maximal planar graphs with vertex resp. edge colouring are naturally casted as (deceiptively similar) institutions. One then tries to embody Tait's equivalence algorithms into morphisms between them, and is lead to a partial redesign of those institutions. This paper aims at introducing a few pragmatic questions which arise in this case study, which also showcases the use of relational concepts and notations in the design of the subject institutions.

Keywords: abstract model theory, institution, institution morphisms, relation algebra, four colour theorem, graph colouring

1 Introduction

Institution morphisms are a lively, albeit controversial subject of debate in the community of researchers who investigate abstract model-theoretic concepts and methods [7] in computing.

The original definition for these structure maps [10] was soon to compete with differently conceived, variously motivated proposals, such as the "maps", "simulations", "transformations", respectively found in [14,6,17], among (several) others. Recent work [11] aims at systematic investigation of properties and interrelations of these notions, that surely is a promising, useful effort.

So far, lesser attention seems to have been attracted by pragmatic questions relating to institution morphisms, whatever sensible kind thereof, such as the understanding of how do those maps affect the design of institutions, meant as formalizations of given logical frameworks. This question is not necessarily to be understood in a "comparative" sense; that is to say, our expectation is that even in straightforward cases where different notions of institution morphism have essentially equivalent instances, it may well happen that institutions designed without taking morphisms into account need to be (partially) redesigned when the problem of mapping (relating, translating, structuring) them comes into play. The present paper is aimed at presenting a little exercise of this kind. We start with introducing and motivating the exercise idea.

* This research has been partially supported by MURST Grant prot. 2001017741 under project "Ragionamento su aggregati e numeri a supporto della programmazione e relative verifiche" at the DMI Department of the University of Catania.

The Four Colour Theorem (4CT) is a paradigmatic case of potential applicability of methods and results that are offspring of research on translations between logical frameworks. Here is why the 4CT offers an interesting case study for translation concepts and methods relating to logical frameworks.

Our starting point is a view of the 4CT as a consistency theorem of finite, ad-hoc logics of graph colouring. The plural form *logics* here is purposeful, since a well-known result by Tait [19,20] proves the equivalence between the 4CT with *vertex colouring* and the 3CT with *edge colouring*. The latter means proper colouring of edges rather than vertices, where "proper" is spelled out as the condition that adjacent edges, i.e. at the border of a same triangular face, must be assigned different colours, whereas adjacent vertices must be assigned different colours by a proper vertex colouring.

Now, Tait's equivalence comes equipped with a constructive proof, whereby algorithms are exhibited that turn any given proper 4-colouring of vertices of any given maximal planar graph into a proper 3-colouring of its edges, and vice versa—see e.g. [8] for an outline of Tait's algorithms. In this paper we use somewhat simpler algorithms for graph colouring conversion, that exploit the nice algebraic properties of the Klein 4-group, as presented in [1].

So, here's our basic idea for an exercise aimed at testing practical impact of institution morphisms, possibly in different flavours, into institution design in the case study in question: 1) formalize maximal planar graph colouring by two distinct institutions, respectively with vertex colourings and edge colourings as models, and 2) (try to) cast Tait's equivalence into a pair of converse morphisms between the two institutions.

The first part of the exercise already raises institution design questions, e.g. the choice of signature morphisms; on pragmatic grounds, one might like to have such morphisms formalize edge contraction, in view of the relevant role played by this operation in reducibility proofs [5], yet contraction doesn't preserve maximality of planar graphs in all cases, which entails that the Set-valued sentence functor ought to map those morphisms to partial functions. One may take the design decision to formulate just vertex, resp. edge permutations as signature morphisms, since these operations are of practical interest, too. This leads to a straightforward solution of the first part of the exercise; in particular, sentences in the institution with edge colouring have an amazing syntactic representation by Matiyasevitch's polynomials [12], whereby the *number* of proper colourings of any given maximal planar graph is quickly found.

The second part of the exercise raises new design questions. Since the solution of the first part was determined without taking mutual interpretability of the two institutions into account, one shouldn't be surprised at finding out that Tait's algorithms prove hard to get embodied into structure-preserving maps between those institutions. Our redesign work in this respect seems quite instructive, but is not included here because of space constraints. Here we work out a solution to the first part of the exercise, where we also showcase the use of relational concepts and notations, whereby one gets a pleasing conciseness and elegance in their presentation.

