Dynamic Systems
The Theory Behind

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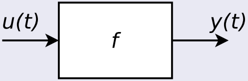
Robotic Systems

A System

Each **real system** (any object belonging to the real world) is an object with finite number of freedom degrees that evolves in time according to a certain law

A real system can be represented as a **black box** that can be externally stimulated from an input (that we call $u(t)$), producing a certan **effect** which is the output (that we call $y(t)$)



$$y(t) = f(u(t))$$

The behaviour is **totally represented** by function $f(\cdot)$

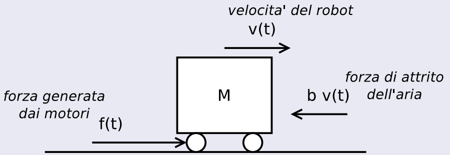
Definition of **Dynamic System**

A **dynamic system** is a (physical) system where, given a certain time instant t, the output $y(t)$ depends on the current value $u(t)$ and the past of $u(t)$ and $y(t)$

A dynamic system (**time-continuous**) is described by a **differential equation** in the time domain:

$$y = f(\dot{y}, \ddot{y}, ..., u, \dot{u}, \ddot{u}, t)$$

The Cart as a **Dynamic System**



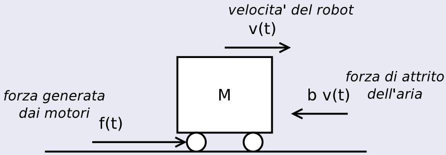
Input = $f(t)$ *Output* =$p(t)$

Let's consider again the cart. If we use the force $f(t)$ as the input and the position $p(t)$ as the output, we can model the cart using the following differential equation:

$$f - b\,\dot{p} = M\,\ddot{p}$$

(recall that $v = \dot{p}$ and $a = \ddot{p}$)

It is a second-order differential equation, that, at first sight, is hard to solve and also to simulate



Input = $f(t)$

Output =$p(t)$

$$f - b\,\dot{p} = M\,\ddot{p}$$

But if we introduce also the v variable, we can rewrite the equation above as a **system** of differential equation:

$$\begin{cases} \dot{v} & = & -\frac{b}{M}v + \frac{1}{M}f \\ \dot{p} & = & v \end{cases}$$

Here we have only first-order differential equations that can be handled more easily

Representation of a **Dynamic System**

A dynamic system described by a n-order **differential equation** as:

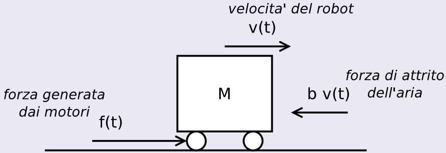
$$y = f(\dot{y}, \ddot{y}, ..., u, \dot{u}, \ddot{u}, t)$$

can be represented as a **system of *n* differential equations** of the first-order by using additional variables that are equal to the first-derivative, second-derivative, etc.

Variables that are derivated are called **state variables** and represent the instantaneous condition of the system

The Cart as a **Dynamic System**

State Variables



Input = $f(t)$

Output = $p(t)$

$$\begin{cases} \dot{v} & = & -\frac{b}{M}v + \frac{1}{M}f \\ \dot{p} & = & v \end{cases}$$

Here p and v are the **state variables** of our cart and the equations above that describe the evolution of such state variables are called **state equations**

Linear Systems

A **linear system** is a system described by a set of first-order linear differential equations

A **linear differential equation** is a differential equation in which a derivated variable **lineary depends** on the variable itself and other variables

$$\begin{cases} \dot{x}_1 & = & a_{11}x_1 + \ldots + a_{1n}x_n \\ \ldots & \ldots & \ldots \\ \dot{x}_n & = & a_{n1}x_1 + \ldots + a_{nn}x_n \end{cases}$$

$$\begin{cases} \dot{v} & = & -\frac{b}{M}v + \frac{1}{M}f \\ \dot{p} & = & v \end{cases}$$

Our cart model is a **linear system** with p and v as state variables

Parts of a linear system model

Given a certain system represented by a linear model, we can identify, in each equations, two basic parts:

- Terms that give dependency of other state variables (in red)
- Terms that give the dependency of the input(s) (in blue)

$$\begin{cases} \dot{v} & = & -\frac{b}{M} v + \frac{1}{M} f \\ \dot{p} & = & v \end{cases}$$

- Terms in red represent the **free evolution** of the system (i.e. the evolution without any input)
- Terms in blue represent the **forced evolution** of the system

Standard Representation of Dynamic Systems

Standard Representation of a Dynamic System

The Cart

- **State Equations** $\begin{cases} \dot{v} & = & -\frac{b}{M}v + \frac{1}{M}f \\ \dot{p} & = & v \end{cases}$
- We know, from algebra, that a set of linear equations can be represented in a **matrix form**
- Let's incapsulate state variables in a (geometric) **vector**:

$$x = \begin{bmatrix} v \\ p \end{bmatrix} \qquad \dot{x} = \begin{bmatrix} \dot{v} \\ \dot{p} \end{bmatrix}$$

