Schemes with Well-Controlled Dissipation: application to nonconservative systems

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Outline

Nonconservative systems: weak solutions

- In Ernest, LeFloch, Mishra SINUM (2015) the authors have introduced a class of schemes, the *schemes with well-controled dissipation* (WCD), which allow one to compute, with robustness and accuracy, small-scale dependent shock wave solutions to nonlinear hyperbolic systems of conservation laws.
- The goal of this work is to extend these schemes to 1d nonlinear hyperbolic models in nonconservative form:

$$U_t + A(U)U_x = 0, \qquad U = U(t,x) \in \mathbb{R}^N.$$

- The definition and the numerical approximation of weak solutions of these systems is particularly challenging. Following the theory developed by Dal Maso, LeFloch, Murat (1995), given a family of paths ϕ, a piecewise smooth function is a weak solution is:
  - It is a classical solution where it is regular.
  - Across a discontinuity the following jump condition is satisfied:

$$\int_0^1 A(\phi(s;W^-,W^+))\frac{\partial \phi}{\partial s}(s;W^-,W^+)\,ds = \sigma(W^+ - W^-),$$

    where σ is the speed of propagation and $W^{\pm}$ the limits to the left and to the right of the discontinuity.

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Nonconservative systems: weak solutions

How to choose the 'good' family of paths?

- When the hyperbolic system is the vanishing diffusion and dispersion limit of a family of problems

$$U_t + A(U)U_x = \varepsilon BU_{xx} + \delta\varepsilon^2 CU_{xxx}. \tag{1}$$

  the adequate family of paths should be consistent with the *traveling waves* of the regularized system (LeFloch (1989)).

- A traveling wave

$$U_\epsilon(x,t) = V\left(\frac{x-\sigma t}{\epsilon}\right), \tag{2}$$

  is a solution of (1) satisfying

$$\lim_{\xi\to-\infty} V(\xi) = U^-, \quad \lim_{\xi\to+\infty} V(\xi) = U^+, \quad \lim_{\xi\to\pm\infty} V'(\xi) = 0, \quad \lim_{\xi\to\pm\infty} V''(\xi) = 0.$$

- If there exists a traveling wave of speed σ linking the states U^-, U^+, the limit when ϵ tends to 0 of U_ϵ is:

$$U(x,t) = \begin{cases} U^- & \text{if } x < \sigma t; \\ U^+ & \text{if } x > \sigma t; \end{cases}$$

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Nonconservative systems: weak solutions

- An easy computation shows that V has to solve the equation

$$-\sigma V' + A(V)\,V' = BV'' + \delta C V'''. \tag{3}$$

- By integrating (3) from $-\infty$ to ∞ and taking into account the boundary conditions, we obtain the jump condition

$$\int_{-\infty}^{\infty} A(V(\xi))\,V'(\xi)\,d\xi = \sigma(U^+ - U^-).$$

- If this jump condition is compared with the generalized Rankine-Hugoniot condition:

$$\int_0^1 \mathcal{A}(\phi(s;U^-,U^+))\frac{\partial \phi}{\partial s}(s;U^-,U^+)\,ds = \sigma(U^+ - U^-),$$

  it is clear that the good choice for the path connecting the states U^- and U^+ would be, after a reparameterization, the viscous profile.

- The computation of viscous profiles may be a very difficult task and, even it they can be computed, a numerical method which is formally consistent with the corresponding definition of weak solution may not converge to the right solutions due to the effects of the numerical viscosity (Castro, LeFloch, Muñoz, CP; JCP (2008)).

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WCD Methods: semi-discrete formulation

- We consider the semi-discrete finite difference scheme:

$$\frac{dU_i}{dt}+\frac{1}{\Delta x}A(U_i)\left(\sum_{j=-p}^{p}\alpha_j U_{i+j}\right)=\frac{c}{\Delta x}B\left(\sum_{j=-p}^{p}\beta_j U_{i+j}\right)+\frac{\delta c^2}{\Delta x}C\left(\sum_{j=-p}^{p}\gamma_j U_{i+j}\right). \tag{4}$$