2 Graph colouring preliminaries

The first proof of the 4CT [2,3,4] raised controversial discussions due to its combinatorial complexity which, for the analysis of the nearly 2000 graphs involved, required the construction of a program—whose correctness was not proven though. A new proof was obtained by [15], keeping the structure of the previous proof but cutting down to 633 the number of graphs involved. The impression yet remains that a simpler reason for the truth of this theorem may exist. One dreams of a logical construction where combinatorics weigh no more than necessary, that is, inherent to the problem rather than to specifics of the proof.

Notation

$\underline{\mathbf{n}}$: the finite ordinal consisting of n elements, viz. the natural numbers from 0 through n-1.

$1_{\underline{\mathbf{n}}}$, $1'_{\underline{\mathbf{n}}}$, $0'_{\underline{\mathbf{n}}}$: resp. the *universal*, *identity* and *diversity* binary relations on $\underline{\mathbf{n}}$, thus $0'_{\underline{\mathbf{n}}} = 1_{\underline{\mathbf{n}}} \backslash 1'_{\underline{\mathbf{n}}}$ or, with standard relation-algebraic notation for the Boolean complement operation: $0'_{\underline{\mathbf{n}}} = {1'_{\underline{\mathbf{n}}}}^{-1}$.

$r^{\smile}$: relation-algebraic converse of r.

$T_V(n)$: the set of $\underline{\mathbf{n+2}}$-labeled n+2-vertex triangulations of the sphere.

$T_E(n)$: the set of $\underline{\mathbf{3n}}$-labeled 3n-edge triangulations of the sphere.

3 Institution preliminaries

The classic definition of institution, already appearing in the paper introducing this concept [9], will suffice for our purposes. Generalizations of this definition were proposed later [10], base on twisted relation categories, that allow one to choose one of set-structure or category-structure for sentences as well as for models, the two choices being independent of each other. This gives rise to four variants of the institution concept, while other variants have been proposed too, referred to as "close variants" in [11].

An *institution* is a 4-tuple $\mathcal{I} = (\mathsf{Sig}, \mathsf{Sen}, \mathsf{Mod}, \models)$, with:

(i) Sig a category, whose objects are called *signatures*,

(ii) $\mathsf{Sen}{:}\mathsf{Sig}{\to}\textsc{Set}$ a functor, sending each signature Σ to the set $\mathsf{Sen}(\Sigma)$ of Σ-sentences, and each signature morphism $\pi{:}\Sigma_1{\to}\Sigma_2$ to the mapping $\mathsf{Sen}(\pi){:}\mathsf{Sen}(\Sigma_1){\to}\mathsf{Sen}(\Sigma_2)$ that translates Σ_1-sentences to Σ_2-sentences,

(iii) $\mathsf{Mod}{:}\mathsf{Sig}^{\mathrm{op}}{\to}\textsc{Cat}$ a contravariant functor, sending each signature Σ to the category $\mathsf{Mod}(\Sigma)$ of Σ-models, and each signature morphism $\pi{:}\Sigma_1{\to}\Sigma_2$ to the π-reduction functor $\mathsf{Mod}(\pi){:}\mathsf{Mod}(\Sigma_2){\to}\mathsf{Mod}(\Sigma_1)$,

(iv) $\models : |\mathsf{Sig}| \to ||\textsc{Rel}||$ a[1] $|\mathsf{Sig}|$-indexed set of binary relations $\models_\Sigma \subseteq |\mathsf{Mod}(\Sigma)| \times \mathsf{Sen}(\Sigma)$, viz. a satisfaction relation between Σ-models and Σ-sentences for each $\Sigma \in |\mathsf{Sig}|$, such that the following *satisfaction condition* holds for all $\pi{:}\Sigma_1{\to}\Sigma_2 \in ||\mathsf{Sig}||$, Σ_2-models M and Σ_1-sentences φ:

$$\mathsf{Mod}(\pi)(\mathrm{M}) \models_{\Sigma_1} \varphi \Leftrightarrow \mathrm{M} \models_{\Sigma_2} \mathsf{Sen}(\pi)(\varphi)$$

[1] Rel is the category of sets with binary relations as morphisms; $|\mathcal{C}|$ is the set of objects of category $\mathcal{C}$, while $||\mathcal{C}||$ is the set of morphisms of category $\mathcal{C}$.

