# Entropy-satisfying well-balanced schemes for balance laws with singular source terms

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In collaboration with B. Andreianov, F. Coquel, J.-M. Hérard, K. Saleh. . .

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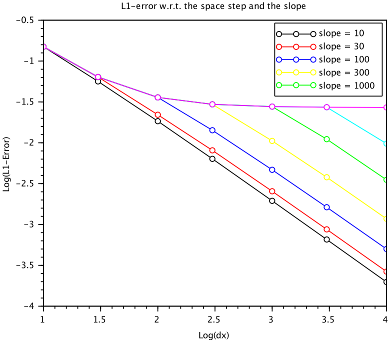
Balance law with a stationary solution, solved by a splitting method

$$\partial_t u + \partial_x \frac{u^2}{2} = -u\partial_x a \quad \text{and} \quad \begin{array}{l} a(x) = \max(-1, \min(1, \text{slope} \times x)) \\ u_0(x) + a(x) = 2 \end{array}$$



# Entropy-satisfying well-balanced schemes for balance laws with singular source terms

## Hyperbolic scalar or systems of balance laws

- Space dependent flux and source term
- Infinite stiffness with respect to $x$
- Resonance: nonlinear interaction between the flux and the source term
- Entropy inequality

## Well-balanced schemes

- First-order explicit finite volume schemes
- Discrete preservation of *some* stationary solutions
- Discrete entropy inequality $\Longrightarrow$ Stability

# Entropy-satisfying well-balanced schemes for balance laws with singular source terms

**Pb 1. Burgers equation with a pointwise friction**

- Scalar equation with a pointwise singular source term
- Definition of stationary solutions
- Construction of well-balanced schemes for any monotone numerical flux
- Analysis: definition of solutions, uniqueness and convergence

with B. Andreianov (and also N. Aguillon and F. Lagoutière)

**Pb 2. Euler equations in a discontinuous nozzle**

- $2 \times 2$ system, discontinuous cross-section (or porosity)
- Approximate Riemann solver, relaxation scheme
- Singular dissipation at the discontinuity to control the CFL
- Positivity and discrete entropy inequalities

with F. Coquel, J.-M. Hérard and K. Saleh

# *Burgers equation with a pointwise friction*

One-dimensional flow in a pipe with a porous grid in $x = 0$:

- Burgers equation for the flow
- Friction term for modeling the grid

Cauchy problem

$$\begin{cases} \partial_t u + \partial_x \dfrac{u^2}{2} = -\lambda \; u \; \delta_0(x) \\ u(0, x) = u_0(x) \end{cases}$$

with $u_0 \in \mathbf{L}^\infty(\mathbb{R})$ and $\lambda > 0$ friction coefficient

Quasilinear form, using Heaviside function $w(0, x) = H(x)$:

$$\partial_t \begin{pmatrix} u \\ w \end{pmatrix} + \begin{pmatrix} u & \lambda u \\ 0 & 0 \end{pmatrix} \partial_x \begin{pmatrix} u \\ w \end{pmatrix} = 0$$

- if $u \neq 0$, then hyperbolicity
- if $u = 0$, then trivial system (easier than [Isaacson, Temple '95])

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# *The singular source term as a coupling problem*

The Burgers equation with the singular source term

$$\partial_t u + \partial_x \frac{u^2}{2} = -\lambda \; u \; \delta_0(x)$$

can be seen as a coupling problem between two Burgers equations:

$$\begin{array}{c|c} x < 0 & x > 0 \\ & \\ \partial_t u + \partial_x \dfrac{u^2}{2} = 0 & \partial_t u + \partial_x \dfrac{u^2}{2} = 0 \end{array}$$

$+$ Coupling conditions between $u(t, 0^-)$ and $u(t, 0^+)$
which have to describe the effects of the singular source term

# *The singular source term as a coupling problem*

The coupling problem:

$$\begin{cases} \partial_t u + \partial_x \dfrac{u^2}{2} = 0 \text{ far from the interface } \{x = 0\} \\ \text{Coupling conditions between } u(t,0^-) \text{ and } u(t,0^+) \end{cases}$$

But:

