Modeling and numerical resolution of hydraulic model partially in charge

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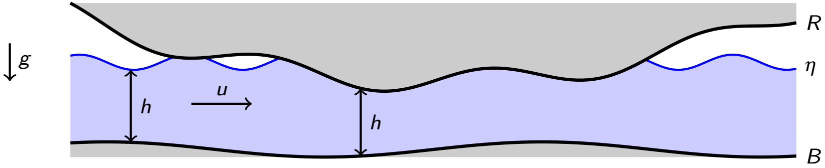
Partially free surface model with friction and viscosity:

Starting from Navier-Stokes equations with gravity and **roof** and following [Gerbeau, Perthame 00], we get

$$\left\{\begin{array}{lll} \partial_t h + \nabla\cdot(hu) & = & 0 \\ \partial_t(hu) + \nabla\cdot\left(hu\otimes u + \frac{g}{2}h^2\mathrm{I_d}\right) & = & -h\nabla\left(gB + \frac{p}{\varrho}\right) - \frac{\kappa_b + \kappa_r\mathbb{1}_{p>0}}{1 + \frac{\kappa_b+\kappa_r\mathbb{1}_{p>0}}{3\mu}h}u + 4\mu\nabla\cdot(h\nabla u) \\ h + B \le R, & & (h+B-R)p = 0. \end{array}\right.$$



$h(t,x)\in\mathbb{R}_+$	water depth	$B(x)\le R(x)\in\mathbb{R}$	bottom level and roof level
$u(t,x)\in\mathbb{R}^d$	averaged horizontal velocity	$(\kappa_b,\kappa_r)\in(\mathbb{R}_+)^2$	Frictions at bottom and at roof
$p(t,x)\in\mathbb{R}_+$	pressure at surface $\eta = h + z_b$	$\mu\in\mathbb{R}_+$	Viscosity of fluid

Many flows are partially in charge:

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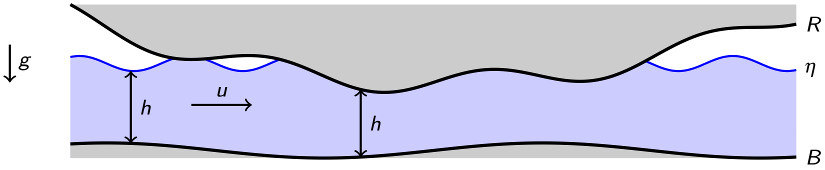
Partially free surface model (SW^R):

Neglecting the dissipative terms, we write

$$\begin{cases} \partial_t h \quad + \nabla \cdot (hu) = 0 \\ \partial_t (hu) + \nabla \cdot \left(hu \otimes u + \frac{g}{2} h^2 \mathrm{I_d} \right) = -h \nabla \left(gB + \frac{p}{\varrho} \right) \\ h + B \leq R, \quad (h + B - R)\, p = 0. \end{cases} \qquad (SW^R)$$



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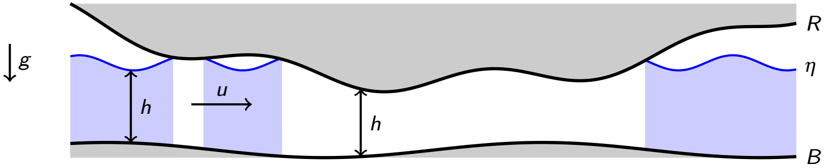


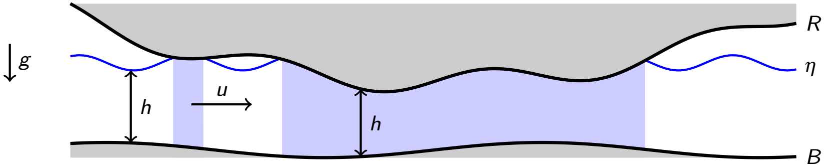
⚠: Coupling approach

- In the part with free surface ⇒ Shallow water equations. **hyperbolic**

Partially free surface model (SW^R):