Notation

A few notational conventions will simplify the presentation.

We shall henceforth adopt the abbreviations: $\pi\varphi$ for $\mathsf{Sen}(\pi)(\varphi)$, and $\mathrm{M}\pi$ for $\mathsf{Mod}(\pi)(\mathrm{M})$, where $\pi{:}\Sigma_1 \rightarrow \Sigma_2$ is a signature morphism, φ is a Σ_1-sentence, and M is a Σ_2-model.

As usual, if a and b are objects in category $\mathcal{C}$, then $\mathsf{Hom}_{\mathcal{C}}(\mathrm{a},\mathrm{b})$ is the subset of $||\mathcal{C}||$ which consists of the morphisms from a to b.

When considering different institutions, it proves convenient to decorate the name of each element of the 4-tuple which an institution consists of, by adding the institution name as first subscript.

4 A vertex colouring institution

Syntax will be abstract, exploiting the fact that institutions do not force one to deal with concrete syntax. Signatures are just positive numbers, ranking maximal planar graphs by their size, and we take the *bijective* relabelings of vertices as signature morphisms. This restriction is a design decision, motivated as follows.

Each n>0 is the rank of the maximal planar graphs, or triangulations of the sphere, that have n+2 vertices. Vertex colouring of such structures require that each vertex be given a unique identity. To this purpose we consider vertices to be uniquely labeled by the elements of finite ordinal $\underline{\mathbf{n+2}}$, for triangulations of rank n. Bijective relabelings are thus just label permutations. The pragmatic question arises as to what purpose could be served by non-bijective maps on finite ordinals. On the one hand, loss of surjectivity appears useless, insofar as it introduces labels in the morphism codomain that are not made use of to label any vertex, according to the morphism image. On the other hand, though, loss of injectivity would seem to be of some use, inasmuch it amounts to identify formerly distinct vertices, thus it could prove useful to formalize edge contraction—whenever an edge connects two such vertices. This operation, however, does not preserve maximality of planar graphs. This happens when a vertex of degree 3 is opposite to the contracted edge, see e.g. fig. 1, where the dashed edge is subject to contraction and its opposite vertices, both of degree 3, are circled.

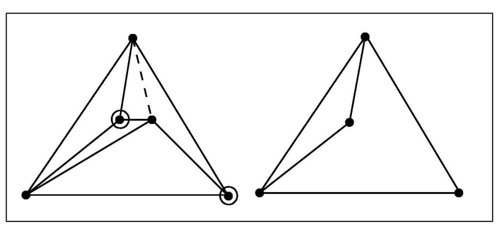

Fig. 1. Edge contraction not preserving maximality of planar graphs

Since edge contraction decreases by one the degree of those vertices which are opposite to the contracted edge, the resulting graph proves maximal in just one case, viz. when it is the smallest triangulation, which consists of three vertices of degree 2 (no other triangulation has vertices of degree 2).

We must conclude that, if non-injective relabelings were admitted as signature morphisms, then the Set-valued sentence functor, giving the set of vertex-labeled triangulations of rank n for each n>0, ought to map those morphisms to *partial* functions, whereas only total functions are available as morphisms in Set.