- the source term contains a product of distributions: $u \times \delta_0$!
- the sign of the velocity $u$ can change (no strict hyperbolicity)
  - $\longrightarrow$ Resonance: nonlinear interaction between the interface and the nonlinear waves of the Burgers equation
  - $\longrightarrow$ The theory of nonconservative products cannot be applied

Idea:

- Regularize the problem to define the coupling between $u(t,0^-)$ and $u(t,0^+)$
- Study the dependence of the coupling conditions w.r.t. the regularization

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# *Coupling conditions and stationary solutions*

Assume that the coupling conditions are defined:

1. Let $\mathcal{G} \subset \mathbb{R}^2$ be the set of all admissible couples of traces, then the coupling problem writes

$$\begin{cases} \partial_t u + \partial_x \dfrac{u^2}{2} = 0 & \text{far from the interface } \{x = 0\} \\ (u(t, 0^-), u(t, 0^+)) \in \mathcal{G} & \text{for a.e. } t > 0 \end{cases}$$

2. Consider any $(\kappa^-, \kappa^+) \in \mathbb{R}^2$ and define the piecewise constant function

$$(\star) \qquad \kappa(x) = \begin{cases} \kappa^- & \text{if } x < 0 \\ \kappa^+ & \text{if } x > 0 \end{cases}$$

**Proposition**

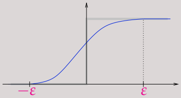
*The function $\kappa(x)$ defined by $(\star)$ is a stationary solution of the coupling problem* **if and only if** $(\kappa^-, \kappa^+) \in \mathcal{G}$

# *Construction of stationary solutions*

Stationary solutions of the Burgers equation with a regularized source term

$H$ is replaced by $H_\varepsilon \in \mathscr{C}^1(\mathbb{R})$, nondecreasing function such that

$$\forall |x| \geqslant \varepsilon \quad H_\varepsilon(x) = H(x)$$




A couple $(\kappa^-, \kappa^+)$ belongs to $\mathcal{G}[H_\varepsilon]$ **if and only if** it exists $u_\varepsilon(x)$ satisfying

$$(S_\varepsilon) \qquad \begin{cases} \dfrac{d}{dx}\dfrac{u_\varepsilon(x)^2}{2} + \lambda\, u_\varepsilon(x)\dfrac{d}{dx}H_\varepsilon(x) = 0, \quad x \in (-\varepsilon, \varepsilon) \\ u_\varepsilon(-\varepsilon) = \kappa^- \\ u_\varepsilon(\varepsilon) = \kappa^+ \end{cases}$$

in the entropy weak sense

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in the entropy weak sense

This problem can be solved by hand!

- smooth parts: $(S_\varepsilon)$ in the classical sense
  Either $u_\varepsilon(x) = 0$
  Or $\frac{d}{dx}\big(u_\varepsilon(x) + \lambda H_\varepsilon(x)\big) = 0$
- Shock waves at $x_0 \in (-\varepsilon, \varepsilon)$:
  $(u_\varepsilon(x_0^-) + u_\varepsilon(x_0^+))/2 = 0$ and $u_\varepsilon(x_0^-) > u_\varepsilon(x_0^+)$

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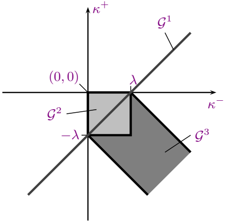
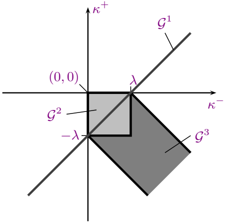


# *Construction of stationary solutions*

The set $\mathcal{G}[H_\varepsilon]$ can be decomposed in 3 parts:

- $\mathcal{G}^1 = \{\kappa^+ = \kappa^- - \lambda\}$
- $\mathcal{G}^2 = [0, \lambda] \times [-\lambda, 0]$
- $\mathcal{G}^3 = ((\mathbb{R}^+ \times \mathbb{R}^-) \setminus \mathcal{G}^2) \cap \{-\lambda \leqslant \kappa^- + \kappa^+ \leqslant \lambda\}$

**NB.** The entropy can be dissipated for $\mathcal{G}^2 \cup \mathcal{G}^3$




### Fundamental remark

The set $\mathcal{G}[H_\varepsilon]$ actually is independent of the regularized function $H_\varepsilon$!

