Roto-translations in 2D and 3D environments

Corrado Santoro

ARSLAB - Autonomous and Robotic Systems Laboratory
Dipartimento di Matematica e Informatica - Università di Catania, Italy
santoro@dmi.unict.it

Robotic Systems

Roto-translations in 2D

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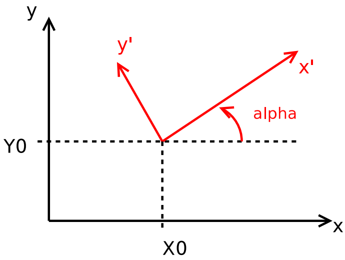
- Let the frame XOY and the frame $X'O'Y'$ **translated** (w.r.t. XOY) to the point (X_0, Y_0) and **rotated** (w.r.t. XOY) of an angle α
- Given a point (x', y') in $X'O'Y'$, its coordinates in XOY will be:

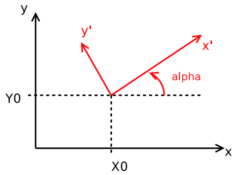
$$
\begin{aligned}
x &= X_0 + x' \cos\alpha + y' \sin\alpha \\
y &= Y_0 - x' \sin\alpha + y' \cos\alpha
\end{aligned}
$$



Roto-translations in 2D

- Let the frame XOY and the frame $X'O'Y'$ **translated** (w.r.t. XOY) to the point (X_0, Y_0) and **rotated** (w.r.t. XOY) of an angle α
- If we adopt **omogeneous coordinates**, we can represent the transformation using a matrix equation:

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\alpha & \sin\alpha & X_0 \\ -\sin\alpha & \cos\alpha & Y_0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix}$$



Roto-translations in 2D

- The matrix:

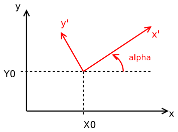
$$A = \begin{bmatrix} \cos\alpha & -\sin\alpha & X_0 \\ \sin\alpha & \cos\alpha & Y_0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} R & & T \\ 0 & 0 & 1 \end{bmatrix}$$

- is the **rototranslation matrix** composed of
- the **rotation matrix**:

$$R = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$$

- and the **translation vector**:

$$T = \begin{bmatrix} X_0 \\ Y_0 \end{bmatrix}$$



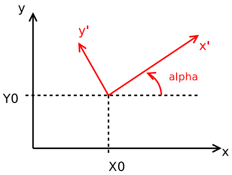
Roto-translations in 2D

- Given a point $P' \equiv (x', y', 1)$ expressed in the $X'O'Y'$ reference frame, its coordinates $P \equiv (x, y, 1)$ in frame XOY can be found by computing

$$P = A\,P'$$

- Conversely, Given a point $P \equiv (x, y, 1)$ expressed in the XOY reference frame, its coordinates $P' \equiv (x', y', 1)$ in frame $X'O'Y'$ can be found by computing

$$P' = A^{-1}\,P$$



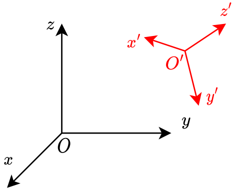
Roto-translations in 3D

Roto-translations in 3D

- Let the frame XYZ and the frame $X'Y'Z'$ **translated** (w.r.t. XYZ) to the point $O' \equiv (X_0, Y_0, Z_0)$ and **rotated** w.r.t. XYZ
- Given a point (x', y', z') in $X'Y'Z'$, its coordinates in XYZ will be:

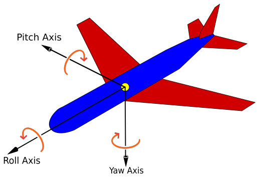
$$\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} & R & & X_0 \\ & & & Y_0 \\ & & & Z_0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix}$$

- The matter is to find the rotation matrix R that is made of **three angles** representing rotations over the three axes