State equations in matrix form

$$\begin{bmatrix} \dot{v} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} -\frac{b}{M} & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} v \\ p \end{bmatrix} + \begin{bmatrix} \frac{1}{M} \\ 0 \end{bmatrix} f$$

$$\dot{x} = \begin{bmatrix} -\frac{b}{M} & 0 \\ 1 & 0 \end{bmatrix} x + \begin{bmatrix} \frac{1}{M} \\ 0 \end{bmatrix} f$$

State Equation in matrix form

$$\begin{bmatrix} \dot{v} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} -\frac{b}{M} & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} v \\ p \end{bmatrix} + \begin{bmatrix} \frac{1}{M} \\ 0 \end{bmatrix} f$$

$$\dot{x} = \begin{bmatrix} -\frac{b}{M} & 0 \\ 1 & 0 \end{bmatrix} x + \begin{bmatrix} \frac{1}{M} \\ 0 \end{bmatrix} f$$

Let:

$$A = \begin{bmatrix} -\frac{b}{M} & 0 \\ 1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} \frac{1}{M} \\ 0 \end{bmatrix} \qquad u = f$$

We obtain the **general state equation** in **matrix form**:

$$\dot{x} = A\ x + B\ u$$

Standard Representation of a Dynamic System

Output Equation in matrix form

- The **output** of our system y is (in general) a linear combination of state variables (state vector) $x = \begin{bmatrix} v \\ p \end{bmatrix}$
- If we consider the position p as output, to obtain it we can simply multiply the state vector with $\begin{bmatrix} 0 & 1 \end{bmatrix}$

$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} v \\ p \end{bmatrix}$$

Generalising we have:

$$y = C\,x$$

$$C = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

Linear Dynamic Systems

Standard Representation of a Linear Dynamic System

In summary, a **linear system time-continuous** is totally specified by the following matrix equations:

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases}$$

- x, state vector (n elements)
- u, input vector (m elements)
- y, output vector (p elements)

- A, **state matrix** $n \times n$
- B, **input matrix** $n \times m$
- C, **output matrix** $p \times n$

Roles of Matrices in the Matrix Form

$$\dot{x} = A\,x + B\,u \quad (1)$$

The state equation has two parts:

- Matrix A characterises the **evolution of the state**
- Matrix B represents the contribution of the input
- When $u = 0$, we have the **free evolution** and the state equation becomes: $\dot{x} = A\,x$
- The **dynamics (behaviour)** of the system is totally specified by matrix A
- In other words, if we have a representation in the form (1) we do not need to solve the differential equations or perform a simulation to understand the behaviour of the system, but it is sufficient to analyse matrix A

Discretization of a Linear System Model

Discretization of a Linear System

- In order to handle/implement a linear system (given its model in matrix form), we can consider the sampling of any variable with prefixed ΔT
- The derivative becomes **incremental ratio** (first-order approximation):

$$
\begin{aligned}
x(t), t \in \mathcal{R} & \Rightarrow x(k\Delta T), k \in \mathcal{N} \\
\dot{x}(t), t \in \mathcal{R} & \Rightarrow \frac{x((k+1)\Delta T) - x(k\Delta T)}{\Delta T}, k \in \mathcal{N} \\
x(t) & \Rightarrow x(k) \\
\dot{x}(t) & \Rightarrow \frac{x(k+1) - x(k)}{\Delta T}
\end{aligned}
$$

Discretization of a Linear System

The system

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases}$$

discretized using a sampling interval ΔT becomes:

$$\begin{cases} \frac{x(k+1)-x(k)}{\Delta T} = Ax(k) + Bu(k) \\ y(k) = Cx(k) \end{cases}$$

$$\begin{cases} x(k+1) - x(k) = A\Delta T x(k) + B\Delta T u(k) \\ y(k) = Cx(k) \end{cases}$$

$$\begin{cases} x(k+1) = A\Delta T x(k) + x(k) + B\Delta T u(k) \\ y(k) = Cx(k) \end{cases}$$

$$\begin{cases} x(k+1) = (A\Delta T + I)x(k) + B\Delta T u(k) \\ y(k) = Cx(k) \end{cases}$$

Discretization of a Linear System

The system:

$$\begin{cases} \dot{x} &= Ax + Bu \\ y &= Cx \end{cases}$$

becomes:

$$\begin{cases} x(k+1) &= \tilde{A}\, x(k) + \tilde{B}\, u(k) \\ y(k) &= \tilde{C}\, x(k) \end{cases}$$

here:

$$\begin{aligned} \tilde{A} &= A\Delta T + I \\ \tilde{B} &= B\Delta T \\ \tilde{C} &= C \end{aligned}$$