### Definition

*A function $u \in \mathbf{L}^\infty(\mathbb{R}^+ \times \mathbb{R})$ is a solution of the Burgers equation with a pointwise friction if (in the entropy weak sense)*

$$\begin{cases} \partial_t u + \partial_x \dfrac{u^2}{2} = 0 & \textit{far from the interface } \{x = 0\} \\ (u(t, 0^-), u(t, 0^+)) \in \mathcal{G} & \textit{for a.e. } t > 0 \end{cases}$$

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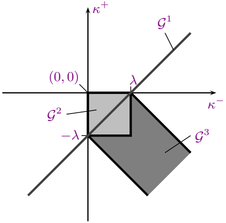


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# *Finite volume schemes*

Discretization

- $t^n = n\Delta t$
- Interfaces $x_{i+1/2} = i\Delta x$ and cells $C_i = (x_{i-1/2}, x_{i+1/2})$
  $\longrightarrow$ The source term is superposed on the interface $x_{1/2} = 0$

First-order explicit monotone schemes:

- Two-point numerical flux $g$:
  - locally Lipschitz
  - Consistency: $g(u,u) = \frac{u^2}{2}$
  - Monotonicity: $g(\nearrow, \searrow)$
- In the cells far from the interface $x_{1/2}$:

$$i \neq 0, 1 \qquad u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x}\big(g(u_i^n, u_{i+1}^n) - g(u_{i-1}^n, u_i^n)\big)$$

# *Partially well-balanced schemes*

Well-balanced schemes for stationary solutions of $\mathcal{G}^1 = \{\kappa^+ = \kappa^- - \lambda\}$:

- Two fluxes at interface $x_{1/2}$: $g^-_{1/2}$ and $g^+_{1/2}$
- Cell formulas near the interface $x_{1/2}$

$$\begin{cases} u_0^{n+1} = u_0^n - \frac{\Delta t}{\Delta x}\left(g^-_{1/2}(u_0^n, u_1^n) - g(u_{-1}^n, u_0^n)\right) \\ u_1^{n+1} = u_1^n - \frac{\Delta t}{\Delta x}\left(g(u_1^n, u_2^n) - g^+_{1/2}(u_0^n, u_1^n)\right) \end{cases}$$

- Well-balanced property. For all $(\kappa^-, \kappa^+) \in \mathcal{G}^1$

$$\begin{cases} u_{-1}^n = u_0^n = \kappa^- \\ u_1^n = u_2^n = \kappa^+ \end{cases} \Longrightarrow \begin{cases} g^-_{1/2}(\kappa^-, \kappa^+) = g(\kappa^-, \kappa^-) \\ g^+_{1/2}(\kappa^-, \kappa^+) = g(\kappa^+, \kappa^+) \end{cases}$$

- Use of reconstructed states:

$$\begin{cases} g^-_{1/2}(u, v) = g(u, v + \lambda) \\ g^+_{1/2}(u, v) = g(u - \lambda, v) \end{cases}$$

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The numerical scheme

$$
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\forall i \neq 0,1 \qquad & u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x}\big(g(u_i^n, u_{i+1}^n) - g(u_{i-1}^n, u_i^n)\big) \\
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\end{aligned}
$$

is well-balanced for stationary solutions of $\mathcal{G}^1$,
but is a priori not well-balanced for stationary solutions of $\mathcal{G}^2 \cup \mathcal{G}^3 \dots$
Reconstructed states are defined since $\mathcal{G}^1$ is the graph of a one-to-one function

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What about the convergence of this numerical scheme?

