On hydrostatic reconstruction schemes for the shallow water equations

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Outline

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2. Semi-discrete finite volume schemes
3. Original hydrostatic reconstruction
   - HR method of Audusse et al.
   - HR method of Morales et. al.
   - HR method of Chen/Noelle
4. Derivation based on subcell reconstructions
5. Stability Properties
6. Numerical experiments

Shallow water equations

$$\partial_t U + \partial_x F(U) = S(U, z)$$

with unknown vector U, flux-vector $F(U)$ and source term $S(U, z)$

$$U = \begin{pmatrix} h \\ hu \end{pmatrix}, \quad F(U) = \begin{pmatrix} hu \\ hu^2 + gh^2/2 \end{pmatrix}, \quad S(U, z) = \begin{pmatrix} 0 \\ s \end{pmatrix}.$$

with

z	bottom topography
h	water height
$w = z + h$	water level
$s = -gh\, z_x$	gravitational acceleration
u	horizontal velocity

Properties of hyperbolic balance laws

- conservative aspects:
  - wavespeeds $u \pm \sqrt{gh}$
  - shocks, weak solutions
- non-conservative aspects:
  - Leroux, Gosse, Seguin: Riemann-problem non-unique
  - Dal Maso, Murat, LeFloch: Path-conservative solutions
  - Pares, Castro, . . . : Path-conservative schemes
  - Near singularity x_*,

$$\int h(x)\,\partial_x z(x)\,dx = \bar{h}(x_*)\,[z]$$

Modeling Assumption determines

average $\bar{h}(x_*)$ and jump $[z]$

This is at the heart of the talk!

Semidiscrete finite volume scheme

Compute evolution of averages

$$U_i(t) \approx \frac{1}{\Delta x} \int_{C_i} U(x,t)\,dx$$

on cells $C_i = \left[x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}\right]$ by method of lines

$$\begin{aligned} \frac{d}{dt} U_i(t) &= R_i(t) \\ &:= -\frac{F_{i+\frac{1}{2}} - F_{i-\frac{1}{2}}}{\Delta x} + \left(S_{i-\frac{1}{2}+} + S_{i+\frac{1}{2}-}\right) \end{aligned}$$

with residuum R_i and

$$\begin{aligned} F_{i+\frac{1}{2}} &\approx F(U(x_{i+\frac{1}{2}}, t)) \\ S_{i-\frac{1}{2}+} &\approx S(U(x_{i-\frac{1}{2}+}, t), z(x_{i-\frac{1}{2}+})) \\ S_{i+\frac{1}{2}-} &\approx S(U(x_{i+\frac{1}{2}-}, t), z(x_{i+\frac{1}{2}-})). \end{aligned}$$

Reconstruction at interfaces

Conservative Step: Given reconstructions $h_{i+\frac{1}{2}\pm}$ and a Riemann solver $\mathcal{F}$, let

$$U_{i-\frac{1}{2}+} = \begin{pmatrix} h_{i-\frac{1}{2}+} \\ h_{i-\frac{1}{2}+} u_i \end{pmatrix}, \quad U_{i+\frac{1}{2}-} = \begin{pmatrix} h_{i+\frac{1}{2}-} \\ h_{i+\frac{1}{2}-} u_i \end{pmatrix}$$

and

$$F_{i+\frac{1}{2}} = \mathcal{F}(U_{i+\frac{1}{2}-}, U_{i+\frac{1}{2}+})$$

Non-Conservative Step: Given reconstruction $z_{i+\frac{1}{2}}$, define $s^{\pm}_{i+\frac{1}{2}}$.

Audusse et al. 2004

Intermediate bottom

$$z_{i+\frac{1}{2}} = z^{\max}_{i+\frac{1}{2}} := \max(z_i, z_{i+1}).$$

One-sided water heights:

$$h_{i+\frac{1}{2}-} = \max(w_i - z_{i+\frac{1}{2}}, 0), \quad h_{i+\frac{1}{2}+} = \max(w_{i+1} - z_{i+\frac{1}{2}}, 0)$$

Source terms:

$$\begin{aligned} \Delta x\, s_{i-\frac{1}{2}+} &= -\frac{g}{2}\left((h_{i-\frac{1}{2}+})^2 - (h_i)^2\right), \\ \Delta x\, s_{i+\frac{1}{2}-} &= -\frac{g}{2}\left((h_i)^2 - (h_{i+\frac{1}{2}-})^2\right). \end{aligned}$$

Morales et. al. 2013

As in Audusse et al., unless water climbs up the hill:

If $w_i < z_{i-1}$, $u_i < 0$, and

$$\frac{(u_i)^2}{2} + g(w_i - z_{i-1}) > \frac{3}{2}\sqrt{g(h_i u_i)^3},$$

then

$$\Delta x \, s_{i-\frac{1}{2}+} = -\frac{g}{2}\left(h_i + h_{i-\frac{1}{2}+}\right)\left(z_i - z_{i-\frac{1}{2}}\right).$$

Chen and Noelle

New bottom:

$$z_{i+\frac{1}{2}} = \min(z^{\max}_{i+\frac{1}{2}}, w^{\min}_{i+\frac{1}{2}})$$

with $w^{\min}_{i+\frac{1}{2}} = \min(w_i, w_{i+1})$.

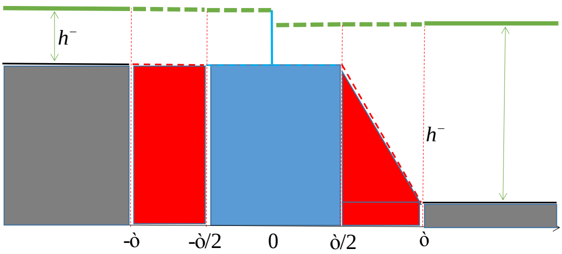
New water heights

$$h_{i+\frac{1}{2}-} = \min(w_i - z_{i+\frac{1}{2}}, h_i), \quad h_{i+\frac{1}{2}+} = \min(w_{i+1} - z_{i+\frac{1}{2}}, h_{i+1}).$$

Source term in natural form:

$$\Delta x\, s_{i-\frac{1}{2}+} = -\frac{g}{2}\left(h_i + h_{i-\frac{1}{2}+}\right)\left(z_i - z_{i-\frac{1}{2}}\right),$$
$$\Delta x\, s_{i+\frac{1}{2}-} = -\frac{g}{2}\left(h_i + h_{i+\frac{1}{2}-}\right)\left(z_{i+\frac{1}{2}} - z_i\right).$$

Preview of subcell reconstruction



- Infinitesimal subcells near $x_{i+\frac{1}{2}}$
- topography and water surface

The subcells

Wlog $x_{i+\frac{1}{2}} = 0$, so

$$C_i \cup C_{i+1} = [-\Delta x, 0] \cup [0, \Delta x].$$

Definition (infinitesimal boundary subcells)

For $\varepsilon \ll \Delta x$ let

$$\begin{array}{ccccccccc} [-\varepsilon, \varepsilon] & = & [-\varepsilon, -\varepsilon/2] & \cup & [-\varepsilon/2, 0] & \cup & [0, \varepsilon/2] & \cup & [\varepsilon/2, \varepsilon] \\ & =: & \widehat{C}_{i+\frac{1}{2}-} & \cup & \widetilde{C}_{i+\frac{1}{2}-} & \cup & \widetilde{C}_{i+\frac{1}{2}+} & \cup & \widehat{C}_{i+\frac{1}{2}+} \end{array}$$

with

singular subcells $\widehat{C}_{i+\frac{1}{2}-}$ *and* $\widehat{C}_{i+\frac{1}{2}+}$

conservative subcells $\widetilde{C}_{i+\frac{1}{2}-}$ *and* $\widetilde{C}_{i+\frac{1}{2}+}$

Piecewise linear subcell reconstruction

Hydrostatic reconstruction /surface gradient method

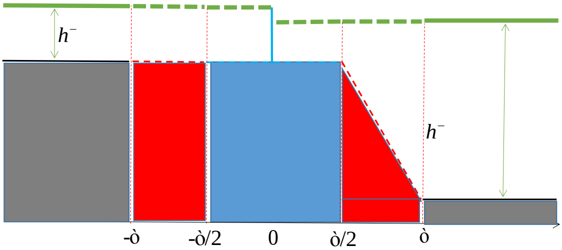
Step 1: bottom $z_\varepsilon(x)$

Step 2: watersurface $w_\varepsilon(x)$

Step 3: waterheight

$$h_\varepsilon(x) := w_\varepsilon(x) - z_\varepsilon(x)$$

Fully wet case (same for all schemes)



- Infinitesimal subcells near $x_{i+\frac{1}{2}}$
- topography and water surface
- singularities separated