Signatures

$|\mathsf{Sig}_{\mathcal{V}}| = \mathbf{N}\backslash\{0\}$
$\mathsf{Hom}_{\mathsf{Sig}_{\mathcal{V}}}(\mathrm{n},\mathrm{n}) = \{\pi{:}\underline{\mathbf{n+2}}\to\underline{\mathbf{n+2}} \mid \pi \text{ is bijective}\}$
$\mathsf{Hom}_{\mathsf{Sig}_{\mathcal{V}}}(\mathrm{m},\mathrm{n}) = \emptyset$ if m≠n

Sentences

Each $\theta\in\mathrm{T_V}(\mathrm{n})$ is represented by the symmetric quotient of a binary relation on vertices, $\epsilon_\theta \overset{def}{=} \eta_\theta/\mathrm{Sym}$, where η_θ is the irreflexive, symmetric edge relation of θ, thus satisfies the relation-algebraic laws $\eta_\theta\leq 0'_{\underline{\mathbf{n+2}}}$, $\eta_\theta = \eta_\theta{}^{\smile}$, while $|\eta_\theta| =$ 6n, but the Sym quotient turns ordered pairs into unordered ones, thus $|\epsilon_\theta| =$ 3n. As a matter of notation, we write $i\ \epsilon_\theta\ j$ or $\{i,j\}\in\epsilon_\theta$, rather than the more cumbersome $\{(i,j),(j,i)\}\in\epsilon_\theta$, whenever $\{(i,j),(j,i)\}\subseteq\eta_\theta$. We thus define:

$$\mathsf{Sen}_{\mathcal{V}}(n) = \{\epsilon_\theta \mid \theta\in\mathrm{T_V}(n)\}$$

Sentence translation

If $\pi\in\mathsf{Hom}_{\mathsf{Sig}_{\mathcal{V}}}(\mathrm{n},\mathrm{n})$ and $\epsilon_\theta\in\mathsf{Sen}_{\mathcal{V}}(\mathrm{n})$, then $\pi\epsilon_\theta\in\mathsf{Sen}_{\mathcal{V}}(\mathrm{n})$, with

$$(\pi i)\ \pi\epsilon_\theta\ (\pi j) \Leftrightarrow i\ \epsilon_\theta\ j$$

Models

The model functor assigns to each signature n>0 the category of 4-colourings of the n+2 vertices, with colour permutations as model morphisms:

$$|\mathsf{Mod}_{\mathcal{V}}(n)| = \underline{\mathbf{4}}^{\underline{\mathbf{n+2}}}, \qquad ||\mathsf{Mod}_{\mathcal{V}}(n)|| = \{\rho\in\underline{\mathbf{4}}^{\underline{\mathbf{4}}}|\rho \text{ is bijective}\}$$

where a colour permutation ρ is a model morphism $\rho{:}\mu\to\mu'$ whenever $\mu'=\rho\circ\mu$.

Model reduction

If $\pi{:}\underline{\mathbf{n+2}}\to\underline{\mathbf{n+2}}\in\mathsf{Hom}_{\mathsf{Sig}_{\mathcal{V}}}(\mathrm{n},\mathrm{n})$ and $\mu{:}\underline{\mathbf{n+2}}\to\underline{\mathbf{4}}\in|\mathsf{Mod}_{\mathcal{V}}(\mathrm{n})|$, then $\mu\pi\in|\mathsf{Mod}_{\mathcal{V}}(\mathrm{n})|$, with $\mu\pi(\mathrm{i}) \overset{def}{=} \mu(\pi\mathrm{i})$, and $\rho\pi \overset{def}{=} \rho$ for all colour permutations $\rho\in||\mathsf{Mod}_{\mathcal{V}}(\mathrm{n})||$. This makes model reduction functorial, with

$$\rho\circ(\mu\pi) = (\rho\circ\mu)\pi$$

Satisfaction

In $\mathcal{V}$, a n-model satisfies a n-sentence iff it is a proper vertex-colouring of that triangulation, that is:

$$\mu \models_{\mathcal{V},n} \epsilon_\theta \text{ iff } \forall i,j{\in}\underline{\mathbf{n+2}}.\ i\epsilon_\theta j \Rightarrow \mu i{\neq}\mu j$$

A relation-algebraic formulation of this definition may exploit the "oriented" edge relation η_θ from which ϵ_θ is obtained as a quotient, and the view of the 4-colouring map as a binary relation $\mu \subseteq \underline{\mathbf{n+2}}{\times}\underline{\mathbf{4}}$. Then we get:

$$\mu \models_{\mathcal{V},n} \epsilon_\theta \text{ iff } \mu^{\smile};\eta_\theta;\mu \leq 0'_{\underline{\mathbf{4}}}$$

This definition complies with the *satisfaction condition*:

$$\mu\pi \models_{\mathcal{V},n} \epsilon_\theta \Leftrightarrow \mu \models_{\mathcal{V},n} \pi\epsilon_\theta$$

therefore $\mathcal{V}$ is an *institution*.