- Definition of solutions of the Burgers equation with pointwise friction
- Uniqueness of the solution
- A priori estimates
- Discrete entropy inequalities

# *Definition of solutions*

Coupling problem for the Burgers equation with pointwise friction:

$$(\mathrm{P}) \qquad \begin{cases} \partial_t u + \partial_x \dfrac{u^2}{2} = 0 & \text{far from the interface } \{x=0\} \\ (u(t,0^-), u(t,0^+)) \in \mathcal{G} & \text{for a.e. } t>0 \end{cases}$$

**Definition ([Andreianov, Karlsen, Risebro '11], [Andreianov, S. '12])**

*A function $u \in \mathbf{L}^\infty(\mathbb{R}^+ \times \mathbb{R})$ is an entropy solution of* (P) *if, for all $(\kappa^-, \kappa^+) \in \mathcal{G}$, it satisfies, for all $(\kappa^-, \kappa^+) \in \mathcal{G}$,*

$$\partial_t |u - \kappa(x)| + \partial_x \Phi(u, \kappa(x)) \leqslant 0$$

*where $\kappa(x) = (1 - H(x))\kappa^- + H(x)\kappa^+$.*

- $\mathbf{L}^1$-stability w.r.t. and of stationary solutions $\kappa(x)$ of $\mathcal{G}^1 \cup \mathcal{G}^2 \cup \mathcal{G}^3$
- Direct extension of Kruzhkov's definition

# *Uniqueness proof*

**Theorem**

*There exists one and only one entropy solution of* (P)*. Let $u_0$ and $v_0$ two initial data and $u$ and $v$ the corresponding solutions, then*

$$\int_{B(0,R)} |u(t,x) - v(t,x)| dx \leqslant \int_{B(0,R+Lt)} |u_0(x) - v_0(x)| dx.$$

- Doubling variable technique far from the interface
- Interfacial terms canceled using the dissipativity of $\mathcal{G}$:

$$\Phi(u(t,0^+), v(t,0^+)) - \Phi(u(t,0^-), v(t,0^-)) \leqslant 0$$

  which leads to

$$\partial_t |u - v| + \partial_x \Phi(u,v) \leqslant 0$$

- Conclude by using the classical appropriate test function

# *Study of the numerical scheme*

Under a small (technical?) restriction on the numerical flux, we have

- Monotonicity of the scheme (Crandall–Tartar lemma does not apply here**!**)
- A priori bounds in $\mathbf{L}^\infty \cap \mathrm{BV}$ are available
- Discrete entropy inequalities. Let $\kappa_i = \kappa(x_i)$.
  If $(\kappa^-, \kappa^+) \in \mathcal{G}^1$, numerical entropy fluxes $(G^n_{i+1/2})$ exist such that

$$\frac{|u_i^{n+1} - \kappa_i| - |u_i^n - \kappa_i|}{\Delta t} + \frac{G^n_{i+1/2} - G^n_{i-1/2}}{\Delta x} \leqslant 0$$

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**Corollary (Numerical stability of discrete stationary solutions)**

*This numerical scheme is stable in $\ell^1$ with respect to stationary solutions of $\mathcal{G}^1$*

Same result for stationary solutions of $\mathcal{G}^2 \cup \mathcal{G}^3$ with more elaborated schemes

# *Numerical simulations: numerical boundary layers*

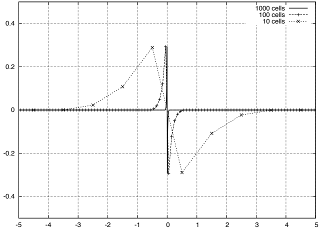


FIGURE 2. Initial datum with $(c_-, c_+) \in \mathscr{G}_\lambda^2$ for several meshes.

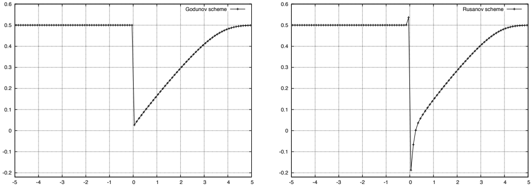


FIGURE 3. Comparison between well-balanced schemes with the Godunov flux (left) and with the Rusanov flux (right).

