Modelling a Cart

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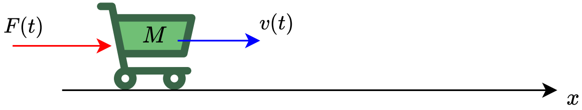
Robotic Systems

A Cart

- Let's consider a **cart** moving over a 1-dimensional space
- We suppose that is has a mass M and is equipped with electric motors
- At the instant $t = 0$ we turn-on the motors, so they provide a certain **traction force** $F(t)$

- What is the behaviour of the cart?
- What is the trend of the speed $v(t)$ over time?
- What is the trend of the position $x(t)$ over time?



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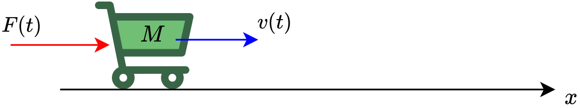

The Second Principle of Dynamics

Let us use the second principle of dynamics: the sum of the forces in a rigid body is equal to the mass' body times the acceleration

$$\sum \vec{f}(t) = M\,\vec{a}(t)$$

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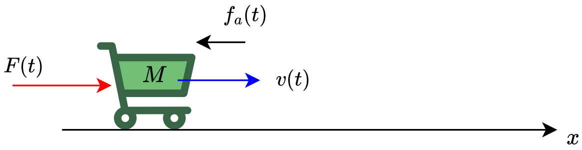

The Second Principle of Dynamics

The forces we consider are:

- The **traction** generated by the motors $F(t)$;
- The **friction of the air** $f_a(t)$ that is dependent of the **speed** and is **opposite to** the speed;
- At low speed regimes, the friction can be modeled with the Stokes law:

$$f_a(t) = -B\,v(t)$$

 were B is a constant that depends on the viscosity of the fluid (air) and the shape of the body

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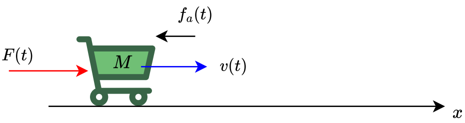


The Second Principle of Dynamics

Therefore the model is:

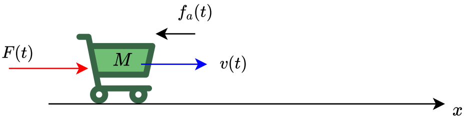
$$F(t) + f_a(t) = M\, a(t)$$
$$F(t) - B\, v(t) = M\, a(t)$$

given that:

$$a(t) = \dot{v}(t)$$

we have:

$$F(t) - B\, v(t) = M\, \dot{v}(t)$$



$$F(t) - B\ v(t) = M\ \dot{v}(t)$$

This a differential equation whose analytical solution strongly depends on the analytical form of $F(t)$, however...

... that analytical form can be implemented using a combination odf proportional and derivative blocks

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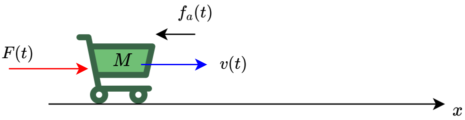

Let's reorder the terms:

$$v(t) = \frac{1}{B}F(t) - \frac{M}{B}\,\dot{v}(t)$$

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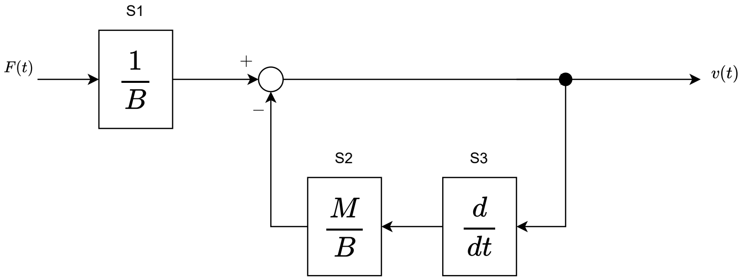
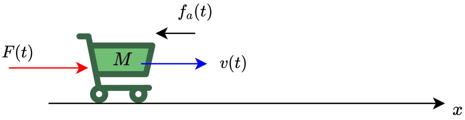


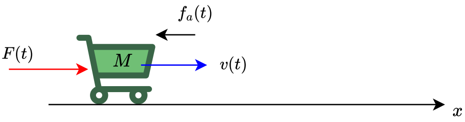


$$F(t) - B\, v(t) = M\, \dot{v}(t)$$

In alternative, we can **discretise** the differential equation:

$$\dot{v}(t) = \frac{1}{M} F(t) - \frac{B}{M}\, v(t)$$

$$\frac{v(t + \Delta T) - v(t)}{\Delta T} = \frac{1}{M} F(t) - \frac{B}{M}\, v(t)$$