5 An edge colouring institution

Syntax will be somewhat more concrete, inspired by Matiyasevich's polynomial representation of triangulations of the sphere [12]. Signatures remain the same, but we now take the bijective relabelings of *edges* as signature morphisms.

Signatures

$|\mathsf{Sig}_{\mathcal{E}}| = |\mathsf{Sig}_{\mathcal{V}}| = \mathbf{N}\backslash\{0\}$
$\mathsf{Hom}_{\mathsf{Sig}_{\mathcal{E}}}(\mathrm{n,n}) = \{\pi{:}\underline{\mathbf{3n}}{\to}\underline{\mathbf{3n}} \mid \pi \text{ is bijective}\}$
$\mathsf{Hom}_{\mathsf{Sig}_{\mathcal{E}}}(\mathrm{m,n}) = \emptyset$ if m$\neq$n

Sentences

Sentences in $\mathsf{Sen}_{\mathcal{E}}(\mathrm{n})$, ranged over by ψ_ϑ, are represented by Matiyasevich's polynomials in product form:

$$\psi_\vartheta \;=\; \prod_{\mathrm{t}_{ijk}\in\vartheta} (x_i - x_j)(x_j - x_k)(x_k - x_i)$$

where $\vartheta{\in}\mathrm{T_E}(\mathrm{n})$ and t_{ijk} is a triangular face of ϑ having edges labeled i,j,k in clockwise order. We thus define:

$$\mathsf{Sen}_{\mathcal{E}}(n) = \{\psi_\vartheta \mid \vartheta{\in}\mathrm{T_E}(n)\}$$

Sentence translation

If $\pi{\in}\mathsf{Hom}_{\mathsf{Sig}_{\mathcal{E}}}(\mathrm{n,n})$ and $\psi_\vartheta{\in}\mathsf{Sen}_{\mathcal{E}}(\mathrm{n})$ represented as above, then $\pi\psi_\vartheta{\in}\mathsf{Sen}_{\mathcal{E}}(\mathrm{n})$, with

$$\pi\psi_\vartheta \;=\; \prod_{\mathrm{t}_{ijk}\in\vartheta} (x_{\pi i} - x_{\pi j})(x_{\pi j} - x_{\pi k})(x_{\pi k} - x_{\pi i})$$

Models

The model functor assigns to each signature n>0 the category of 3-colourings of the 3n edges, with colour permutations as model morphisms:

$$|\mathsf{Mod}_{\mathcal{E}}(n)| = \underline{\mathbf{3}}^{\underline{\mathbf{3n}}}, \qquad ||\mathsf{Mod}_{\mathcal{E}}(n)|| = \{\rho\in\underline{\mathbf{3}}^{\underline{\mathbf{3}}} | \rho \text{ is bijective}\}$$

where a colour permutation ρ is a model morphism $\rho{:}\nu{\to}\nu'$ whenever $\nu'{=}\ \rho{\circ}\nu$.

Model reduction

If $\pi{:}\underline{\mathbf{3n}}{\to}\underline{\mathbf{3n}}{\in}\mathsf{Hom}_{\mathsf{Sig}_{\mathcal{E}}}(\text{n,n})$ and $\nu{:}\underline{\mathbf{3n}}{\to}\underline{\mathbf{3}}{\in}|\mathsf{Mod}_{\mathcal{E}}(\text{n})|$, then $\nu\pi{\in}|\mathsf{Mod}_{\mathcal{E}}(\text{n})|$, with $\nu\pi(\text{i}) \stackrel{def}{=} \nu(\pi\text{i})$, and $\rho\pi \stackrel{def}{=} \rho$ for all colour permutations $\rho{\in}||\mathsf{Mod}_{\mathcal{E}}(\text{n})||$. This makes model reduction functorial, with

$$\rho{\circ}(\nu\pi) = (\rho{\circ}\nu)\pi$$

Satisfaction

In $\mathcal{E}$, a n-model satisfies a n-sentence iff it is a proper edge-colouring of that triangulation, that is:

$$\nu \models_{\mathcal{E},n} \psi_\vartheta \text{ iff } \forall i,j{\in}\underline{\mathbf{3n}}.(x_i - x_j) \text{ occurs in } \psi_\vartheta \Rightarrow \nu i{\neq}\nu j$$

