JOHANNES GUTENBERG UNIVERSITÄT MAINZ

Well-balanced asymptotic preserving methods for the Euler equations with the gravity

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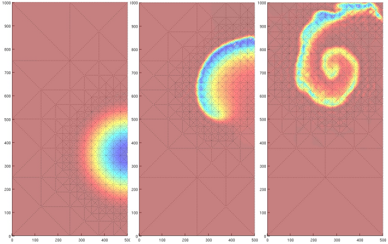
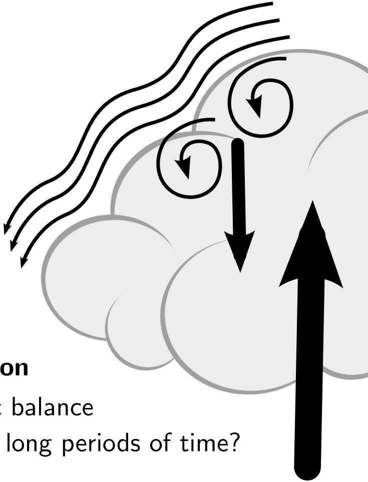
Application





Meteorology: Cloud Simulation

Gravity induces hydrostatic balance
How do clouds evolve over long periods of time?

Multiscale phenomena of oceanographical, atmospherical flows

-wave speeds differ by several orders: $\|\mathbf{u}\| << c \Rightarrow \mathbf{M}, \mathbf{Fr} := \frac{\|\mathbf{u}\|}{\mathbf{c}} << \mathbf{1}$

-typically $\mathbf{M} \approx \mathbf{10^{-2}}$

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$$\frac{\max(|u| + c, |v| + c)\Delta t}{\Delta x} \leq 1$$

$$\max\left(\left(1 + \frac{1}{\mathbf{M}}\right)\ \sqrt{u^2 + v^2}\right)\frac{\Delta t}{\Delta x} \leq 1$$

- number of time steps $\mathcal{O}(1/\mathbf{M})$

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- number of time steps $\mathcal{O}(1/\mathbf{M})$

-low Mach / low Froude number problem

[Bijl & Wesseling ('98), Klein et al.('95, '01), Meister ('99,01), Munz &Park ('05), Degond, S. Jin, J.-G. Liu ('07), Degond, Tang ('11), Haack, S. Jin, J.-G. Liu ('12), Bispen, Arun, ML, Noelle ('14), Noelle, Bispen, Arun, ML, Munz ('14), . . .]

Cancelation problem

- Sesterhenn et al. ('99)

- "pressure term" $\frac{1}{2\mathbf{M^2}}\nabla h^2 \Longrightarrow$

but round off errors can yield the cancelation effects

leading order error in the pressure term $\approx \frac{1}{\mathbf{M^2}}\epsilon_1 \mathcal{O}(\frac{1}{\mathbf{M^2}}) \approx \mathcal{O}(1)\,!$

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Accuracy problem

- numerical viscosity of upwind methods depends on $\mathbf{M}$
- truncation error grows as $\mathbf{M} \to \mathbf{0}$ [Guillard, Viozat ('99), Rieper ('10)]

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- **AIM:**
  - large time step scheme: Δt does not depends on $\mathbf{M}$
  - efficient scheme for advection effects
  - stability and accuracy of the scheme is independent on $\mathbf{M} \longrightarrow$ AP

Asymptotic preserving schemes

S.Jin&Pareschi('01), Gosse&Toscani('02), Degond et al.('11), ...