- $U_i(t)=U(x_i,t)$ represents the nodal value and $c=c(t)\geq 0$ is a time-dependent parameter to be determined.
- In order to have a $2p$-th order consistent scheme, the coefficients α_i, β_i, γ_i have to satisfy the following order conditions:

$$\sum_{j=-p}^{j=p} j\alpha_j=1; \qquad \sum_{j=-p}^{j=p} j^s\alpha_j=0, \qquad s\neq 1 \text{ and } 0\leq s\leq 2p,$$

$$\sum_{j=-p}^{j=p} j^2\beta_j=2; \qquad \sum_{j=-p}^{j=p} j^s\beta_j=0, \qquad s\neq 2 \text{ and } 0\leq s\leq 2p,$$

$$\sum_{j=-p}^{j=p} j^3\gamma_j=6; \qquad \sum_{j=-p}^{j=p} j^s\gamma_j=0, \qquad s\neq 3 \text{ and } 0\leq s\leq 2p.$$

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WCD Methods: equivalent equation

- A formal Taylor expansion in (4) allows us to derive the equivalent equation:

$$\begin{aligned} \frac{dU}{dt} + A(U)U_x &= c\Delta x B U_{xx} + \delta c^2 \Delta x^2 C U_{xxx} \\ &- A(U)\left(\sum_{k=2p+1}^{\infty} \frac{\Delta x^{k-1}}{k!} A_k^p U^{(k)} \right) \\ &+ cB \left(\sum_{k=2p+1}^{\infty} \frac{\Delta x^{k-1}}{k!} B_k^p U^{(k)} \right) \\ &+ \delta c^2 C \left(\sum_{k=2p+1}^{\infty} \frac{\Delta x^{k-1}}{k!} C_k^p U^{(k)} \right), \end{aligned} \tag{5}$$

what shows that (4) is a first order method for the hyperoblic system and a $2p$-order method for the regularized one, where:

$$A_k^p = \sum_{j=-p}^{j=p} \alpha_j j^k, \quad B_k^p = \sum_{j=-p}^{j=p} \beta_j j^k, \quad C_k^p = \sum_{j=-p}^{j=p} \gamma_j j^k.$$

WCD Methods: choice of c

- At a discontinuity the weak solution formally satisfies:

$$U^{(k)} = O\left(\frac{[U]}{\Delta x^k}\right).$$

Substituting $U^{(k)}$ by $[U]/\Delta x^k$ in (5) we obtain:

$$\begin{aligned}\frac{dU}{dt} + A(U^-,U^+)\frac{[U]}{\Delta x} &= cB\frac{[U]}{\Delta x} + \delta c^2 C\frac{[U]}{\Delta x} \\ &-S_p^A A(U^-,U^+)\frac{[U]}{\Delta x} + cS_p^B B\frac{[U]}{\Delta x} + \delta c^2 S_p^C C\frac{[U]}{\Delta x},\end{aligned} \qquad (6)$$

where $A(U^-,U^+)$ represents some intermediate matrix and

$$S_p^A = \sum_{k=2p+1}^{\infty} \frac{A_k^p}{k!}, \quad S_p^B = \sum_{k=2p+1}^{\infty} \frac{B_k^p}{k!}, \quad S_p^C = \sum_{k=2p+1}^{\infty} \frac{C_k^p}{k!}.$$

WCD Methods: choice of c

- If $A(U^-, U^+)$ satisfies the Roe property

$$A(U^-, U^+)[U] = \int_0^1 A(\Phi(s; U^-, U^+))\partial_s \Phi(s; U^-, U^+)\, ds,$$

then (6) can be written as follows:

$$\begin{aligned} \frac{dU}{dt} + \sigma \frac{[U]}{\Delta x} &= cB\frac{[U]}{\Delta x} + \delta c^2 C \frac{[U]}{\Delta x} \\ &- S_p^A A(U^-, U^+)\frac{[U]}{\Delta x} + cS_p^B B \frac{[U]}{\Delta x} + \delta c^2 S_p^C C \frac{[U]}{\Delta x}. \end{aligned} \quad (7)$$

- Ideally the high order terms

$$h.o.t. = S_p^A A(U^-, U^+)\frac{[U]}{\Delta x} - cS_p^B B\frac{[U]}{\Delta x} - \delta c^2 S_p^C C\frac{[U]}{\Delta x},$$

should be dominated in amplitude by the leading order terms

$$l.o.t. = A(U^-, U^+)\frac{[U]}{\Delta x} - cB\frac{[U]}{\Delta x} - \delta c^2 C\frac{[U]}{\Delta x}.$$

- In order to achieve this correct balance, a tolerance parameter $\tau << 1$ is fixed and c is chosen so that

$$\frac{|h.o.t.|}{|l.o.t.|} < \tau.$$

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WCD Methods: implementation

- **Stability condition:** if p is hight enough and $A(U)$, B, C commute, the von Neumann analysis leads to the condition:

$$\frac{\Delta t}{\Delta x} \leq \frac{2c\mu_k^B}{c^2\pi^2(\mu_k^B)^2 + (\mu_k^A + \delta c^2\pi^2\mu_k^C)^2}$$

where μ_k^A, μ_k^B, μ_k^C are the eigenvalues of A, B, and C respectively.