# *Conclusion and perspectives*

[Andreianov, S. '12] (following [Andreianov, Karlsen, Risebro '11])

- Well-posed balance law with a singular source term
- Construction of the set of admissible traces
- Convergence of partially well-balanced schemes
- Adapted entropy inequalities: stability with respect to stationary solutions
- Numerical stability according to exactly preserved stationary solutions

[Aguillon, Lagoutière, S. '15]

- well-balanced scheme for more steady states
- Extension to a moving singular source term (Burgers + pointwise particle)

**Rq.** Most of the previous results fail for

$$dv_t u + \partial_x \frac{u^2}{2} = +\lambda \ u \ \delta_0$$

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**Pb 2. Euler equations in a discontinuous nozzle**

- $2 \times 2$ system, discontinuous cross-section (or porosity)
- Approximate Riemann solver, relaxation scheme
- Singular dissipation at the discontinuity to control the CFL
- Positivity and discrete entropy inequalities

with F. Coquel, J.-M. Hérard and K. Saleh

# *Euler equations with porosity*

**Gas dynamics in a porous medium, with a discontinuous porosity $\alpha$**

$$\begin{cases} \partial_t(\alpha\rho) + \partial_x(\alpha\rho u) = 0 \\ \partial_t(\alpha\rho u) + \partial_x(\alpha\rho u^2 + \alpha p(\rho)) = p(\rho)\partial_x\alpha \end{cases}$$

where $\alpha(x) = \alpha_l(1 - H(x)) + \alpha_r H(x)$ with $\alpha_l, \alpha_r > 0$

Setting $a = \log\alpha$: $\begin{cases} \partial_t\rho + \partial_x(\rho u) + \rho u\partial_x a = 0 \\ \partial_t(\rho u) + \partial_x(\rho u^2 + p(\rho)) + \rho u^2\partial_x a = 0 \end{cases}$

Setting $m = \alpha\rho$: $\begin{cases} \partial_t m + \partial_x(mu) = 0 \\ \partial_t(mu) + \partial_x(mu^2 + \alpha p(m/\alpha)) - p(m/\alpha)\partial_x\alpha = 0 \end{cases}$

# *General setting*

System of balance laws with singular source term

$$\begin{cases} \partial_t U + \partial_x f(U,\alpha) + s(U,\alpha)\partial_x \alpha = 0 \\ \partial_t \alpha = 0 \quad \text{with } \alpha(x) = \alpha_l(1 - H(x)) + \alpha_r H(x) \end{cases}$$

Quasilinear form:

$$\partial_t \begin{pmatrix} U \\ \alpha \end{pmatrix} + \begin{pmatrix} \nabla_U f(U,\alpha) & \nabla_\alpha f(U,\alpha) + s(U,\alpha) \\ 0 & 0 \end{pmatrix} \partial_x \begin{pmatrix} U \\ \alpha \end{pmatrix} = 0$$

Non-strict hyperbolicity: $\nabla_U f(U,\alpha) = 0 \not\Rightarrow \nabla_\alpha f(U,\alpha) + s(U,\alpha) = 0$
(Euler equations with porosity, shallow-water equations with bathymetry...)

**Entropy inequality**

It exists $(\eta, F) = (\eta, F)(U,\alpha)$ with $\eta$ strictly convex w.r.t. $U$ and

$$(\nabla_U \eta \cdot \nabla_U f,\ \nabla_U \eta \cdot (\nabla_\alpha f + s)) = \nabla_{(U,\alpha)} F^\top$$

i.e.

$$\partial_t \eta(U,\alpha) + \partial_x F(U,\alpha)\ [=\ \text{or}\ \leqslant]\ 0$$

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# *Coupling problem*

Here again, such singular system can be seen as a coupling problem

$$\begin{cases} \partial_t U + \partial_x f(U, \alpha_l) = 0 & \text{for } x < 0 \\ \partial_t U + \partial_x f(U, \alpha_r) = 0 & \text{for } x > 0 \\ \text{Coupling conditions between } U(t, 0^-) \text{ and } U(t, 0^+) \end{cases}$$

But, here again:

- the source term contains a product of distributions: $g(U, \alpha)\partial_x \alpha$!
- the sign of the eigenvalues can change (no strict hyperbolicity)

$\longrightarrow$ Resonance: nonlinear interaction between the interface and the nonlinear waves of the left and right systems

$\longrightarrow$ The theory of nonconservative products cannot be applied

Same idea as before:

- Regularize the problem to define the coupling between $U(t, 0^-)$ and $U(t, 0^+)$
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# *Coupling conditions and stationary solutions*