A relation-algebraic formulation of this definition may use the binary relation of "occurrence in ψ_ϑ", $\xi_{\psi_\vartheta}{\leq}1_{\underline{\mathbf{3n}}}$: $i\ \xi_{\psi_\vartheta}\ j$ iff $(x_i - x_j)$ occurs in ψ_ϑ. Then, by using the view of a 3-colouring map as a binary relation $\nu \subseteq \underline{\mathbf{3n}}{\times}\underline{\mathbf{3}}$, we get:

$$\nu \models_{\mathcal{E},n} \psi_\vartheta \text{ iff } \nu^{\smile};\xi_{\psi_\vartheta};\nu \leq 0'_{\underline{\mathbf{3}}}$$

This definition complies with the *satisfaction condition*:

$$\nu\pi \models_{\mathcal{E},n} \psi_\vartheta \Leftrightarrow \nu \models_{\mathcal{E},n} \pi\psi_\vartheta$$

therefore $\mathcal{E}$, too, is an *institution.*

6 Tait's equivalence

A triangulation admits a proper 4-colouring of its vertices if, and only if, it admits a proper 3-colouring of its edges. This is Tait's classical result [19,20], albeit here stated in graph-theoretic terms rather than, as in its original formulation, in terms of cubic map colourings. The equivalence is shown by exhibiting two algorithms, which we are going to recast in graph-theoretic terms, that for any given triangulation respectively turn any proper 4-colouring of its vertices into a proper 3-colouring of its edges, and vice versa.

We take $\underline{\mathbf{4}}$ as the set of colours for vertex-colouring and $\underline{\mathbf{4}}\backslash\underline{\mathbf{1}}$ that for edge-colouring. Taking the latter rather than $\underline{\mathbf{3}}$ somewhat simplifies the presentation of Tait's algorithms, thanks to the properties of an elegant, algebraic construction which uses the *Klein 4-group*, as provided in [1]. We take $\underline{\mathbf{4}}$ as the group carrier, with 0 as its neutral element. Every element is self-inverse, and the binary group operation $\oplus$ further satisfies $x \oplus y = z$ whenever $\{x, y, z\} = \underline{\mathbf{4}}\backslash\underline{\mathbf{1}}$. This defines $\oplus$, since $0 \oplus x = x \oplus 0 = x \oplus x = 0$ for all $x{\in}\underline{\mathbf{4}}$, by the previous conditions.

4CT ⇒ 3CT

Let μ:$\underline{\mathbf{n+2}}\to\underline{\mathbf{4}}$ be a proper 4-colouring of given triangulation $\theta\in\mathrm{T_V}(\mathrm{n})$. For each edge x in θ, let $\mu i\neq\mu j$ be the colours assigned by μ to the vertices connected by x. Then their Klein sum $\mu i \oplus \mu j$ is the colour assigned to edge x.

By the properties of the Klein 4-group, this colour is never 0 insofar as $\mu i\neq\mu j$ (by assumption, μ is a proper colouring of the vertices of θ). Furthermore, any two edges sharing a face get different colours since they share one vertex (thus one addend of the Klein sums yielding their respective colours), whereas the other two vertices they resp. join are coloured differently by μ, as they are the ends of the third edge sharing the same face. We thus have a proper 3-colouring of the edges of θ, with colours out of $\underline{\mathbf{4}}\backslash\underline{\mathbf{1}}$.

3CT ⇒ 4CT

The construction in the converse direction is a bit more complex. Let ν:$\underline{\mathbf{3n}}\to\underline{\mathbf{4}}\backslash\underline{\mathbf{1}}$ be a proper 3-colouring of given triangulation $\vartheta\in\mathrm{T_E}(\mathrm{n})$. Choose a vertex in the triangulation as start-vertex, and assign it colour 0. Every other vertex is then coloured by the Klein sum of the colours assigned by ν to the edges of any path from the start-vertex to that vertex.