Neglecting the dissipative terms, we write

$$\left\{\begin{array}{ll} \partial_t h + \nabla\cdot((R-B)\,u) & = 0 \\ \partial_t u + (u\cdot\nabla)\,u + \nabla\left(gR + \dfrac{p}{\varrho}\right) & = 0 \\ \qquad h+B\leq R, & \quad (h+B-R)\,p = 0. \end{array}\right. \qquad (SW^R)$$



⚠: Coupling approach

- In the part with free surface ⇒ Shallow water equations. **hyperbolic**
- In the part in charge ⇒ Lake equations **not hyperbolic**

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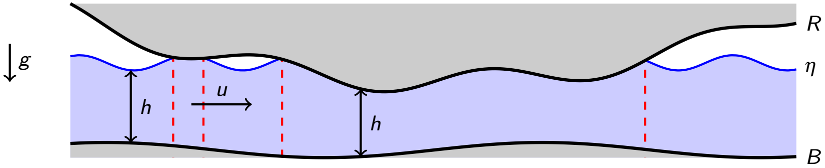


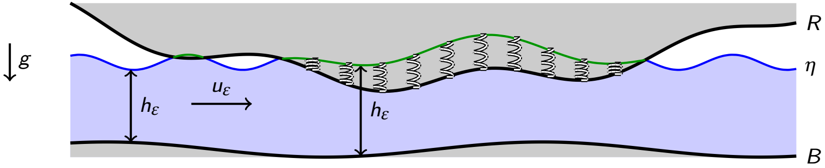
⚠: Coupling approach

- In the part with free surface ⇒ Shallow water equations. **hyperbolic**
- In the part in charge ⇒ Lake equations **not hyperbolic**
- What boundary condition at the interface ? Dynamic of the interface ?

Regularized partially free surface model (SW_ε^R):

Let us consider the **hyperbolic** model for any parameter $\varepsilon > 0$:

$$\begin{cases} \partial_t h_\varepsilon + \nabla\cdot(h_\varepsilon u_\varepsilon) = 0 \\ \partial_t(h_\varepsilon u_\varepsilon) + \nabla\cdot\left(h_\varepsilon u_\varepsilon \otimes u_\varepsilon + \frac{g}{2}h_\varepsilon^2 \mathrm{I_d}\right) = -h_\varepsilon \nabla\left(gB + \frac{p_\varepsilon}{\varrho}\right) \\ p_\varepsilon = \frac{\varrho g (h_\varepsilon + B - R)_+}{\varepsilon^2} \end{cases} \qquad (SW_\varepsilon^R)$$



Theorem: Energy balance

The mechanic energy $E = \mathscr{E} + \mathscr{K}$ is decreasing, i.e. $\partial_t \int_{\mathbb{R}^d} E \,\mathrm{d}x \le 0$

with the kinetic energy $$\mathscr{K} = \frac{1}{2} h_\varepsilon |u_\varepsilon|^2,$$

and the potential energy $$\mathscr{E} = \frac{g}{2}h_\varepsilon^2 + gBh_\varepsilon + \left(\frac{g}{2}h_\varepsilon^2 + g(B-R)h_\varepsilon\right)\mathbb{1}_{h_\varepsilon + B \ge R}.$$

In addition, the equality case holds for smooth solutions.

THEOREM: Convergence $(SW_\varepsilon^R) \underset{\varepsilon\to 0}{\longrightarrow} (SW^R)$

formal proof

Following the argument in [Klainerman, Majda 82] and $s \geq 2 + \left[\frac{d}{2}\right]$,
in the part of the domain in charge, i.e. $h_\varepsilon + B \geq R$, we get the error order

$$\|h_\varepsilon - h,\ p_\varepsilon - p,\ u_\varepsilon - u\|_{L^\infty(0,T;H^s(\mathbb{1}_{h_\varepsilon+B\geq R}))} = \|h_\varepsilon - h,\ p_\varepsilon - p,\ u_\varepsilon - u\|_{L^\infty(0,T;H^s(\mathbb{1}_{h_\varepsilon+B\leq R}))} + O\left(\varepsilon^2\right)$$

⚠: Hyperbolic equation with **stiff conservative source term**

On a cartesian grid, the explicit Godunov-type solver

- is stable under restrictive CFL condition $\left(u_\varepsilon + \sqrt{\left(1+\frac{1}{\varepsilon^2}\right) g h_\varepsilon}\right) \Delta t \leq C$
- diverge when the regularization parameter ε vanishes, i.e. $Err = O\left(\frac{\mathrm{d}x}{\varepsilon}\right)$

Q1. How produce numerical scheme **accurate** when $\varepsilon \ll 1$?