Assume that the coupling conditions are defined:

1. Let $\mathcal{G} \subset \Omega^2$ be the set of all admissible couples of traces, then the coupling problem writes

$$\begin{cases} \partial_t U + \partial_x f(U, \alpha_l) = 0 & \text{for } x < 0 \\ \partial_t U + \partial_x f(U, \alpha_r) = 0 & \text{for } x > 0 \\ (U(t, 0^-), U(t, 0^+)) \in \mathcal{G} & \text{for a.e. } t > 0 \end{cases}$$

2. Consider any $(\kappa^-, \kappa^+) \in \Omega^2$ and define the piecewise constant function

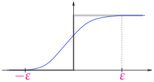
$$(\star) \qquad \kappa(x) = \begin{cases} \kappa^- & \text{if } x < 0 \\ \kappa^+ & \text{if } x > 0 \end{cases}$$

**Proposition**

*The function $\kappa(x)$ defined by $(\star)$ is a stationary solution of the coupling problem* **if and only if** $(\kappa^-, \kappa^+) \in \mathcal{G}$

# *Construction of stationary solutions*

1. Replace $\alpha(x)$ by $\alpha_\varepsilon(x) = \alpha_l(1 - H_\varepsilon(x)) + \alpha_r H_\varepsilon(x)$ with




2. Solve the boundary value problem to define $\mathcal{G}[a_\varepsilon]$

A couple $(\kappa^-, \kappa^+)$ belongs to $\mathcal{G}[a_\varepsilon]$ **if and only if** it exists $U_\varepsilon(x)$ satisfying

$$(S_\varepsilon) \qquad \begin{cases} \dfrac{d}{dx} f(U_\varepsilon, a_\varepsilon) + s(U_\varepsilon, a_\varepsilon) \dfrac{d}{dx} a_\varepsilon(x) = 0, \quad x \in (-\varepsilon, \varepsilon) \\ U_\varepsilon(-\varepsilon) = \kappa^- \\ U_\varepsilon(\varepsilon) = \kappa^+ \end{cases}$$

in the entropy weak sense

3. Piecewise smooth solutions:
   - Smooth parts: $(S_\varepsilon)$ in the classical sense (Riemann invariants)
   - Shock waves at $x_0 \in (-\varepsilon, \varepsilon)$:
     $f(U_\varepsilon, a_\varepsilon)(x_0^+) = f(U_\varepsilon, a_\varepsilon)(x_0^-)$ and $F(U_\varepsilon, a_\varepsilon)(x_0^+) \leqslant F(U_\varepsilon, a_\varepsilon)(x_0^-)$

# *Case of Euler equations with porosity*

- Riemann invariants

$$\begin{cases} I_1(U,\alpha) = \alpha\rho u & \text{(mass conservation)} \\ I_2(U,\alpha) = \alpha u(\rho E + p) & \text{(energy/entropy conservation)} \end{cases}$$

  do not provide neither an injective nor an surjective relation

- Shock waves inside $[-\varepsilon, \varepsilon]$ lead to a strict decay of the entropy

$$F(U_\varepsilon, a_\varepsilon)(x_0^+) < F(U_\varepsilon, a_\varepsilon)(x_0^-)$$

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**Theorem (See for instance [Goatin, LeFloch '04])**

- *The set $\mathcal{G}[a_\varepsilon]$ can be constructed by hand and is independent of $a_\varepsilon$*
- *The Riemann problem for the Euler equations with a discontinuous porosity (seen as a coupling problem) admits between one and three solutions*

# *Finite volume schemes*

Discretization

- $t^n = n\Delta t$
- Interfaces $x_{i+1/2} = i\Delta x$ and cells $C_i = (x_{i-1/2}, x_{i+1/2})$
  $\longrightarrow$ The source term is superposed on the interface $x_{1/2} = 0$

First-order explicit entropy-satisfying schemes:

- Two-point numerical flux $g$
  - Consistency: $g(U, U, \alpha) = f(U, \alpha)$
  - Entropy stability: see [Tadmor '03] or [Bouchut '04] for details
- In the cells far from the interface $x_{1/2}$:

$$i \neq 0, 1 \qquad U_i^{n+1} = U_i^n - \frac{\Delta t}{\Delta x}\big(g(U_i^n, U_{i+1}^n) - g(U_{i-1}^n, U_i^n)\big)$$