Equilibrium

Equilibrium

Generic Definition of Equilibrium state

Given a dynamic system defined by state equation $\dot{x} = f(x, u)$, with $u = u(t)$ as input, the state $\overline{x} = x(T^*)$ is an **equilibrium** if

$$f(\overline{x}, u(t)) = 0, \forall t \geq T^*$$

that is, no state variations, thus $\dot{x} = 0$

In other words, an equilibrium state means no further state variations for $t \geq T^*$

Equilibrium

Equilibrium in Continuous Linear Systems

Given a linear system defined by the state equation

$$\dot{x} = Ax + Bu$$

the state $\overline{x}$ is an **equilibrium** if

$$0 = A\overline{x} + Bu$$

Searching for equilibrium states in a linear system implies to solve the (linear) equation:

$$x = -A^{-1}Bu$$

if A is *not invertible* then there are either **no equilibrium states** or **infinite equilibrium states**

Equilibrium in Discrete Linear Systems

Given a linear system defined by the state equation

$$x(k+1) = \widetilde{A}\, x(k) + \widetilde{B}\, u$$

the state $\overline{x}$ is an **equilibrium** if

$$\overline{x} = \widetilde{A}\, \overline{x} + \widetilde{B}\, u$$

Searching for equilibrium states in a linear system implies to solve the (linear) equation:

$$x = -(A - I)^{-1} Bu$$

if $A - I$ is *not invertible* then there are either **no equilibrium states** or **infinite equilibrium states**

Equilibrium

Equilibrium in the Cart

Let's consider the cart: $\begin{cases} \dot{v} = -\frac{b}{M}v + \frac{1}{M}f \\ \dot{p} = v \end{cases}$

$$\begin{bmatrix} \dot{v} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} -\frac{b}{M} & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} v \\ p \end{bmatrix} + \begin{bmatrix} \frac{1}{M} \\ 0 \end{bmatrix} f$$

Let's consider a **constant** input F; here matrix A is singular so we have:

$$\begin{cases} 0 = -\frac{b}{M}v + \frac{1}{M}F \\ 0 = v \end{cases}$$

and:

$$\begin{cases} v = \frac{F}{b} \\ v = 0 \end{cases}$$

Relations above are in contrast and this comes from the singularity of A

Indeed the solution is $v = \frac{F}{b}$ and $p = any$ (so infinite equilibrium points)

Equilibrium in the Arm

Let's consider the arm: $\left\{ \begin{array}{lcl} \dot{\omega} & = & -\frac{br}{M}\omega - g\theta + \frac{1}{Mr}M_o \\ \dot{\theta} & = & \omega \end{array} \right.$

Given a **constant** input M_o we have:

$$\left\{ \begin{array}{lcl} 0 & = & -\frac{b\ r}{M}\omega - g\theta + \frac{1}{M\ r}M_o \\ 0 & = & \omega \end{array} \right.$$

and:

$$\left\{ \begin{array}{lcl} \theta & = & \frac{M_o}{g\ M\ r} \\ \omega & = & 0 \end{array} \right.$$

(for small oscillations)

Equilibrium without Input

Equilibrium in the Cart

Let's consider the cart in **free evolution**: $\left\{ \begin{array}{rcl} \dot{v} & = & -\frac{b}{M} v \\ \dot{p} & = & v \end{array} \right.$

If we determine the equilibrium points we have:

$$\left\{ \begin{array}{rcl} v & = & 0 \\ p & = & any \end{array} \right.$$

Indeed, if we have **no inputs**, there are **infinite** equilibrium points where the speed is 0

Equilibrium without Input

Equilibrium in the Arm

Let's consider the arm in **free evolution**: $\left\{ \begin{array}{lcl} \dot{\omega} & = & -\frac{br}{M}\omega - g\theta \\ \dot{\theta} & = & \omega \end{array} \right.$

we have:

$$\left\{ \begin{array}{lcl} \theta & = & 0 \\ \omega & = & 0 \end{array} \right.$$

However, if we consider a **real system** (without the sin approximation):

$$\left\{ \begin{array}{lcl} 0 & = & -\frac{br}{M}\omega - g\sin\theta \\ 0 & = & \omega \end{array} \right.$$

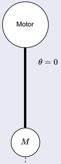
we have:

$$\left\{ \begin{array}{lcl} \theta & = & k\pi \\ \omega & = & 0 \end{array} \right.$$

