Modelling a Rotating Arm

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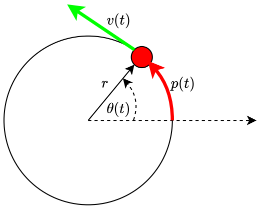
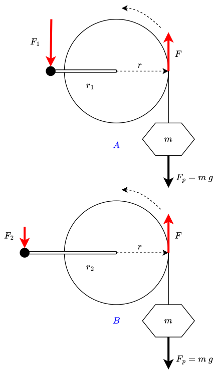
Robotic Systems



Kinematics

- In rotating systems **cinematic quantities** are usually expressed in term of **angles**
- $\theta(t)$: angle
- $\omega(t)$: angular velocity
- $\alpha(t)$: angular acceleration

$$
\begin{aligned}
p(t) &= r\,\theta(t) & [rad] \\
v(t) &= r\,\omega(t) & [rad/s] \\
a(t) &= r\,\alpha(t) & [rad/s^2]
\end{aligned}
$$



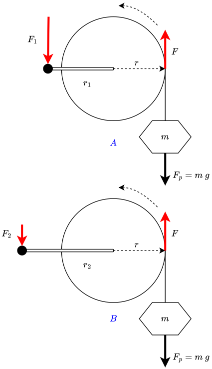
Dynamics

- In order to pull the mass, we must apply a force, at the left side, that results, at the right side in $F > m\,g$
- According to the principles of levers, we have:

$$\begin{aligned} r_1\,F_1 &= r\,F \\ r_2\,F_2 &= r\,F \end{aligned}$$

- In the case *A* we need a higher force that case *B*, because:

$$\begin{aligned} r_1 &< r_2 \\ F_1 &> F_2 \end{aligned}$$



Torques

- In order to avoid the dependency to distance (r), rotating systems are not treated using forces but with **torques**, i.e. the cross product between the distance and the force:

$$\vec{T} = \vec{r} \times \vec{F} \quad [N\,m]$$

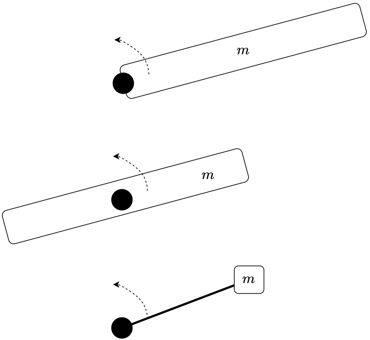
- In rotating systems, distance and force are perpendicular, thus:

$$T = r\,F$$

- Therefore, levers are based on the equilibrium of torques:

$$T_1 = r_1\,F_1 = r\,F = T$$
$$T_2 = r_2\,F_2 = r\,F = T$$

Rotating Systems



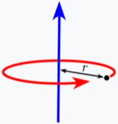
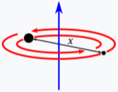
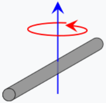
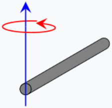
Moment of Inertia

- The contribution of inertia in rotating systems depends on the **distribution of the mass** around the center of rotation
- The cases in figure, even if they feature the same quantity of mass m, have different inertial behaviours
- To consider this aspect, physics defines the **moment of inertia I**, that is a mass-equivalent quantity however considers the distribution of the mass
- The moment of inertia is defined as a volume integral:

$$I = \iiint_V \rho(x, y, z) |\vec{r}|^2 dV$$

- The measure unit is $kg\ m^2$

Moments of Inertia

Figure	Moment(s) of inertia
	$I = Mr^2$
	$I = \frac{m_1 m_2}{m_1 + m_2} x^2 = \mu x^2$
	$I_{\mathrm{center}} = \frac{1}{12} mL^2$ [1]
	$I_{\mathrm{end}} = \frac{1}{3} mL^2$ [1]

(source: Wikipedia)

Rotating Systems

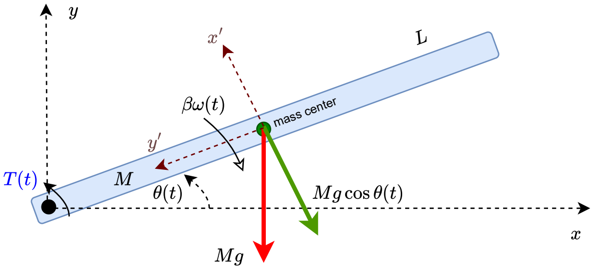
Newton's Second Law

- In rotating systems, the Newton's second law becomes:

$$\sum \overrightarrow{T} = I \, \overrightarrow{\alpha}$$

- The sum of torques is the product of moment of inertia and angular acceleration