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Goal 1: *Derive a scheme, which gives a consistent approximation of the limiting equations for any* $\varepsilon = \mathbf{M} \to \mathbf{0}$

Goal 2: *Derive a scheme, which is uniformly accurate with respect to* $\varepsilon = \mathbf{M}$*, if* $\varepsilon = \mathbf{M} \to \mathbf{0}$

Euler equations with gravity

Compressible Euler equations

$$\begin{aligned} \partial_t \rho' \quad &+ \nabla \cdot (\rho \mathbf{u}) &&= 0 \\ \partial_t (\rho \mathbf{u}) &+ \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u} + p' \, \mathrm{Id}) &&= -\rho' g \mathbf{k} \equiv -\rho' g \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} \\ \partial_t (\rho\theta)' &+ \nabla \cdot (\rho\theta \mathbf{u}) &&= 0 \end{aligned}$$

with background state $\bar{p}$, $\bar{\rho}$, $\bar{\theta}$ in **hydrostatic balance**

$$\boxed{\partial_z \bar{p} = -\bar{\rho} g}$$

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State variables: $\mathbf{w} = [\rho', \rho u_1, \rho u_2, \rho u_3, (\rho\theta)']^{\mathsf{T}}$

State equation

$$\boxed{p = p_0 \left(\frac{R\rho\theta}{p_0} \right)^{\gamma}}$$

- Potential temperature $\theta := T/\pi$
- *Exner-pressure* $\pi(z) := 1 - \dfrac{gz}{c_p \theta}$

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Non-dimensional formulation in $(\rho', \mathbf{q} \equiv (\rho u_1, \rho u_2, \rho u_3), (\rho\theta)')$

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- following Klein we consider $\gamma M^2 = Fr^2$; and set $\varepsilon := \sqrt{\gamma} M$

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Asymptotic analysis as $\varepsilon \to 0$

- Hilbert expansion

$$f = (f^{(0)} + \varepsilon f^{(1)}) + \varepsilon^2 f^{(2)} = \bar{f} + f'$$

$\Rightarrow$ plug into the Euler eqs. in $(\rho, \rho\mathbf{u}, \rho\theta)$

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- momentum eqs. $\Longrightarrow$

$$p_z^{(0)} = -\rho^{(0)}, \quad p_z^{(1)} = -\rho^{(1)} \tag{1}$$

- due to the state eq. $\Longrightarrow \rho^{(0)} = \rho^{(0)}(t,z)$, $\theta^{(0)} = \theta^{(0)}(t,z)$,
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- non-dimensional state eq.

$$p = (\rho\theta)^\gamma \tag{2}$$

$\Longrightarrow$

$$(\rho\theta^{(i)})^{\gamma-1}(\rho\theta^{(i)})_z = -\frac{1}{\gamma}\rho^{(i)} \tag{3}$$

for $i = 0, 1$

Limiting system as $\varepsilon \to 0$

If $\theta^{(0)}, \ \theta^{(1)} =$ const., then the leading order terms $i = 0, 1$

$$\rho^{(i)} = \frac{1}{\theta^{(i)}} \left(1 - \frac{(\gamma - 1) z}{\gamma \theta^{(i)}} \right)^{\frac{1}{\gamma - 1}} \cong \frac{p_0}{R \bar{\theta}} \pi^{c_v / R} \quad \pi(z) = 1 - \frac{g z}{c_p \theta} \text{ Exner pressure}$$

$\rho^{(0)} + \varepsilon \rho^{(1)} \ldots$ constant in time

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$\rho^{(0)} + \varepsilon\rho^{(1)}$ ... constant in time
$\Longrightarrow$ the limiting system $\varepsilon \to 0$

$$\begin{aligned} \nabla\cdot(\rho^{(0)}\mathbf{u}^{(0)}) &= 0 \\ (\rho^{(0)}\mathbf{u}^{(0)})_t + \nabla\cdot\left(\rho^{(0)}\mathbf{u}^{(0)}\otimes\mathbf{u}^{(0)}\right) + \nabla p^{(2)} &= -\rho^{(2)}\mathbf{k} \end{aligned} \qquad (4)$$

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Does a numerical scheme give a consistent and order preserving approximation of (4) ?