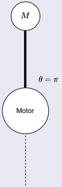
Equilibrium without Input

Equilibrium in the Arm

Basically there are **two** equilibrium points:



$$\begin{cases} \theta & = & 0 \\ \omega & = & 0 \end{cases}$$



$$\begin{cases} \theta & = & \pi \\ \omega & = & 0 \end{cases}$$

But it's clear that the two points are **somewhat different**

Stability

Stability

Simple Stability

An equilibrium point $x_0 = x(T^*)$ is **stable**, when, after a perturbation of the state at time $\bar{t} \geq T^*$, the resulting state evolution (trajectory) is **always near** point $x_0, \forall t \geq \bar{t}$:

$$\exists M > 0 : \|x(t) - x_0\| < M, \forall t \geq \bar{t}$$

Asymptotical Stability

An equilibrium point $x_0 = x(T^*)$ is **asymptotically stable**, when, after a perturbation of the state at time $\bar{t} \geq T^*$, the resulting state evolution (trajectory) **tends to** x_0:

$$\lim_{t \to \infty} x(t) = x_0$$

Instability

An equilibrium point $x_0 = x(T^*)$ is **instable**, when, after a perturbation of the state at time $\bar{t} \geq T^*$, the resulting state evolution (trajectory) **always steps away from** x_0:

$$\lim_{t \to \infty} x(t) = \infty \quad \text{or it does not exists}$$

Stability in the Cart

The equilibrium is:

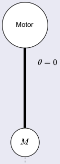
$$\begin{cases} v & = & 0 \\ p & = & any \end{cases}$$

It is **simply stable**

Given a position p_1, if perturbate it as p_2, the cart remains in p_2 thus “near” p_1

Stability in the Arm

The point:



$$\begin{cases} \theta = 0 \\ \omega = 0 \end{cases}$$

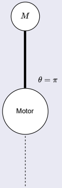
is **asymptotically stable**.

If we move the mass to a position $\overline{\omega} \neq 0$, it returns to 0

Stability

Stability in the Arm

The point:



$$\begin{cases} \theta = \pi \\ \omega = 0 \end{cases}$$

is **simply stable**.

If we move the mass to a position $\varpi \neq \pi$, it returns to 0, thus "near" to π

Stability in a Linear System

A **linear system** is said to be **stable** if all state trajectories are **stable**

Trajectories of a Linear System

Trajectories of a Linear System

Given a linear system in an equilibrium point, if we perturbate the state, the resulting trajectory, in **free running state**, and for each **state variable** has the following form:

$$x_j(t) = K_{1,j}e^{\lambda_{1,j}t} + ... + K_{m,j}e^{\lambda_{m,j}t} + K_{m+1,j}e^{\sigma_{m+1,j}t} \sin \omega_{m_1,j}t + ...$$

In other words, it's a weighted sum of:

Exponential Terms	$e^{\lambda t}$
Exponential-Sinusoidal Terms	$e^{\sigma t} \sin \omega t$

This result comes from the theorems on the integration of linear differential equations

Trajectories of a Linear System

Terms of the trajectory in **free running state** of a linear system:

Exponential Terms	$e^{\lambda t}$
Exponential-Sinusoidal Terms	$e^{\sigma t} \sin \omega t$

The exponentials play a fundamental role in understanding stability, in particular according to **sign** of the exponent, i.e. parameters λ and σ (given that we always consider $t \geq 0$)

Trajectories of a Linear System

$$\lambda > 0, \sigma > 0$$

		The term
Exponential Terms	$e^{\lambda t}$	**diverges**
Exponential-Sinusoidal Terms	$e^{\sigma t} \sin \omega t$	**diverges**

The system is **instable**

Trajectories of a System

Trajectories of a Linear System

$$\lambda = 0, \sigma = 0$$

		The term
Exponential Terms	$e^{\lambda t}$	**is** 1
Exponential-Sinusoidal Terms	$e^{\sigma t} \sin \omega t$	**becomes** $\sin \omega t$ and oscillates between $[-1, 1]$

The system is **simply stable**

Trajectories of a Linear System

$$\lambda < 0, \sigma < 0$$

		The term
Exponential Terms	$e^{\lambda t}$	**tends to 0**
Exponential-Sinusoidal Terms	$e^{\sigma t} \sin \omega t$	**tends to 0**

The system is **asymptotically stable**

Trajectories of a System

Trajectories and Stability

Given a trajectory:

$$x_j(t) = K_{1,j}e^{\lambda_{1,j}t} + \ldots + K_{m,j}e^{\lambda_{m,j}t} + K_{m+1,j}e^{\sigma_{m+1,j}t}\sin\omega_{m_1,j}t + \ldots$$

the stability **depends only on the sign** of parameters $\lambda_{i,j}$ and $\sigma_{i,j}$

Condition	Stability
All $\lambda_{i,j} < 0, \sigma_{i,j} < 0$	**Asymptotical**
All $\lambda_{i,j} < 0, \sigma_{i,j} < 0$ but a $\lambda_{i*,j} = 0, \sigma_{i*,j} = 0$	**Simple**
At least one $\lambda_{i*,j} > 0, \sigma_{i*,j} > 0$	**Instability**

Trajectories, Stability and Eigenvalues

Roots of a Polynomial

Given of polynomian in x with degree n with real coefficients:

$$a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n \qquad a_i \in \mathcal{R}$$

then the **roots** will be:

- either **all** real ($\in \mathcal{R}$)
- or, if some of them are complex, they will be **complex and conjugate**