Of course, the specified construction is only sound if Klein summation of the colours assigned by ν proves invariant for all paths joining any given pair of vertices. This holds because (i) every element is self-inverse in the Klein group and (ii) Klein summation of the colours assigned by ν along every circuit turns out to be 0. A proof of this fact is worked out in [1] (pp. 22–23), for the colouring of cubic maps, but it can be readily interpreted in our present setting, just as follows.

Let $\mathrm{S}=\sum \nu x_i$ be the Klein sum of the colours assigned by ν to the edges of a given circuit K, and consider those triangular faces which belong to one of the two regions of the sphere having K as border (no matter which one). Since ν is a proper 3-colouring with colours out of $\underline{\mathbf{4}}\backslash\underline{\mathbf{1}}$, the Klein sum of the colours of the edges of any given triangular face is always $1\oplus 2\oplus 3=0$. Summation over all faces considered above must obviously yield 0 as well. Now, each edge in K is counted only once in this summation, whereas each other edge is counted twice, giving thus a null contribution to the summation since $x\oplus x=0$ for all $x\in\underline{\mathbf{4}}$. We must thus conclude that summation over the edges in K alone must yield 0, viz. S=0.

Finally, it is immediately seen that the 4-colouring of vertices specified above is proper, since adjacent vertices have paths from the start-vertex that differ by one edge only, and ν assigns a non-zero colour to this edge, whence the two vertices get different colours.

One may wonder whether the present construction has *exactly* the previous one as its inverse, for every given triangulation. That is to say, if one starts with a proper 3-colouring ν of edges, gets a proper 4-colouring of vertices out of it as specified here above, and then gives this as input to the previous algorithm, does then this yield back the 3-colouring ν one started with?

The answer is positive, and an *almost* similar exactness holds in the converse direction, where one starts with a proper 4-colouring μ of vertices, and gets it back *if* in the second, 3CT→4CT stage the start-vertex is chosen among those which are assigned colour 0 by μ. The proof of these facts exploits the algebraic properties of the Klein 4-group, and is left to the reader as an exercise.

7 Morphism-driven redesign of institutions

A basic obstacle makes it impossible to embody Tait's algorithms into (whatever kind of) morphism between the $\mathcal{V}$ and $\mathcal{E}$ institutions presented above, and that is: the lack of a non-trivial functorial mapping between their categories of signatures. Although those categories share their objects, their signature morphisms differ, and these prove hard to map. It's seems worthwhile to review the implicit reason for the choice of different signature morphisms in the design of the aforementioned institutions.

The choice of signature morphisms for $\mathcal{V}$ was just the obvious one, as far as abstract syntax for vertex colourings is concerned. Similarly, that for $\mathcal{E}$ was inspired by Matiyasevich's polynomial representation of triangulations, where only the naming of edges matter, thus it seemed fairly natural to take edge renamings as the edge colouring counterpart of vertex renamings for vertex colouring, as far as abstract syntax for edge colourings is concerned. This choice is actually sentence independent, in that it only depends on the rank of the triangulation (since every triangulation of given rank n has the same number of edges, that is 3n), therefore it was appropriate as a design choice for signature morphisms.

Our "local" design choices of signature morphisms prove no longer appropriate when a wider perspective is taken, that is to say, as soon as one needs to know which vertices are connected by which edges—as it happens to be the case with Tait's algorithms, for example.

Now, as a concluding remark, we note that the situation whereby a structure is first designed according to a local view of its purpose, and only later a wider context of its operation comes into play, seems to be a fairly general trait of human design activities. Since the main vehicle of context interaction on abstract algebraic structures is the concept of structure-preserving map, or morphism, it is hardly surprising to find out that context-driven redesign becomes morphism-driven redesign in the case under study. A solution to this problem is worked out in [18], where an isomorphism between the redesigned institutions is obtained.

8 Further work

Further exercise ideas, on the theme considered here, may be inspired by the vast literature on graph colouring, that over the years has resulted in several equivalent reformulations of the 4CT, see e.g. [16,13].

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