- **Time stepping:** 3d order TVD RK method Shu, Osher (1988)).

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Coupled Burgers equation

- We consider the following model problem for nonconservative hyperbolic systems:

$$\partial_t u + u\,\partial_x(u+v) = 0,$$
$$\partial_t v + v\,\partial_x(u+v) = 0.$$

- In order to set unambiguously the jump conditions across a shock, the small scale effects have to be taken into account. Following Berthon CRAS (2002) we consider the vanishing diffusion term given by:

$$\partial_t u + u\,\partial_x(u+v) = \varepsilon_1 \partial^2_{xx}(u+v),$$
$$\partial_t v + v\,\partial_x(u+v) = \varepsilon_2 \partial^2_{xx}(u+v).$$

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- We consider first the Riemann problem with initial data

$$U_l = \begin{bmatrix} 1 \\ 0.25 \end{bmatrix}, \quad U_r = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \tag{8}$$

and apply the WCD problem with constant $c = 0.1$ and different values of Δx, p.

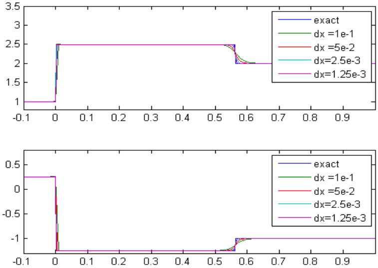

Figure : Numerical results of the WCD method with $c = 0.1$, $p = 8$ for the coupled Burgers system with initial condition (8): (up for the variable u and down for the variable v).

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and apply the WCD problem with constant $c = 0.1$ and different values of Δx, p.

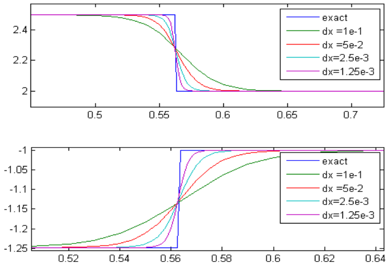

Figure : Numerical results of the WCD method $c = 0.1$, $p = 8$ for the coupled Burgers system with initial condition (8) (up for the variable u and down for the variable v): zoom of the solution.

Coupled Burgers equation

- We consider first the Riemann problem with initial data

$$U_l = \begin{bmatrix} 1 \\ 0.25 \end{bmatrix}, \quad U_r = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \tag{8}$$

and apply the WCD problem with constant $c = 0.1$ and different values of Δx, p.

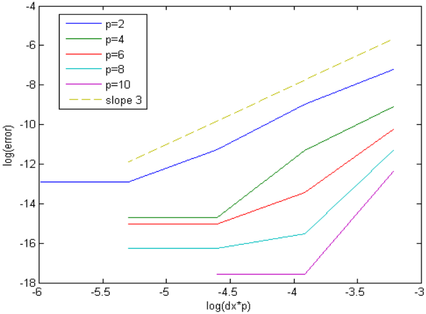

Figure : Errors corresponding to the intermediate state of the Riemann problem corresponding to (8) in logarithmic scale. Horizontal axis: $\log(p\Delta x)$. Vertical axis: $\log(error)$.

Coupled Burgers equation

- We consider first the Riemann problem with initial data

$$U_l = \begin{bmatrix} 1 \\ 0.25 \end{bmatrix}, \quad U_r = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \qquad (8)$$

and apply the WCD problem with constant $c = 0.1$ and different values of Δx, p.