# *Partially well-balanced schemes*

Well-balanced schemes for stationary solutions corresponding to some $\mathcal{G}^0 \subset \mathcal{G}$:

- Two fluxes at interface $x_{1/2}$: $g^-_{1/2}$ and $g^+_{1/2}$
- Cell formulas near the interface $x_{1/2}$

$$\begin{cases} U_0^{n+1} = U_0^n - \frac{\Delta t}{\Delta x}\left(g^-_{1/2}(U_0^n, U_1^n) - g(U_{-1}^n, U_0^n)\right) \\ U_1^{n+1} = U_1^n - \frac{\Delta t}{\Delta x}\left(g(U_1^n, U_2^n) - g^+_{1/2}(U_0^n, U_1^n)\right) \end{cases}$$

- Well-balanced property. For all $(\kappa^-, \kappa^+) \in \mathcal{G}^0$

$$\begin{cases} U_{-1}^n = U_0^n = \kappa^- \\ U_1^n = U_2^n = \kappa^+ \end{cases} \implies \begin{cases} g^-_{1/2}(\kappa^-, \kappa^+) = g(\kappa^-, \kappa^-) \\ g^+_{1/2}(\kappa^-, \kappa^+) = g(\kappa^+, \kappa^+) \end{cases}$$

- **But**, for using reconstructed states, $\mathcal{G}^0$ should be the graph of a monotone function, difficult in general…

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# *Approximate Riemann solvers*

In order to construct the fluxes $g^{\pm}_{1/2}$, we use approximate Riemann solvers [Harten, Lax, van Leer '83]

$$\begin{cases} \partial_t U + \partial_x f(U, \alpha_l) = 0 & \text{for } x < x_{1/2} \\ \partial_t U + \partial_x f(U, \alpha_r) = 0 & \text{for } x > x_{1/2} \\ \text{Approximate coupling conditions} & \\ \qquad \text{for } (U(t, x^-_{1/2}), U(t, x^+_{1/2})) & \end{cases}$$

Approximate coupling conditions for $(U(t, x^-_{1/2}), U(t, x^+_{1/2}))$ such that

- Preservation of stationary solutions associated with $\mathcal{G}^0$
- Dissipation of the entropy through the interface

But in practice, problems with non-ordered waves, resonance...

Following [Coquel *et al.* '99] (linearly degenerate extension of [Jin, Xin '95]), we propose a relaxation approximation of the coupling problem

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# *Relaxation models for approximate Riemann solvers*

Relaxation approximation

$$\begin{cases} \partial_t V_\varepsilon + \partial_x \tilde{f}(V_\varepsilon, \alpha_l) = \text{Relax.} & \text{for } x < x_{1/2} \\ \partial_t V_\varepsilon + \partial_x \tilde{f}(V_\varepsilon, \alpha_r) = \text{Relax.} & \text{for } x > x_{1/2} \\ \text{Approximate coupling conditions} & \\ \qquad \text{for } (V_\varepsilon(t, x_{1/2}^-), V_\varepsilon(t, x_{1/2}^+)) & \end{cases}$$

In general, the relaxation approximation is based on LD systems of the form

$$\partial_t V_\varepsilon + \partial_x \tilde{f}(V_\varepsilon, \alpha) + \tilde{s}(V_\varepsilon, \alpha)\partial_x \alpha = \text{Relax.}$$

Euler equations with porosity [Coquel, Saleh, S. '14]

- Classical entropy decay (sub-characteristic condition) and robustness
- Preservation of stationary solutions associated with $\mathcal{G}^0 = \{u \equiv 0, p(\rho) \equiv \text{Cst}\}$
- ⋆ Dissipation of the entropy through the interface