THEOREM: Low-Mach number flow [Dellacherie 10]

At the asymptotic regime $\varepsilon = 0$, the acoustic operator have to be discretized with a **centered scheme**.

Q2. How produce numerical scheme **stable** when $\varepsilon \ll 1$?

[MP, Vila 15] Centered-Potential Regularization of Advection Upstream Splitting Method. *SIAM Journal on Numerical Analysis*

Q1. How produce numerical scheme **accurate** when $\varepsilon \ll 1$?

R1a. We use a **AUSM** based scheme

$$\partial_t \begin{pmatrix} h_k \\ h_k u_k \end{pmatrix} + \frac{1}{\Delta x_k} \sum_{f \in \mathbb{F}_k} \begin{pmatrix} \mathscr{F}_f^h \\ \mathscr{F}_f^{hu} \end{pmatrix} \cdot N_f^k \mu_f^k = \begin{pmatrix} 0 \\ \mathscr{Q}_k \end{pmatrix}$$

with $\mathscr{F}_f^h = \int_f h_\varepsilon u_\varepsilon \, d\sigma$, $\mathscr{F}_f^{hu} = \int_f h_\varepsilon u_\varepsilon \otimes u_\varepsilon \, d\sigma$, $\mathscr{Q}_k = -\frac{1}{|V_k|} \int_{V_k} g h_\varepsilon \nabla \phi_\varepsilon \, dx$

and the **potential** $\phi_\varepsilon(x, h_\varepsilon) = h_\varepsilon + B + \frac{(h_\varepsilon + B - Z)_+}{\varepsilon^2}$.

R1b. with a **centered discretization** of the potential for any $\varepsilon > 0$

$$\mathscr{Q}_k \approx h_k \sum_{f \in \mathbb{F}_k} \phi_f N_f^k |f| = h_k \sum_{f \in \mathbb{F}_k} \delta\phi_f |f|$$

with $\phi_f = \dfrac{\phi(x_k, h_k) + \phi\left(x_{k_f}, h_{k_f}\right)}{2}$ and $\delta\phi_f = \dfrac{\phi\left(x_{k_f}, h_{k_f}\right) - \phi(x_k, h_k)}{2} N_f^k$.

- Leads to a **consistent** numerical scheme when ε goes to 0 [Dellacherie 10].

Q2a. How to control the **space discretization** error?

R2a. We use an **up-wind** scheme in velocity

$$\partial_t (h_k u_k) + \frac{1}{\Delta x_k} \sum_{f \in \mathbb{F}_k} \left(u_k \left(\mathscr{F}_f^\rho \cdot N_f^k \right)_+ - u_{k_f} \left(\mathscr{F}_f^\rho \cdot N_f^k \right)_- \right) \mu_f^k = \mathscr{Q}_k$$

- Ensure the **dissipation of the discrete kinetic energy**

We introduce a **regularization** using the potential jump (τ : time scale; γ : regu. param.)

$$\mathscr{F}_f^h = h_f \left(u_f - \gamma \frac{\tau}{\Delta \mathbf{x}_f} \delta\phi_f \right) \qquad \text{with} \qquad 2\frac{h_f}{\Delta \mathbf{x}_f} = \frac{h_k}{\Delta x_k} + \frac{h_{k_f}}{\Delta x_{k_f}}$$

$$h_f = \frac{h_k + h_{k_f}}{2} \qquad \text{and} \qquad h_f u_f = \frac{h_k u_k + h_{k_f} u_{k_f}}{2}$$

- Ensure :
  - the **dissipation of the discrete potential energy**
  - the **steady state at rest**

Q2b. How to control the **time discretization** error?