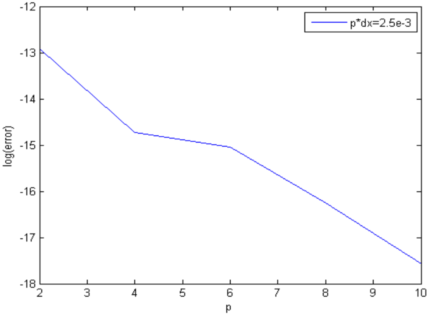

Figure : Errors corresponding to the intermediate state of the Riemann problem corresponding to (8) in logarithmic scale with different values of p and $p\Delta x = 2.5 \cdot 10^{-3}$. Horizontal axis: p. Vertical axis: $\log(error)$.

Coupled Burgers equation

- We consider now the Riemann problem with initial data

$$U_l = \begin{bmatrix} 1 \\ 0.25 \end{bmatrix}, \quad U_r = \begin{bmatrix} -2 \\ -1 \end{bmatrix} \tag{9}$$

and apply the WCD problem with constant $c = 1$, and $p\Delta x = cte$.

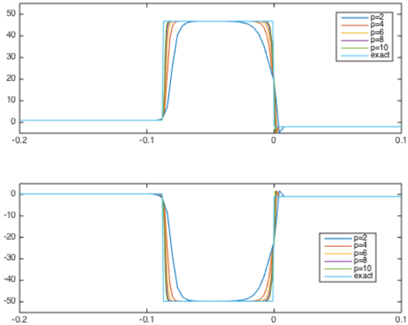

Figure : Numerical results of the WCD method at $t = 0.1$ for the coupled Burgers system with initial condition (9) with different values of p and $p\Delta x = 0.004$(up for the variable u and down for the variable v).

Coupled Burgers equation

- Approximation of the Rankine-Hugoniot curve of a given left state.

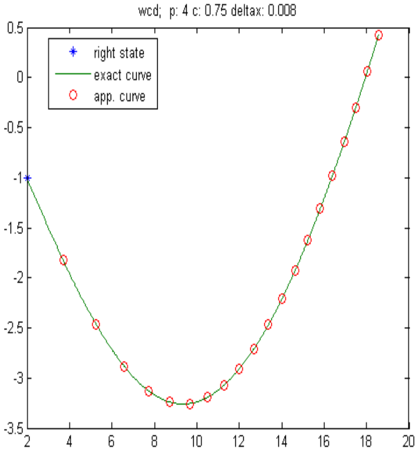

Figure : Approximation of the Hugoniot curve of the right state $(2, -1)$ and the approximation obtained with $c = 0.75$, $p = 4$, and $\Delta x = 0.08$).

Coupled cubic equation

- We consider next the system

$$\begin{aligned} \partial_t u + 2(u+v)^2 u_x + (u+v)^2 v_x &= 0, \\ \partial_t v + (u+v)^2 u_x + 2(u+v)^2 v_x &= 0. \end{aligned} \tag{10}$$

- If the component equations of this system are added, then a scalar conservation law for $w := u + v$ with cubic flux is obtained:

$$\partial_t w + \partial_x w^3 = 0.$$

- In order to set unambiguously the jump conditions, we consider the following regularized system:

$$\begin{aligned} \partial_t u + 2(u+v)^2 u_x + (u+v)^2 v_x &= \varepsilon_1 (u+v)_{xx} + \delta_1 \varepsilon_1^2 (u+v)_{xxx}, \\ \partial_t v + (u+v)^2 u_x + 2(u+v)^2 v_x &= \varepsilon_2 (u+v)_{xx} + \delta_2 \varepsilon_1^2 (u+v)_{xxx}, \end{aligned} \tag{11}$$

where δ_i, $i = 1, 2$.

- If these two constants are such that:

$$\gamma := \delta_1 r_1^2 + \delta_2 r_2^2 = \frac{\delta_1 r_1^2 - \delta_2 r_2^2}{r_1 - r_2}, \tag{12}$$

where

$$r_i = \frac{\varepsilon_i}{\varepsilon_1 + \varepsilon_2}, \quad i = 1, 2,$$

then the traveling waves of the regularized system can be explicitly computed.

Coupled cubic equation

- We consider next the system

$$\partial_t u + 2(u+v)^2 u_x + (u+v)^2 v_x = 0,$$
$$\partial_t v + (u+v)^2 u_x + 2(u+v)^2 v_x = 0. \tag{10}$$

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then the traveling waves of the regularized system can be explicitly computed.