HR scheme based on subcell reconstructions

Definition

The semidiscrete HR method based on subcell reconstructions is given by

$$\Delta x \frac{d}{dt} U_i(t) = \Delta x\, R_i(t)$$
$$= \left(F_{i-\frac{1}{2}} - F_{i+\frac{1}{2}}\right) - g\left(\bar{h}\,[z]\right)_{i-\frac{1}{2}+} - g\left(\bar{h}\,[z]\right)_{i+\frac{1}{2}-}$$

with

$$\bar{h}_{i-\frac{1}{2}+} := \lim_{\varepsilon\to 0}\left(\frac{1}{\varepsilon/2}\int_{\widehat{C}_{i-\frac{1}{2}+}} h_\varepsilon(x)\,dx\right), \quad [z]_{i-\frac{1}{2}+} := z_i - z_{i-\frac{1}{2}}$$
$$\bar{h}_{i-\frac{1}{2}+} := \lim_{\varepsilon\to 0}\left(\frac{1}{\varepsilon/2}\int_{\widehat{C}_{i+\frac{1}{2}-}} h_\varepsilon(x)\,dx\right), \quad [z]_{i+\frac{1}{2}-} := z_{i+\frac{1}{2}} - z_i$$

This answers a key question of the talk

Main Theorem

Theorem (Chen, Noelle 2015)

All three original HR schemes coincide with the corresponding new HR schemes based on suitable subcell reconstructions given below.

Key difficulties:

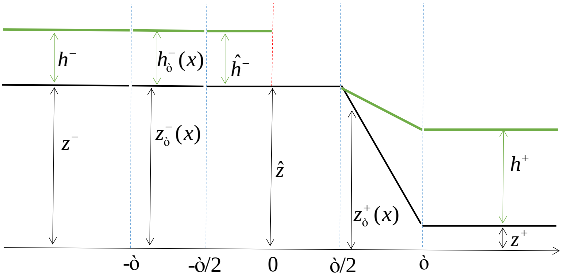
- the almost dry case

$$z_i > w_{i+1} > z_{i+1}$$

- the wet-dry front

$$z_i > w_{i+1} > z_{i+1} \quad \text{and} \quad h_i = 0$$

Almost dry interface: Audusse et al.

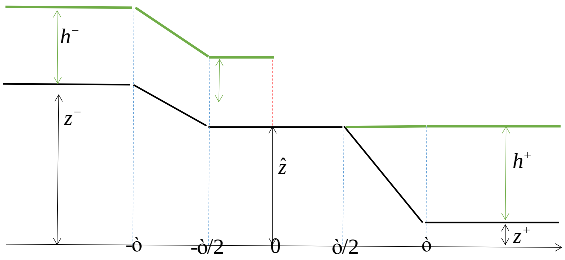


$z_{i+\frac{1}{2}} = z_i$, and

$$\Delta x \left(R_i^{(2)} + R_{i+1}^{(2)} \right)^{\text{Aud}}$$

$$= - \left(F_{i+1}^{(2)} - F_i^{(2)} \right) - \frac{g\, h_{i+1}}{2} \left(z_{i+1} - z_i \right)$$

Almost dry interface: Chen/Noelle

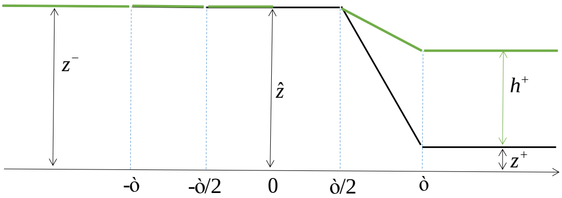


$z_{i+\frac{1}{2}} = w_{i+1}$, and

$$\Delta x \left(R_i^{(2)} + R_{i+1}^{(2)}\right)^{\text{CN}}$$

$$= -\left(F_{i+1}^{(2)} - F_i^{(2)}\right) - gh_i \left(w_{i+1} - z_i\right) + \frac{g\,h_{i+1}}{2}\,h_{i+1}$$

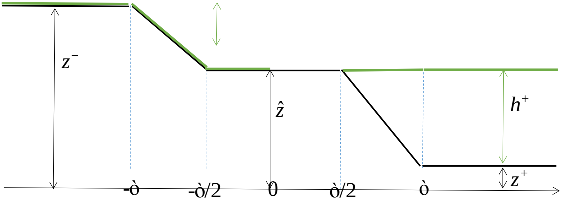
Wet-dry front: Audusse



$$\Delta x\left(R_i^{(2)} + R_{i+1}^{(2)}\right)^{\text{Aud}} = -\left(hu^2\right)_{i+1} - \frac{gh_{i+1}}{2}\left(w_{i+1} - w_i\right)$$

additional downhill acceleration

Wet-dry front: Chen/Noelle



$$\Delta x \left(R_i^{(2)} + R_{i+1}^{(2)}\right)^{\mathrm{CN}} = -\left(hu^2 + \frac{g}{2}h^2\right)_{i+1} - \frac{g\,h_{i+1}}{2}\left(z_{i+1} - w_{i+1}\right)$$
$$= -\left(hu^2\right)_{i+1}$$

Acceleration only by outflow!