The Characteristic Polynomial

Given a square matrix $A = \{a_{i,j}\} \in \mathcal{R}^{n \times n}$, then its **eigenvalues** will be:

- either **all** real ($\in \mathcal{R}$)
- or, if some of them are complex, they will be **complex and conjugate**

Let's recall that the eigenvalues of a matrix A are the roots of the characteristic polynomial:

$$|\lambda I - A|$$

Merging all the worlds

Given a linear system in free running defined by

$$\dot{x} = Ax$$

given that the eigenvalues of A are:

$$\lambda_1, \lambda_2, ..., \sigma_k \pm i\omega_k, \sigma_{k+1} \pm i\omega_{k+1} ...$$

with $\lambda_i \in \mathcal{R}$ and and $\sigma_k \pm i\omega_k \in \mathcal{C}$ then the trajectory will have the form:

$$x(t) = K_1 e^{\lambda_1 t} + ... + K_{k-1} e^{\lambda_{k-1} t} + K_k e^{\sigma_k t} \sin \omega_k t + ...$$

(the explaination will be given below)

Eigenvalues and Stability

Given a linear system defined by

$$\dot{x} = Ax + Bu$$

given that the eigenvalues of A are:

$$\lambda_i \in \mathcal{C} \qquad \text{(remind that } \mathcal{R} \subset \mathcal{C}\text{)}$$

i.e. λ_i may be either **real** or **complex and conjugate**, we have

Condition	System Stability
All $Re(\lambda_i) < 0$	**Asymptotical**
At least $Re(\lambda_{i^*}) = 0$, but, if λ_{i^*} is compl-and-conj, it is simple (mutiplicity = 1)	**Simple**
All other cases	**Instability**

Link between Response and Eigenvalues

Given a square matrix $A = \{a_{i,j}\} \in \mathcal{R}^{n \times n}$ whose eigenvalues are λ_i, then the following holds:

$$A = T^{-1} E \; T$$

where:

- T is a $n \times n$ invertible matrix
- E is a $n \times n$ **diagonal** matrix whose diagonal elements are the eigenvalues of A

Link between Response and Eigenvalues

Given a linear system defined by

$$\dot{x} = Ax + Bu$$

and given that the eigenvalues of A are: λ_i, we can write:

$$\begin{aligned} \dot{x} &= T^{-1} E\, T\, x + B\, u \\ T\dot{x} &= T\, T^{-1} E\, T\, x + T\, B\, u \\ T\dot{x} &= E\, T\, x + T\, B\, u \end{aligned}$$

Let's assume that $\tilde{x} = T\dot{x}$ and $\tilde{B} = T\, B$, we have:

$$\dot{\tilde{x}} = E\, \tilde{x} + \tilde{B}\, u$$

Link between Response and Eigenvalues

The system:

$$\dot{x} = Ax + Bu$$

and

$$\dot{\tilde{x}} = E\ \tilde{x} + \tilde{B}\ u$$

are **equivalent**, i.e. **the same**, so they feature the same response and the same stability conditions

We have only applied a **change of reference frame** for state variables defined by matrix T

Link between Response and Eigenvalues

Let's consider:

$$\dot{\tilde{x}} = E\ \tilde{x}$$

that can be rewritten as:

$$\begin{cases} \dot{\tilde{x}}_1 & = & \lambda_1\ \tilde{x}_1 \\ \dot{\tilde{x}}_2 & = & \lambda_2\ \tilde{x}_2 \\ & \ldots & \\ \dot{\tilde{x}}_n & = & \lambda_n\ \tilde{x}_n \end{cases}$$

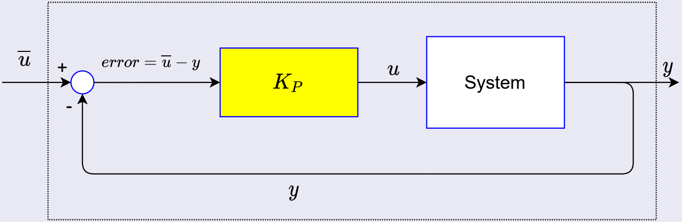
If $\lambda_i \in \mathcal{R}$ then each differential equation can be easily integrated as:

$$\tilde{x}_i\ (t) = e^{\lambda_i t}$$

C.V.D.