Coupled cubic equation

- We consider next the system

$$\begin{aligned} \partial_t u + 2(u+v)^2\, u_x + (u+v)^2\, v_x &= 0, \\ \partial_t v + (u+v)^2\, u_x + 2(u+v)^2\, v_x &= 0. \end{aligned} \tag{10}$$

- If the component equations of this system are added, then a scalar conservation law for $w := u + v$ with cubic flux is obtained:

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- In order to set unambiguously the jump conditions, we consider the following regularized system:

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Coupled cubic equation

- The Rankine-Hugoniot curve of a given left state $U_l = [u_l, v_l]^T$ with $w_l = u_l + v_l > 0$ has a singularity at $w = 0$

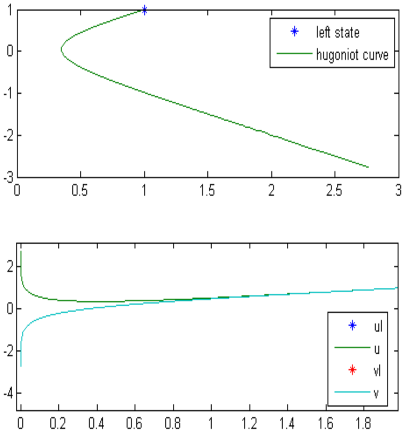

Figure : Rankine-Hugoniot curve of the state $U_L = [1, 1]$ (left); u and v components of the right states as a function of w_r (right).

Coupled cubic equation

- The Riemann problem with initial conditions

$$U_l = \begin{bmatrix} 1.5 \\ 0.5 \end{bmatrix}, \quad U_r = \begin{bmatrix} 0.75 \\ 0.25 \end{bmatrix} \tag{13}$$

is first considered. The solution consists of a contact discontinuity traveling at speed 4 and a shock whose speed is 7. The numerical method WCD is used with $dx = 1/250$

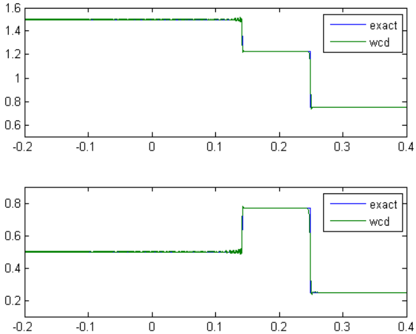

Figure : Numerical results of the WCD method with $dx = 1/2500$, $\tau = 1e-1$, $p = 4$ for the non-convex coupled cubic system with initial conditions (13) : up u, down v.

Coupled cubic equation

- Approximation of the Rankine-Hugoniot curve of a given left state.

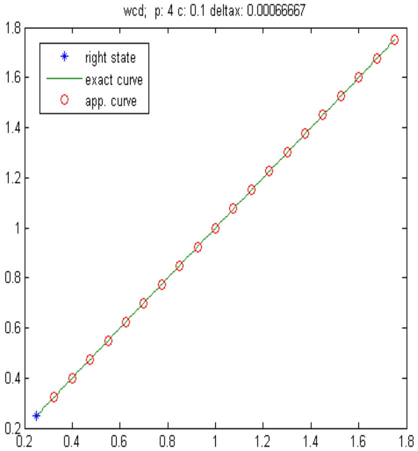

Figure : Approximation of the Hugoniot curve of the right state $(.5, .5)$ for the cubic coupled system and the approximation obtained with $\tau = 0.1$, $p = 4$, and $\Delta x = 1/1500$).

Coupled cubic equation

- In order to test the ability of the numerical methods to correctly capture non-classical shocks we consider the following Riemann problem for the cubic flux conservation law:

$$\begin{cases} w_t + (w^3)_x = 0, \\ w(x,0) = \begin{cases} w_l = 3 & \text{if } x < 0, \\ w_r = -3 + \sqrt{2}/\sqrt{3\delta} & \text{otherwise} \end{cases} \end{cases}$$

  The solution of this Riemann problem consistent with the considered regularization consists of a non-classical shock linking w_l to w_r: see LeFloch (2002).

- If we consider now the coupled cubic system the Riemann problem with left and right states equal to:

$$U_L = \begin{bmatrix} w_l/2 \\ w_l/2 \end{bmatrix}, \quad U_R = \begin{bmatrix} w_r/2 \\ w_r/2 \end{bmatrix},$$

  the solution will not be a single shock, as Hugoniot curves cannot pass through $w = 0$.

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  the solution will not be a single shock, as Hugoniot curves cannot pass through $w = 0$.