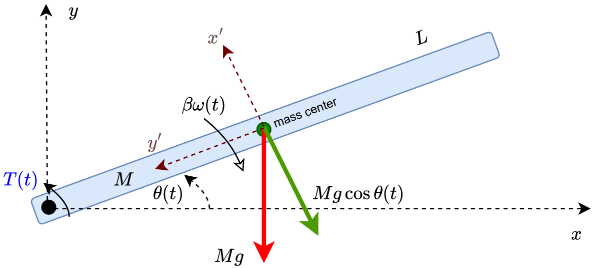
The Arm



Model

- We consider a bar with uniform mass distribution, of mass M and length L
- $T(t)$ is torque generated by the motor: it is the input of the system
- β is air friction coefficient
- g is the gravity acceleration

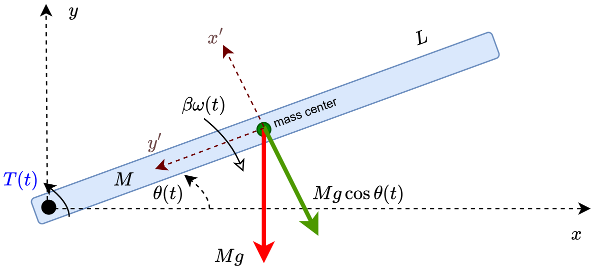
The Arm



Model

$$T(t) - \beta\omega(t)\frac{L}{2} - Mg\cos\theta(t)\frac{L}{2} = I\dot{\omega}(t)$$

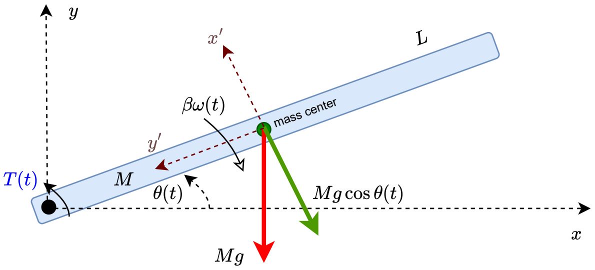
The Arm



Model

$$T(t) - \beta\frac{L}{2}\omega(t) - Mg\frac{L}{2}\cos\theta(t) = \frac{1}{3}ML^2\dot{\omega}(t)$$

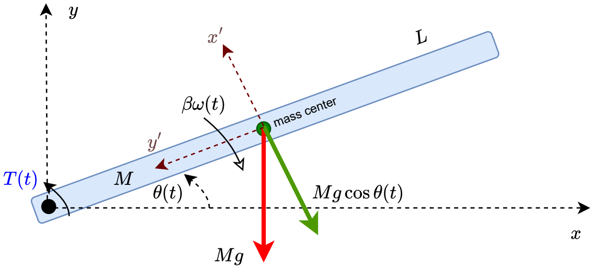
The Arm



Model

$$\dot{\omega}(t) = -\frac{3}{2}\frac{1}{ML}\beta\omega(t) - \frac{3}{2}\frac{1}{L}g\cos\theta(t) + \frac{3}{ML^2}T(t)$$

$$\dot{\theta}(t) = \omega(t)$$

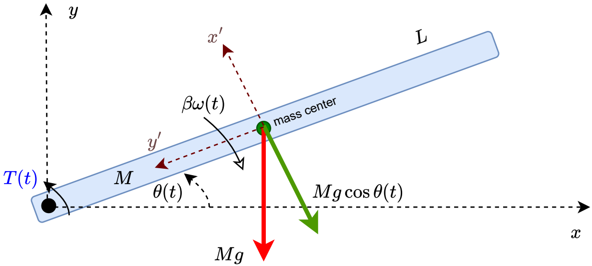


Discretisation

$$\omega(k+1) = (1 - \frac{3}{2}\frac{\Delta T}{ML}\beta)\omega(k) - \frac{3}{2}\frac{\Delta T}{L} g\cos\theta(k) + \frac{3\Delta T}{ML^2} T(k)$$

$$\theta(k+1) = \theta(k) + \omega(k)\Delta T$$

The Arm



Implementation

See:
examples/arm/arm_test.ipynb
lib/system/arm.py

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