Dyamic Systems and Control Systems

Linear Systems and Control Systems

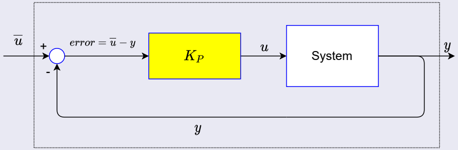


Let's consider a linear system

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases}$$

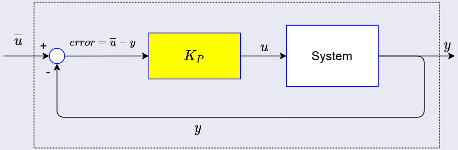
in feedback loop with a proportional controller

Is the resulting system a linear system as well?



$$\begin{cases} \dot{x} &= Ax + Bu \\ y &= Cx \end{cases}$$

$$\begin{aligned} u &= K_P error \\ &= K_P(\overline{u} - y) \\ &= K_P\overline{u} - K_P y \end{aligned}$$

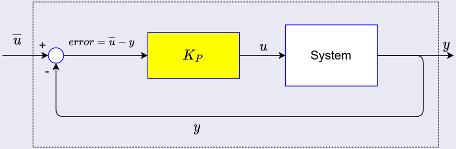


$$\begin{cases} \dot{x} & = & Ax + B(K_P\overline{u} - K_P y) \\ y & = & Cx \end{cases}$$

$$\begin{cases} \dot{x} & = & Ax + BK_P\overline{u} - BK_P y \\ y & = & Cx \end{cases}$$

$$\begin{cases} \dot{x} & = & Ax + BK_P\overline{u} - BK_P Cx \\ y & = & Cx \end{cases}$$

$$\begin{cases} \dot{x} & = & (A - BK_P C)x + BK_P\overline{u} \\ y & = & Cx \end{cases}$$

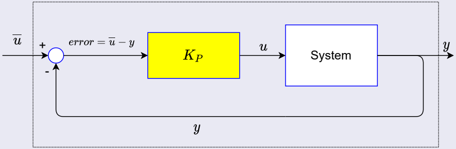


The resulting system is

$$\begin{cases} \dot{x} & = & \overline{A}x + \overline{B}\,\overline{u} \\ y & = & Cx \end{cases}$$

with

$$\begin{aligned} \overline{A} &= A - BK_P C \\ \overline{B} &= BK_P \end{aligned}$$

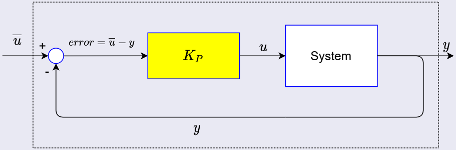


Given that:

$$\overline{A} = A - BK_PC$$
$$\overline{B} = BK_P$$

we have:

- The **closed loop** system is a linear system
- The **dynamics** (behaviour) of the closed loop system depends of A, B, C, K_P (while the dependency of the orginal system were only w.r.t to A)
- The parameter K_P **strongly affects** both the dynamics and the **stability** of the closed loop system



$$\overline{A} = A - BK_P C$$
$$\overline{B} = BK_P$$

By acting on K_P we can:

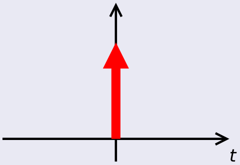
- Make **instable** a stable system
- Make **stable** an instable system
- **Completely change** the system's response, e.g. by introducing or removing oscillation
- ...

Canonical Signals

Canonical Signals

System Theory often uses some specific input signals, called **canonical signals**, to study the behaviour of a system:

- **Impulse** or **Dirac Delta**
- **(Unitary) Step**
- **Ramp**

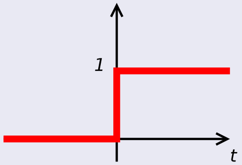


The **Dirac Delta** $\delta(t)$ is an impulsive signal that, from the mathematical point of view, is defined as:

$$\begin{cases} \delta(t) = 0, \forall t \neq 0 \\ \delta(t) = +\infty, t = 0 \\ \int_{-\infty}^{+\infty} \delta(t) dt = 1 \end{cases}$$

It is used to represent a physical fenomena with a great intensity but with an infinitesimal duration

Unitary Step

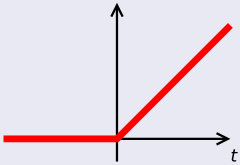


The **step** $u(t)$ is a signal defined as follows:

$$\begin{cases} u(t) & = & 0, \forall t < 0 \\ u(t) & = & 1, \forall t \geq 0 \end{cases}$$

It is used to model the application, at time 0, of a constant stimulus to a system

Ramp



The **ramp** $r(t)$ is an increasing signal defined as follows:

$$\begin{cases} r(t) = 0, \forall t < 0 \\ r(t) = t, \forall t \geq 0 \end{cases}$$

It is used to model the application to a system, at time 0, of a simulus that grows indefinitely

Relationship between Canonical Signals

Signals

$$u(t) = \int_0^t \delta(\tau)d\tau \qquad \frac{du(t)}{dt} = \delta(t)$$

$$r(t) = \int_0^t u(t)d\tau \qquad \frac{dr(t)}{dt} = u(t)$$

Responses

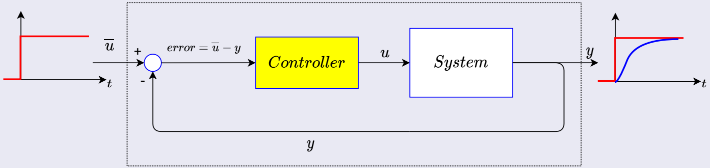
- Given a **linear system**, if $y_d(t)$ is the impulse response, then the **step response** is:

$$y_s(t) = \int_0^t y_d(\tau)d\tau$$

- and the **ramp response** is:

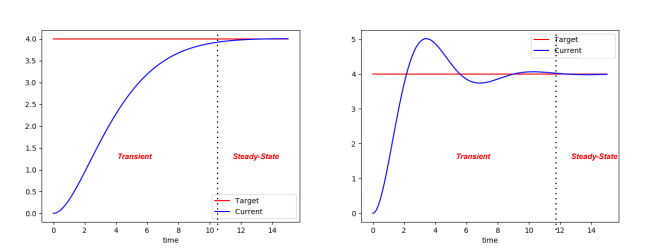
$$y_r(t) = \int_0^t y_s(\tau)d\tau$$

Canonical Signals and Control Systems



- Given a control system, its performaces are measured on the basis of **canonical inputs**
- The step represents a **constant reference** that is suddenly applied
- The ramp represents a **moving reference**, thus making it possible to measure the ability of the control system to follow changing references

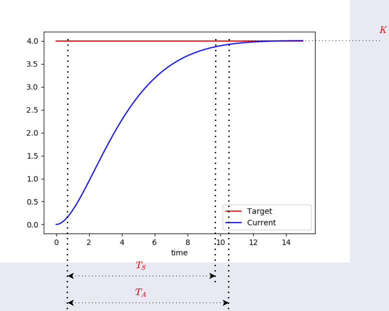
Transient and Steady-State Regimes



The response of a system to a step (or a pulse is composed of two parts:

- **Transient**: initial part of the response; the output changes substantially during time
- **Steady-State**: when the transient is over, the output features small or no changes and stabilise to a specific value
- According to the response type (left or right figures above), the transient features some specific characteristics

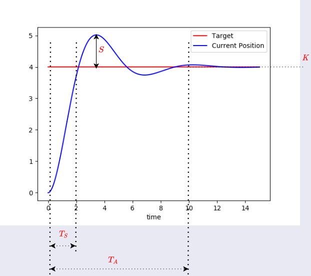
Transient Characteristics



- **Steady-State Value**: $K = \lim_{t\to\infty} y(t)$
- **Rise Time** T_S ("tempo di salita"): the time required to go from 10% of K to 90% of K
- **Set-up Time** T_A ("tempo di assestamento"): the time required to have the output around the 98% of K

Canonical Signals and Control Systems

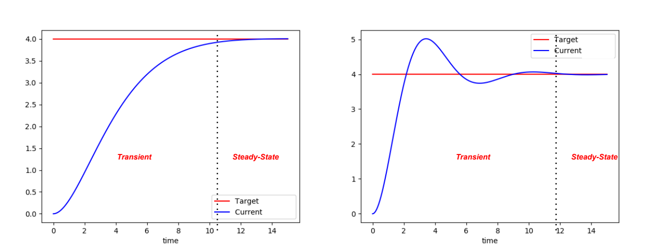
Transient Characteristics



- **Steady-State Value**: $K = \lim_{t \to \infty} y(t)$
- **Rise Time** T_S ("tempo di salita"): the time required to go from 10% of K to 90% of K
- **Set-up Time** T_A ("tempo di assestamento"): the time required to have the output around the 98% of K
- **Overshot** S ("sovraelongazione"): the percentage w.r.t. K of the first peak $S = \frac{peak - K}{K}$

System Response and Eigenvalues

- Eigenvalues of state matrix *A* not only are able to determine system stability but also **encode** important information about the transient



- Left-side response represents a system with **real and negative** eigenvalues
- Right-side response represents a system with at least a couple of **complex and conjugate** eigenvalues with **negative real part**

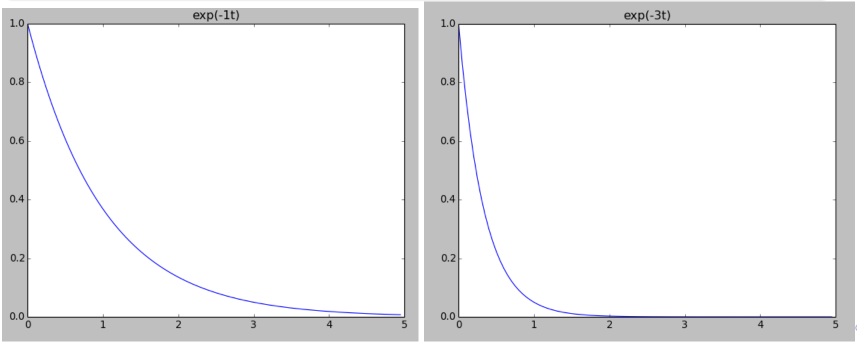
Real Negative Eigenvalues

Let us consider a system with $\lambda_1 = -1$ e $\lambda_2 = -3$, then the response will be of type:

$$e^{-t} + e^{-3t}$$

From the plots we see that the duration of the transient for λ_1 is greater than the one of λ_2