3D Rotation: Euler Angles

- There are several ways of reprenting a rotation on 3D space, all of them are based on the composition of **elementary rotations** on the axis of the reference frame
- **Euler Angles =** $\{\phi, \theta, \psi\}$ represent three elementary rotations of the (rotated) frame on one of each axis:
  - ϕ, roll: it is the rotation of $y'z'$ plane along x' axis;
  - θ, pitch: it is the rotation of $x'z'$ plane along y' axis;
  - ψ, yaw: it is the rotation of $x'y'$ plane along z' axis.
- They are used in **avionics** to represent the **attitude** of a plane in the space

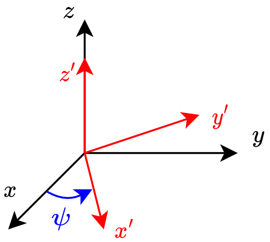


3D Rotation: Yaw

- The **yaw** ψ represent the rotation along z' axis of the $x'y'$ frame
- It is expressed by the **rotation matrix**:

$$R_\psi = \begin{bmatrix} cos\psi & -sin\psi & 0 \\ sin\psi & cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- Only x and y coordinates **are affected**, while z remains the **same**

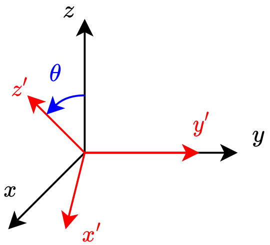


3D Rotation: Pitch

- The **Pitch** θ represent the rotation along y' axis of the $x'z'$ frame
- It is expressed by the **rotation matrix**:

$$R_\theta = \begin{bmatrix} cos\theta & -sin\theta & 0 \\ 0 & 0 & 1 \\ sin\theta & cos\theta & 0 \end{bmatrix}$$

- Only x and z coordinates **are affected**, while y remains the **same**

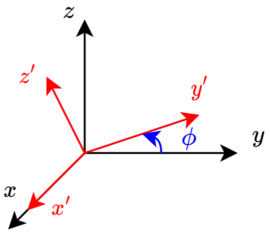


3D Rotation: Roll

- The **Roll** ϕ represent the rotation along x' axis of the $y'z'$ frame
- It is expressed by the **rotation matrix**:

$$R_\phi = \begin{bmatrix} 0 & 0 & 1 \\ cos\phi & -sin\phi & 0 \\ sin\phi & cos\phi & 0 \end{bmatrix}$$

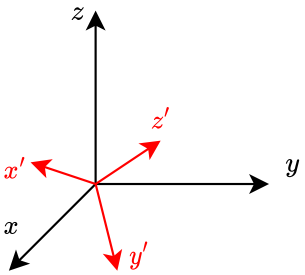
- Only y and z coordinates **are affected**, while x remains the **same**



3D Rotation

- A **complete 3D rotation** can be represented by combining the ϕ, θ, ψ rotations in sequence
- This leads to the complete **rotation matrix** that is given by multiplying the three single rotation matrices:

$$R_{\psi\theta\phi} = R_\psi R_\theta R_\phi$$



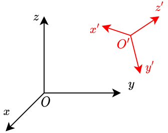
Roto-translations in 3D

- Let the frame XYZ and the frame $X'Y'Z'$ **translated** (w.r.t. XYZ) to the point $O' \equiv (X_0, Y_0, Z_0)$ and **rotated** w.r.t. XYZ by Euler angles **roll, pitch, yaw**
- Given a point (x', y', z') in $X'Y'Z'$, its coordinates in XYZ will be:

$$\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} & R_{\psi\theta\phi} & & \begin{matrix} X_0 \\ Y_0 \\ Z_0 \end{matrix} \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix}$$

- where:

$$R_{\psi\theta\phi} = \begin{bmatrix} cos\theta cos\psi & sin\phi sin\theta cos\psi - cos\phi sin\psi & cos\phi sin\theta cos\psi + sin\phi sin\psi \\ cos\theta sin\psi & sin\phi sin\theta sin\psi + cos\phi cos\psi & cos\phi sin\theta sin\psi - sin\phi cos\psi \\ -sin\theta & sin\phi cos\theta & cos\phi cos\theta \end{bmatrix}$$



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