R2b. Using an **IMEX** scheme: implicit for the water level h / explicit for the velocity u

- **water level: implicit** scheme of type **non-linear** advection-diffusion

$$h_k^{n+1} - h_k^n + \frac{\Delta t}{\Delta x_k} \sum_{f \in \mathbb{F}_k} h_f^{n+1} \left(u_f^n - \gamma \frac{\Delta t}{\Delta \mathbf{x}_f^{n+1}} \delta\phi_f^{n+1} \right) \cdot N_f^k \mu_f^k = 0$$

- **velocity: explicit** upwind scheme with source term.

$$h_k^{n+1} u_k^{n+1} - h_k^n u_k^n + \frac{\Delta t}{\Delta x_k} \sum_{f \in \mathbb{F}_k} \left(u_k^n \left(\mathscr{F}_f^{n+1} \cdot N_f^k \right)^+ - u_{k_f}^n \left(\mathscr{F}_f^{n+1} \cdot N_f^k \right)^- \right) \mu_f^k = -\frac{h_k^{n+1}}{\Delta x_k} \sum_{f \in \mathbb{F}_k} \delta\phi_f^{n+1} \mu_f^k$$

THERORЕМ: Entropy dissipation

Let $\gamma \geq 1$ and assume the following **CFL-type condition** is satisfies

$$\left(\left| u_f^n \cdot N_f^k \right| + \sqrt{\frac{\gamma}{2}} \sqrt{\left| \delta\phi_f^{n+1} \right|} \right) \Delta t \leq \frac{\min\left(h_k^{n+1}, h_{k_f}^{n+1} \right)}{h_k^{n+1} + h_{k_f}^{n+1}} \min\left(\Delta x_k, \Delta x_{k_f} \right)$$

then
- the solution is **positive**, i.e. $h_k^n > 0$
- the discrete mechanic energy is **decreasing**

Non-linear fixed point :

→ We set $h_k^{n,0} = h_k^n$

- → we compute $\phi_{i,k}^{n,q} = \phi_i\left(x_k, h_k^{n,q}\right)$
- we compute $\Delta t^{n,q} = f\left(\phi_k^{n,q}, u_k^n\right)$

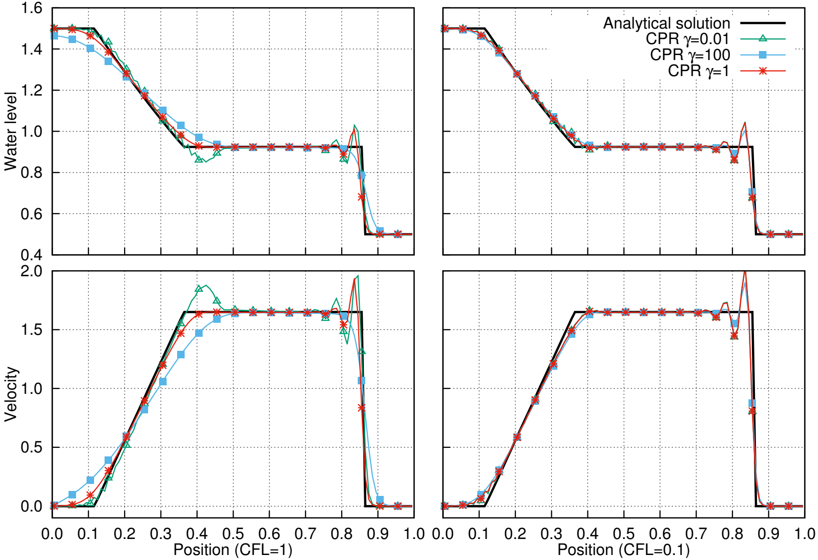
$$\left(\left|u_f^n \cdot N_f^k\right| + \sqrt{\frac{\gamma}{2}}\sqrt{\left|\delta\phi_f^{n,q}\right|}\right)\Delta t \le \frac{\min\left(h_k^{n,q}, h_{k_f}^{n,q}\right)}{h_k^{n,q} + h_{k_f}^{n,q}} \min\left(\Delta x_k, \Delta x_{k_f}\right)$$