Coupled cubic equation

- Nevertheless, the sum of the two equations of the numerical method provides a WCD method in nononservative form for the cubic flow scalar law.

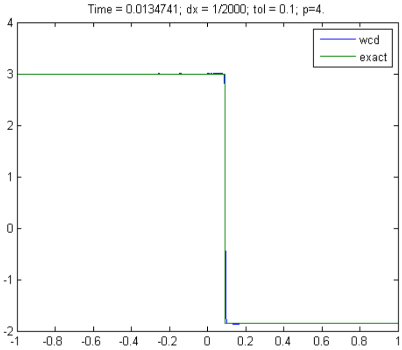

Figure : Numerical results of the WCD method for the nonclassical shcok of the cubic flux conservation law.

Coupled cubic equation

- If instead we apply to the coupled cubic some other numerical method, as the path-conservative Lax-Friedrichs scheme, the sum is again a WCD in nonconservative form for the cubic flux scalar law. The numerical solution converges to the classical solution of the Riemann problem consisting of a classical shock and plus a rarefaction wave.

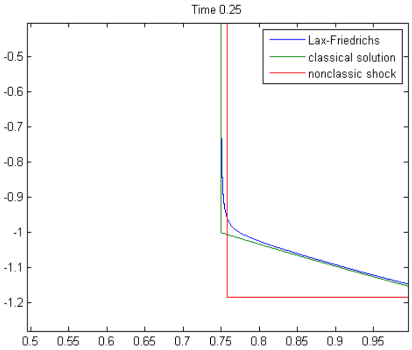

Figure : Numerical results of the WCD method for the nonclassical shcok of the cubic flux conservation law.

The modified shallow water system

- In Castro, LeFloch, Muñoz, CP; JCP (2008) the following system was considered

$$h_t + q_x = 0,$$
$$\partial q_t + \left(\frac{q^2}{h}\right)_x + qhh_x = 0.$$

- The system can be written as follows:

$$h_t + q_x = 0,$$
$$\left(q + \frac{h^3}{6}\right)_t + \left(\frac{q^2}{h} + q\frac{h^2}{2}\right)_x = 0.$$

what allows one to derive the correct Rankine-Hugoniot conditions.

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what allows one to derive the correct Rankine-Hugoniot conditions.

The modified shallow water system

- We consider Riemann problem with initial condition:

$$h(0,x) = \begin{cases} 3 & x \leq 0 \\ .5 & \text{otherwise}, \end{cases}$$

and $u(0,x) = 1$. We consider $\Delta x = 1/500$, $\tau = 0.01$, and $p = 12$.

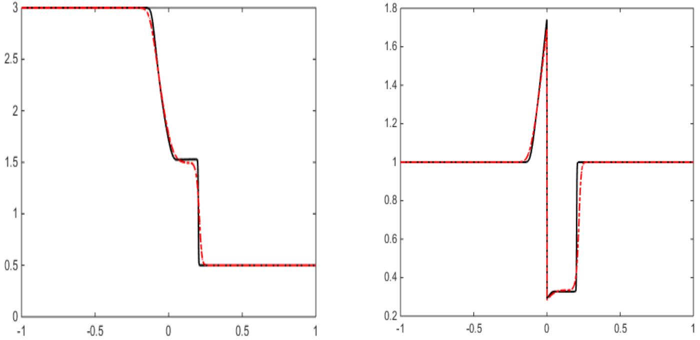

Figure : Solution with at time 0.1. Left column: h. Right column: u. Black lines: solutions using the WCD scheme for the nonconservative system. Red lines: solutions using the WCD scheme for conservative systems.

The modified shallow water system

- Hugoniot curve for a given left state. The numerical curves are obtained by using the WCD scheme for nonconservative systems using $p = 4$, $p = 12$, $p = 20$ and $p = 50$ with $\Delta x = 1/500$.

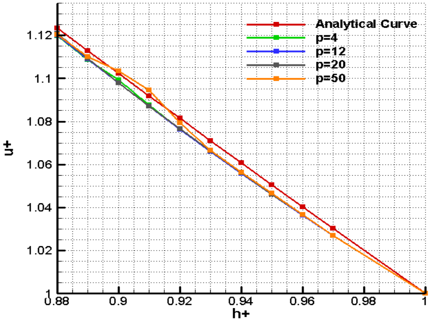

Figure : 1-shock curves corresponding to the left state $(h_-, u_-) = (1, 1)$.