In other words, λ_1 is "slower" than λ_2

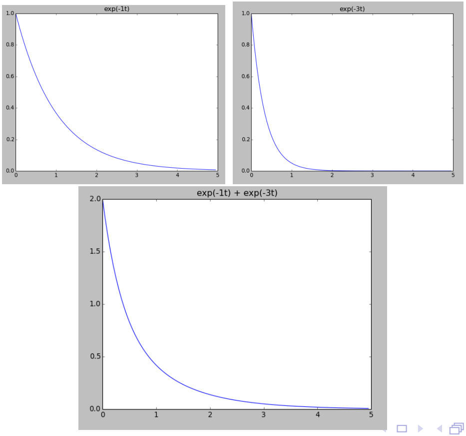


System Reponse and Eigenvalues

Real Negative Eigenvalues

If we combine the plots and see the complete response, we observe that its transient is more influenced by λ_1 rather than λ_2

In other words, the overall response is as “slow” as the one with only λ_1

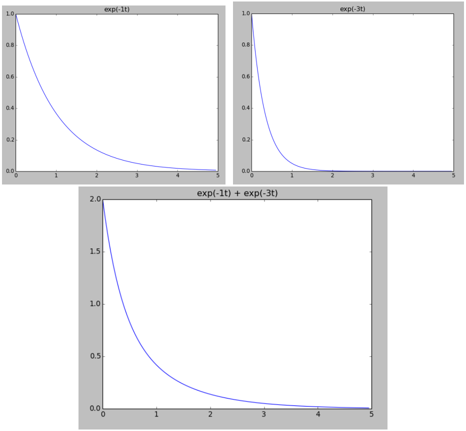


System Reponse and Eigenvalues

Dominant Eigenvalues

In other words, the overall response is as “slow” as the one with only λ_1

We say that λ_1 **dominates** λ_2, or that λ_1 is a **dominant eigenvalue**

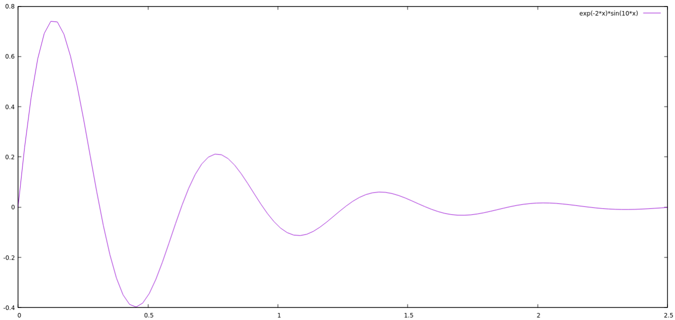


Complex and Conjugate Eigenvalues

If we have a couple of complex and conjugate eigenvalue, i.e. $\sigma \pm i\omega$ the response has the form

$$e^{\sigma t} \sin \omega t$$

and the value of σ (the real part) characterised the duration of the transient



Dominant Eigenvalue

Let's consider a asymptotically stable system with eigenvalue:

$$\lambda_1, \lambda_2, ..., \sigma_1 \pm i\omega_1, \sigma_2 \pm i\omega_2, ...$$

The **dominant eigenvalue** is $\lambda^* = \max(\lambda_1, \lambda_2, ..., \sigma_1, \sigma_2, ...)$, i.e. the highest value of the real parts

System Reponse and Eigenvalues

Natural Frequencies or Modes

Let's consider a asymptotically stable system with eigenvalue:

$$\lambda_1, \lambda_2, ..., \sigma_1 \pm i\omega_1, \sigma_2 \pm i\omega_2, ...$$

The real parts contribute to the response as an exponential term:

$$e^{\lambda_i t} \qquad e^{\sigma_i t}$$

The real part of an eigenvalue is a **frequency** and its measure unit is:

$$Hz = s^{-1} = \frac{1}{s}$$

For this reason, eigenvalues are called **natural frequencies** or **modes** of the system

System Reponse and Eigenvalues

Dominant Eigenvalue and Transient Duration

Let's consider a asymptotically stable system with eigenvalue:

$$\lambda_1, \lambda_2, ..., \sigma_1 \pm i\omega_1, \sigma_2 \pm i\omega_2, ...$$

The real parts contribute to the response as an exponential term:

$$e^{\lambda_i t} \qquad e^{\sigma_i t}$$

The inverse-absolute of a real part of an eigenvalue is called **time constant**

$$T_i = \frac{1}{|\lambda_i|} \qquad T_i = \frac{1}{|\sigma_i|}$$

Given that λ_* is the dominant eigenvalue (real part), the duration of the transient (set-up time) is approximately:

$$T_A \simeq \frac{3}{|\lambda_*|}$$

Dynamic Systems
The Theory Behind

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Robotic Systems