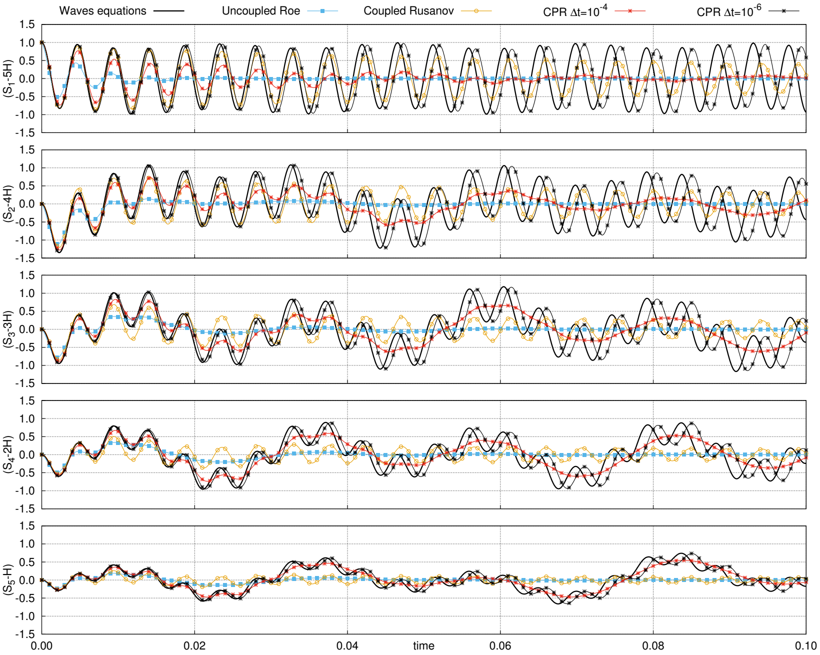
- we compute implicitly the scheme for $h_k^{n,q+1}$

$$h_k^{n,q+1} - h_k^n + \frac{\Delta t}{\Delta x_k}\sum_{f\in\mathbb{F}_k}\left(h_f^{n,q+1} u_{i,f}^n - \gamma\frac{\Delta t}{\Delta x_{i,f}^{n,q}} h_f^{n,q}\delta\phi_{i,f}^{n,q+1}\right)\cdot N_f^k \mu_f^k = 0$$

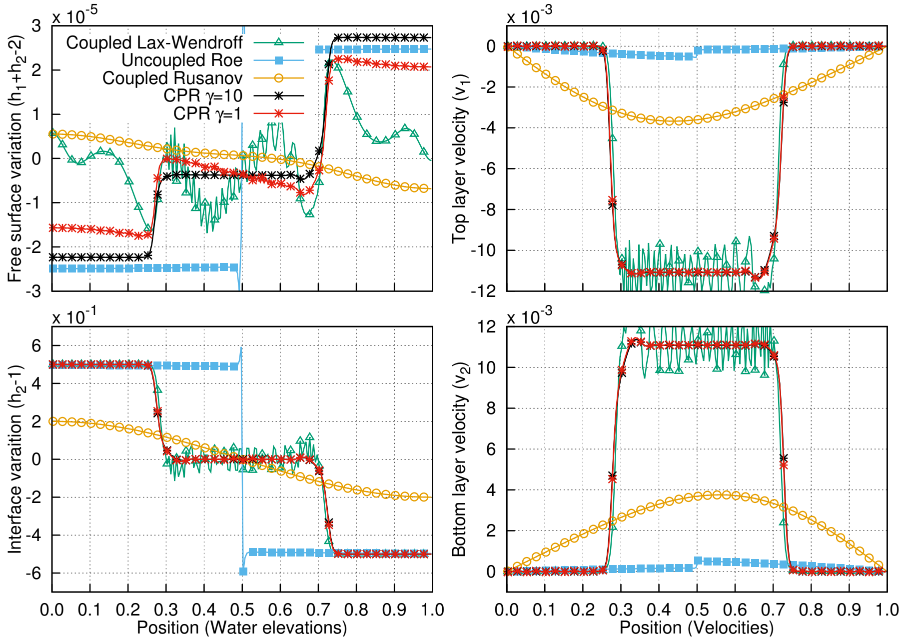
- we compute explicitly the scheme for $u_{i,k}^{n,q+1}$
- we estimate the variation of entropy $E_k^{n,q+1} = f\left(h_k^{n,q+1}, u_k^{n,q+1}\right)$
- we test $\left(E_k^{n,q+1} \le E_k^n - \Delta t \ \mathrm{flux}\right)$ — *a posteriori* **stop test**
  - ← if not — **new iteration**
  - ← if yes, we set $\left(h_k^{n+1}, u_k^{n+1}\right) = \left(h_k^{n,q+1}, u_k^{n,q+1}\right)$ — **next time step**