Let's reorder the terms:

$$\frac{v(t+\Delta T) - v(t)}{\Delta T} = \frac{1}{M} F(t) - \frac{B}{M}\, v(t)$$

$$v(t+\Delta T) - v(t) = \frac{\Delta T}{M}\, F(t) - \frac{B\,\Delta T}{M}\, v(t)$$

$$v(t+\Delta T) = v(t) - \frac{B\,\Delta T}{M}\, v(t) + \frac{\Delta T}{M}\, F(t)$$

$$v(t+\Delta T) = v(t)\,(1 - \frac{B\,\Delta T}{M}) + \frac{\Delta T}{M}\, F(t)$$

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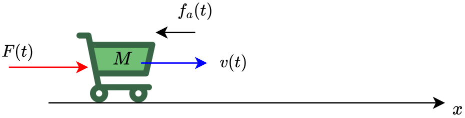

$$v(t + \Delta T) = v(t)\,(1 - \frac{B\,\Delta T}{M}) + \frac{\Delta T}{M}\,F(t)$$

This equation is interpreted as follows:

The new value of the speed depends on the previous value of the speed and the input applied

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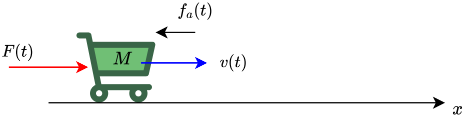

Let's include also the **position** which is the integral of the speed (or the speed is the derivative of the position)

$$\dot{x}(t) = v(t)$$

$$\frac{x(t+\Delta T) - x(t)}{\Delta T} = v(t)$$

$$x(t+\Delta T) - x(t) = v(t)\Delta T$$

$$x(t+\Delta T) = x(t) + v(t)\Delta T$$

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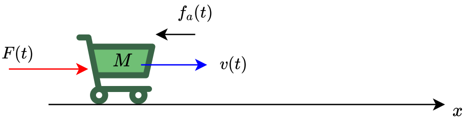


Final Model

$$v(t+\Delta T) = v(t)\,(1 - \frac{B\,\Delta T}{M}) + \frac{\Delta T}{M}\,F(t)$$
$$x(t+\Delta T) = x(t) + v(t)\Delta T$$

Implementing the Cart

Final Model

$$v(t+\Delta T) = v(t)\,(1 - \frac{B\,\Delta T}{M}) + \frac{\Delta T}{M}\,F(t)$$
$$x(t+\Delta T) = x(t) + v(t)\Delta T$$

```
class Cart:
    def __init__(self, _mass: float, _friction: float):
        self.M: float = _mass
        self.B: float = _friction
        self.speed: float = 0
        self.position: float = 0

    def evaluate(self, delta_t: float, _force: float) -> tuple:
        new_speed: float = (1 - self.B * delta_t / self.M) * self.speed
                           + delta_t * _force / self.M
        new_position: float = self.position + self.speed * delta_t
        self.speed = new_speed
        self.position = new_position
        return (self.position, self.speed)
```

Cart Implementation and Test

(examples/cart/test_cart_1.ipynb)
(examples/cart/test_cart_2.ipynb)

The Model and the Reality

Is the model accurate?

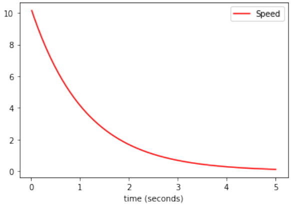
If we compute the analytical form of the speed we have:

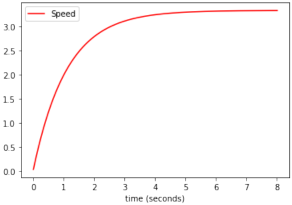
- Impulse Response:

$$v(t) \simeq 10e^{-\frac{3}{4}t}$$

- Step Response:

$$v(t) \simeq 0.3(1 - e^{-\frac{3}{4}t})$$





The Model and the Reality

Is the model accurate?

- In reality, after a “kick”, the cart will stop in a finite time
- In the model, the speed is zero only for $t \to +\infty$:

$$v(t) \simeq 10e^{-\frac{3}{4}t}$$

- In reality, the cart moves only if the input force is greater than a certain value (due to mass value, static frictions, etc.)
- In the model, the cart moves even if the input force is very low

The model is only for “reference”, i.e. to have an understanding of the **form/trend** of the behaviour of the system.

But we are not interested in the exact analytical form or exact values of parameters, because the control algorithms will work anyway

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