Modelling an Arm

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Robotic Systems
Let’s consider an arm with a single joint and a bar

At the end of the bar there is a mass $M$

We can ignore the mass of the bar since we suppose it is sensibly less that $M$

We surely consider the friction

The arm is moved by a motor that generates a certain torque $M_o(t)$

We want to determine the angle $\theta(t)$ and the (angular) velocity $\omega(t)$
Why the **Torque**?

- In rotational systems, it’s better to consider the concept of **couple of forces** rather than a single force.
- In the figure below, given that the mass is the same, for the case 2 we need a motor with a **power greater** than the motor used in case 1.
- This is because, in case 2, the distance of the mass from the rotation point is greater than the one in case 1: \( r_2 > r_1 \).
- Therefore, in rotational system, in determining the **force** needed to move a certain mass **the distance matters**!

\[
\begin{align*}
T_1 &= \text{massa} \ M \ r_1 \\
T_2 &= \text{massa} \ M \ r_2
\end{align*}
\]
Why the **Torque**?

- Therefore, in rotational system, in determining the force needed to move a certain mass **the distance matters!**

- For this reason, in rotation systems, we consider the **mechanical momentum** or **torque** which is the cross-product of the force with the distance: \( \vec{M}_o = \vec{F} \times \vec{r} \)

- In scalar terms, given that often the force and the distance are perpendicular, we have: \( M_o = F r \)
We consider the reference system \( xy \) centered in the mass.

The weight of the mass is split in the two components, according to the angle \( \theta(t) \) (in radians).

Along \( y \)-axis, the component is compensated by reaction \( R \):

\[
R = Mg \cos(\theta)
\]

Along \( x \)-axis, we apply the second Newton's law:

\[
-Mg \sin(\theta) + \frac{M_o}{r} - br\omega = M\ddot{\omega}
\]

Here \( \theta \) is the angular position, \( \omega = \dot{\theta} \) is the angular speed and \( \ddot{\omega} \) is the angular acceleration.
An Arm

Arm Model

Given that:

\[-Mg \sin(\theta) + \frac{M_o}{r} - br\omega = M\dot{\omega}\]

the system model is:

\[
\begin{align*}
\dot{\omega} &= -\frac{br}{M} \omega - g \sin \theta + \frac{1}{Mr} M_o \\
\dot{\theta} &= \omega
\end{align*}
\]

It is non linear!!
Arm Model Linearization

\[
\begin{align*}
\dot{\omega} &= -\frac{br}{M} \omega - g \sin \theta + \frac{1}{Mr} M_o \\
\dot{\theta} &= \omega
\end{align*}
\]

If we consider small values of \( \theta(t) \), we can apply the following approximation:

\[ \sin \theta \simeq \theta \]

therefore:

\[
\begin{align*}
\dot{\omega} &= -\frac{br}{M} \omega - g \theta + \frac{1}{Mr} M_o \\
\dot{\theta} &= \omega
\end{align*}
\]
An Arm

Arm Model Discretization

\[
\begin{align*}
\dot{\omega} &= -\frac{b r}{M} \omega - g \theta + \frac{1}{M_r} M_o \\
\dot{\theta} &= \omega
\end{align*}
\]

\[
\begin{align*}
\frac{\omega(t+\Delta T)-\omega(t)}{\Delta T} &= -\frac{b r}{M} \omega(t) - g \theta(t) + \frac{1}{M_r} M_o(t) \\
\frac{\theta(t+\Delta T)-\theta(t)}{\Delta T} &= \omega(t)
\end{align*}
\]

\[
\begin{align*}
\omega(t + \Delta T) &= \omega(t) - \frac{b r}{M} \Delta T \omega(t) - g \Delta T \theta(t) + \frac{\Delta T}{M_r} M_o(t) \\
\theta(t + \Delta T) &= \theta(t) + \omega(t) \Delta T
\end{align*}
\]
Implementing the Arm

G = 9.81

class Arm:

    def __init__(self, _mass, _friction, _length):
        self.M = _mass
        self.b = _friction
        self.r = _length
        self.omega = 0
        self.theta = 0

    def evaluate(self, delta_t, _torque):
        new_omega = self.omega - ((self.b * self.r) / self.M) * delta_t * self.omega - G * delta_t * self.theta + delta_t/(self.M * self.r) * _torque
        new_theta = self.theta + self.omega * delta_t
        self.omega = new_omega
        self.theta = new_theta
Simulating the Arm

\[ M = 1 \text{ Kg}, \ b = 0.8, \ r = 0.6 \text{ m}, \ M_o = 3 \text{ Nm} \]
Arm Model Discretization without approximation (non-linear)

\[
\begin{align*}
\dot{\omega} & = -\frac{br}{M} \omega - g \sin \theta + \frac{1}{Mr} M_o \\
\dot{\theta} & = \omega
\end{align*}
\]

\[
\begin{align*}
\frac{\omega(t+\Delta T) - \omega(t)}{\Delta T} & = -\frac{br}{M} \omega(t) - g \sin \theta(t) + \frac{1}{Mr} M_o(t) \\
\frac{\theta(t+\Delta T) - \theta(t)}{\Delta T} & = \omega(t)
\end{align*}
\]

\[
\begin{align*}
\omega(t + \Delta T) & = \omega(t) - \frac{br}{M} \Delta T \omega(t) - g \Delta T \sin \theta(t) + \frac{\Delta T}{Mr} M_o(t) \\
\theta(t + \Delta T) & = \theta(t) + \omega(t) \Delta T
\end{align*}
\]
Simulating the Arm without Approximation

\[ M = 1 \text{ Kg}, \quad b = 0.8, \quad r = 0.6 \text{ m}, \quad M_o = 3 \text{ Nm} \]
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