Comments on the wet-dry front

still water:

$$u = 0 \quad \text{and} \quad \partial_x w = 0$$

lake at rest:

$$u = 0 \quad \text{and} \quad h\, \partial_x w = 0$$

Typical example: wet-dry interface

$$\begin{aligned} u &= 0 \quad \text{in} \quad C_i \cup C_{i+1} \\ h &= 0 \quad \text{in} \quad C_i \\ \partial_x w &= 0 \quad \text{in} \quad C_{i+1} \end{aligned}$$

Well-balancing

Theorem (Audusse et al. 2004)

The HR^{Aud} scheme guarantees

- *positivity*
- *still water equilibrium*
- *a semi-discrete entropy inequality*

Theorem (Chen/Noelle 2015)

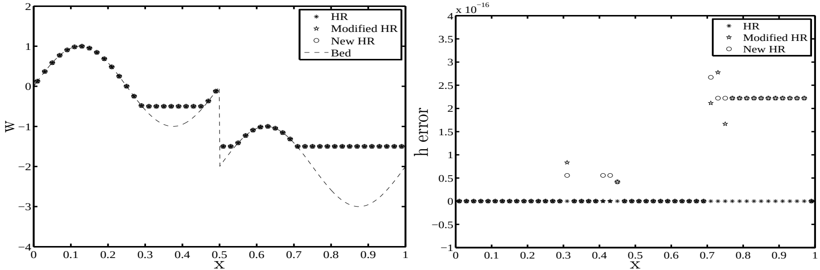
The HR^{Aud} scheme guarantees

- *positivity*
- *lake at rest equilibrium*

Conjecture

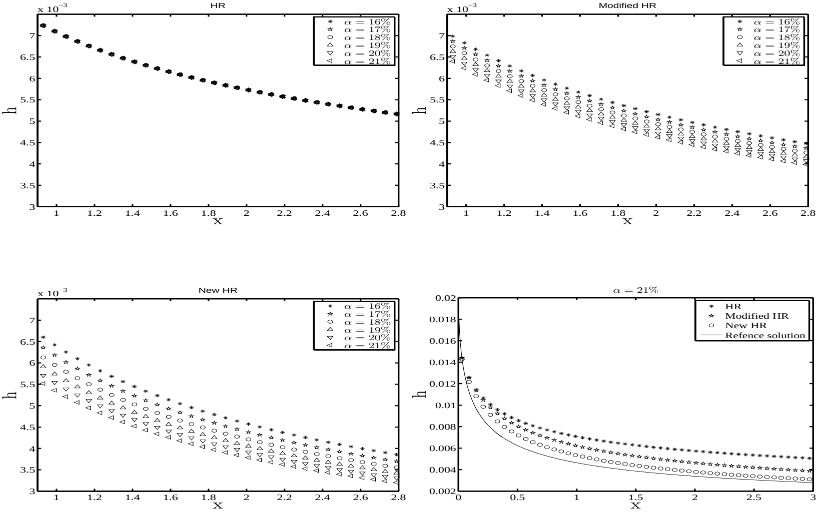
The HR^{CN} scheme satisfies a semi-discrete entropy inequality

Lakes at rest in the mountains



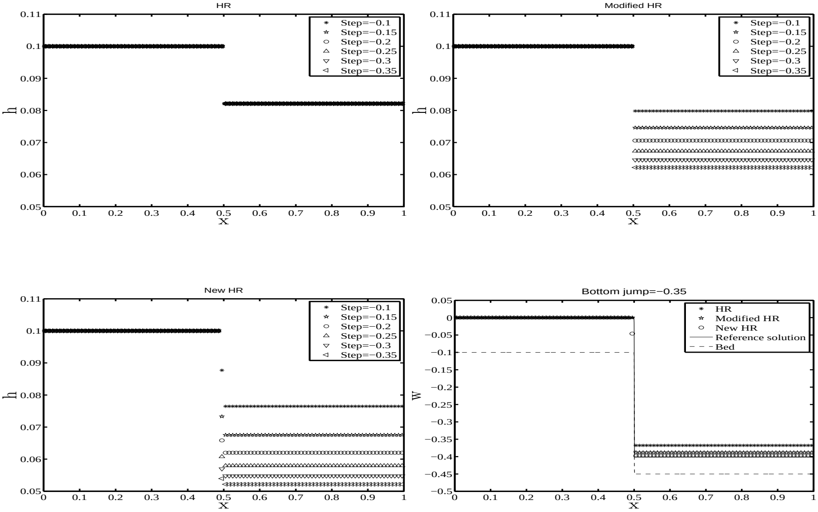
Left: surface w; Right: error of water hight h. (50 cells).