Converge with **few iterations** when the potential is **regular** enough.


Water level
Velocity
Position (CFL=1)
Position (CFL=0.1)
Analytical solution
CPR γ=0.01
CPR γ=100
CPR γ=1
1.6
1.4
1.2
1.0
0.8
0.6
0.4
2.0
1.5
1.0
0.5
0.0
0.0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1.0



Waves equations
Uncoupled Roe
Coupled Rusanov
CPR Δt=10^-4
CPR Δt=10^-6
(S1-5H)
(S2-4H)
(S3-3H)
(S4-2H)
(S5-H)
1.5
1.0
0.5
0.0
-0.5
-1.0
-1.5
0.00
0.02
0.04
time
0.06
0.08
0.10


x 10^-5
Coupled Lax-Wendroff
Uncoupled Roe
Coupled Rusanov
CPR γ=10
CPR γ=1
Free surface variation (h1+h2-2)
x 10^-3
Top layer velocity (v1)
x 10^-1
Interface varaition (h2-1)
Position (Water elevations)
x 10^-3
Bottom layer velocity (v2)
Position (Velocities)

▸ Partially free surface

Centered-Potential regularization scheme:

- The CPR scheme is very robust:
  - non-conservative product
  - stiff conservative source term (low-Mach)
  - easily adaptable to several physics
- The CPR scheme is very stable:
  - well-balanced for steady state at rest
  - not restrictive CFL condition
  - weak numerical dissipation

Prospects:
- asymptotic scheme for $\varepsilon = 0$
- wet/dry transition
- non-conservative forces for stability of steady state

Partially free surface flows:

- Derivation of a shallow water type model for partially free surface flows
- Analysis and numerical resolution for regular solution

Prospects:
- Propagation of discontinuities

Thank you for your attention

FORMAL PROOF: Convergence $(SW_\varepsilon^R) \underset{\varepsilon \to 0}{\longrightarrow} (SW^R)$

$$\begin{cases} \partial_t h_\varepsilon + \nabla \cdot (h_\varepsilon u_\varepsilon) = 0 \\ \partial_t (h_\varepsilon u_\varepsilon) + \nabla \cdot \left(h_\varepsilon u_\varepsilon \otimes u_\varepsilon + \frac{g}{2} h_\varepsilon^2 \mathrm{I_d} \right) = -h_\varepsilon \nabla \left(gB + \frac{p_\varepsilon}{\varrho} \right) \\ p_\varepsilon = \frac{\varrho g (h_\varepsilon + B - R)_+}{\varepsilon^2} \end{cases} \qquad (SW_\varepsilon^R)$$

- In the part **in charge**, i.e. $(h_\varepsilon + B - R) \geq 0$:
  - The **main term** of the momentum balance leads to the **constrain** $(h_\varepsilon + B - R)_+ = O\left(\varepsilon^2\right)$.
  - Since the solution is regular, the **main term** of the mass conservation leads to the **divergence free constrain** $\nabla \cdot (h_\varepsilon u_\varepsilon) = O\left(\varepsilon^2\right)$.
  - Considering the second term of the momentum balance, we conclude that the **main term** of $(h_\varepsilon, u_\varepsilon)$ satisfy the **lake equation**.
- In the part with **free surface**, i.e. $(h_\varepsilon + B - R)_+ = 0$ the pressure vanishes and we recover the **shallow water equations**.

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