$\Longrightarrow$ Full control of the CFL condition

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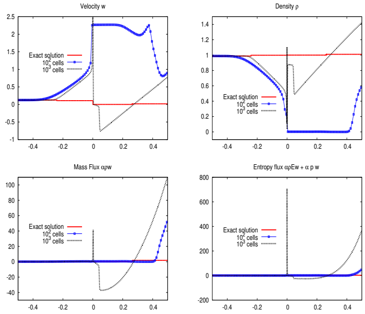
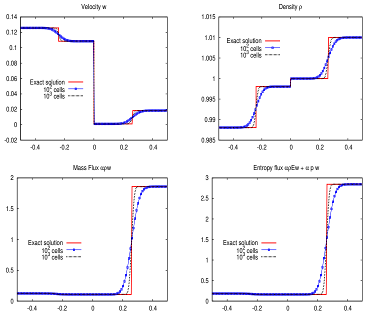
# *Comparison with Rusanov scheme + splitting*

$\alpha_l = 1$, $\alpha_r = 100$

## Rusanov scheme




## Relaxation scheme



# *Relative entropy for stationary solutions*

Continuous entropy inequality

$$\begin{cases} \partial_t U + \partial_x f(U,\alpha) + s(U,\alpha)\partial_x \alpha = 0 \\ \partial_t \eta(U,\alpha) + \partial_x F(U,\alpha) \leqslant 0 \end{cases}$$

Relative entropy to compare $U$ and $V$ for the same $\alpha$ ($H(U,V,\alpha) \simeq |U-V|^2$)

- $H(U,V,\alpha) := \eta(U,\alpha) - \eta(V,\alpha) - \nabla_U \eta(V,\alpha) \cdot (U-V)$

$\rightarrow$ $H(\cdot,V,\alpha)$ nonnegative strictly convex function, $H(U,V,\alpha) = 0$ iff $U = V$

Compare an entropy weak solution $U$ and a stationary solution $V$:

$$\begin{aligned} \partial_t H &= \partial_t \eta(U,\alpha) - \nabla_U \eta(V,\alpha) \cdot \partial_t U \\ &\leqslant -\partial_x F(U,\alpha) - \nabla_U \eta(V,\alpha) \cdot \big(\partial_x f(U,\alpha) + s(U,\alpha)\partial_x \alpha\big) \\ &\leqslant -\partial_x \big[F(U,\alpha) + \nabla_U \eta(V,\alpha) \cdot \partial_x f(U,\alpha)\big] \\ &\qquad + \big(\partial_x \nabla_U \eta(V,\alpha)\big) \cdot f(U,\alpha) - \nabla_U \eta(V,\alpha) \cdot s(U,\alpha)\partial_x \alpha \end{aligned}$$

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# *Relative entropy for stationary solutions*

**Proposition**

*Let $U$ be an entropy weak solution and $V$ a stationary solution. If*

- $\big(\partial_x \nabla_U \eta(V,\alpha)\big) \cdot f(U,\alpha) \equiv 0$
- $\nabla_U \eta(V,\alpha) \cdot s(U,\alpha) \equiv 0$

*then*

$$\frac{d}{dt}\int H(U,V,\alpha)\,dx \leqslant 0$$

*In other words, $V$ is a stable stationary state.*

- Shallow-water equations with topography:
  $\eta(U,\alpha) = hu^2/2 + gh^2/2 + gh\alpha$,
  $\nabla_U \eta(U,\alpha) = (-u^2/2 + g(h+\alpha), u)^\top$
- Euler equations with porosity:
  $\eta(U,\alpha) = \alpha\rho u^2/2 + \alpha\rho e(\rho)$,
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These two conditions are (only) satisfied for

- Shallow-water equations with topography: "lake at rest" states
- Euler equations with porosity: null-velocity states

**NB.** Asymptotic stability cannot be generally expected (periodic solutions exist)

# *Relative entropy for well-balanced schemes*

**Proposition**

*Assume the two previous assumptions. Consider an entropy-satisfying scheme which exactly preserves a stationary state $(V_i)_i$. Then,*

$$\Delta x \sum_i H(U_i^{n+1}, V_i, \alpha_i) \leqslant \Delta x \sum_i H(U_i^n, V_i, \alpha_i)$$

*In other words, the numerical scheme is stable in $\ell^2$ with respect to the stationary discrete state $(V_i)_i$.*

- No smoothness assumption on $\alpha$ and $V$, valid in multi-D
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