Sequence of linear downhill flows



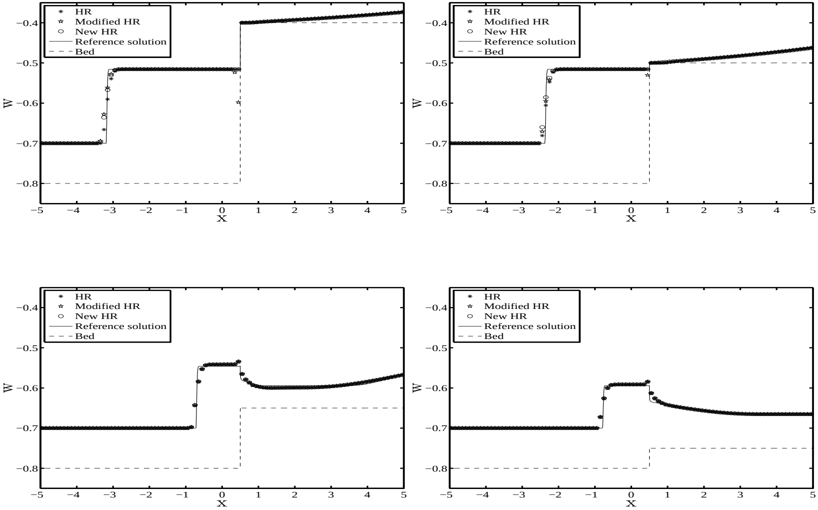
Linear downhill flows (16, 17, 18, 19, 20, 21%). 50 cells

Sequence of dambreaks over various step



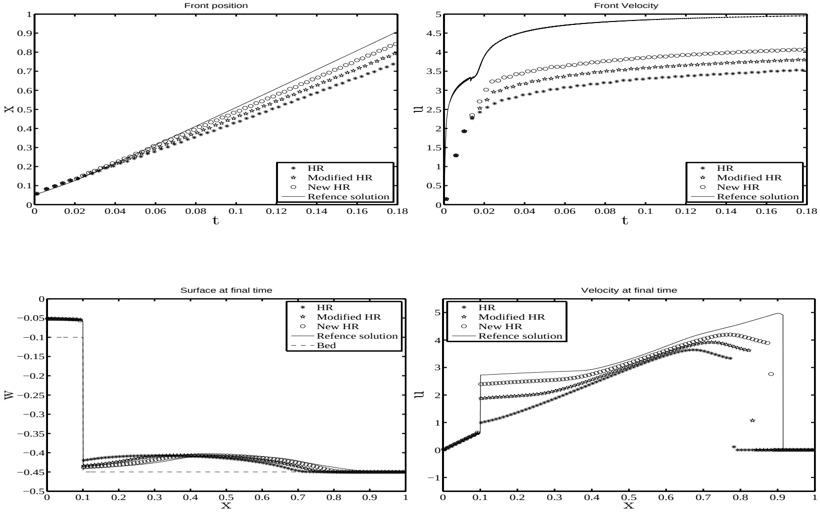
Downhill step sizes 0.1, 0.15, 0.2, 0.25, 0.3, 0.35 (100 cells)

Sequence of flows up a step



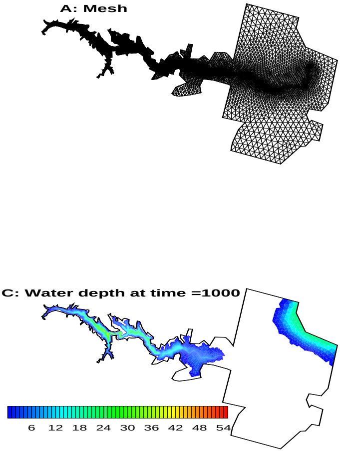
Uphill step sizes 0.4, 0.3, 0.15, 0.05 (100 cells)

Dam-break problem over discontinuous dry bottom



Front position and velocity, and final solution

Malpasset dam-break event



Mesh: 2600 elements and 13541 points;

Malpasset dam-break event

Flood arrival times at three electric transformers.

trans formers	x	y	Meas	HR	Modified HR	New HR
A	5550	4400	100	124	128	125
B	11900	3250	1240	1317	1310	1313
C	13000	2700	1420	1431	1423	1425