Time discretization

Key idea:

- semi-implicit time discretization: splitting into the linear and nonlinear part
- linear operator models gravitational and acoustic waves are treated implicitly
- rest nonlinear terms are treated explicitly

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$$\frac{\partial \mathbf{w}}{\partial t} = -\nabla \cdot \mathbf{F}(\mathbf{w}) + \mathbf{S}(\mathbf{w}) \equiv \mathcal{L}(\mathbf{w}) + \mathcal{N}(\mathbf{w})$$

$$\mathbf{w} = (\rho', \rho\mathbf{u}, \rho\theta')^T,$$

[Restelli ('07), Giraldo & Restelli ('10)]
[Yelash, Müller, M.L., Giraldo, Wirth (JCP'14)] ... Euler eq. DG-method
[Bispen, Arun, M.L., Noelle (CiCP'14)] ... shallow water eq.

Linear/Non-linear splitting

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\partial_t \rho' \quad &+ \ \nabla \cdot (\rho \mathbf{u}) + \nabla \cdot \mathbf{0} &&= \ 0 \\
\partial_t (\rho \mathbf{u}) \ &+ \ \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u} + \frac{1}{\varepsilon^2} p' \, \mathrm{Id}) &&= \ -\frac{1}{\varepsilon^2} \rho' \mathbf{k} \\
\partial_t (\rho \theta)' \ &+ \ \nabla \cdot (\bar{\theta} \rho \mathbf{u}) + \nabla \cdot (\theta' \rho \mathbf{u}) &&= \ 0
\end{aligned}$$

linear part for the Euler system

$$\partial_t \mathbf{w} = \mathcal{L}(\mathbf{w})$$

$$\mathbf{w} := \begin{pmatrix} \rho' \\ \rho u \\ \rho v \\ \rho w \\ (\rho\theta)' \end{pmatrix} \qquad \mathcal{L}(\mathbf{w}) := - \begin{pmatrix} \mathrm{div}(\rho \mathbf{u}) \\ \frac{1}{\varepsilon^2} \partial p' / \partial x \\ \frac{1}{\varepsilon^2} \partial p' / \partial y \\ \frac{1}{\varepsilon^2} \partial p' / \partial z + \frac{1}{\varepsilon^2} \rho' \\ \mathrm{div}(\overline{\theta} \rho \mathbf{u}) \end{pmatrix}$$

- linearized version of p': $p' = \frac{c_p \overline{p}}{c_v \overline{\rho} \overline{\theta}} (\rho\theta)'$

[Yelash, Müller, M.L., Giraldo, Wirth (JCP'14)]

Properties of the linear subsystem

$$\mathbf{w}'_t + \nabla \cdot \mathcal{F}_L(\mathbf{w}', \bar{\mathbf{w}}) = S(\mathbf{w}'), \quad \mathcal{F}_L = \begin{bmatrix} \mathbf{q}^T \\ \frac{\bar{c}^2}{\bar{\theta}\varepsilon^2}(\rho\theta)'\mathbb{1} \\ \mathbf{q}^T\bar{\theta} \end{bmatrix}, \quad S = \begin{bmatrix} 0 \\ -\frac{\rho'}{\varepsilon^2}\mathbf{k} \\ 0 \end{bmatrix} \tag{5}$$

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- matrix pencil:

$$\mathbb{P}(\mathbf{w}', \mathbf{n}) = \sum_{i=1}^{3} \frac{d\mathcal{F}_{L_i}}{d\mathbf{w}'} n_i = \begin{bmatrix} 0 & \mathbf{n}^T & 0 \\ 0 & 0 & \frac{\bar{c}^2}{\bar{\theta}\varepsilon^2}\mathbf{n} \\ 0 & \bar{\theta}\mathbf{n}^T & 0 \end{bmatrix}$$

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- eigenvalues:

$$\lambda_1 = -\frac{\bar{c}}{\varepsilon}, \quad \lambda_2 = \lambda_3 = \lambda_4 = 0, \quad \lambda_5 = \frac{\bar{c}}{\varepsilon}$$

$\Longrightarrow$ **stiff linear subsystem**

Properties of the non-linear subsystem

$$\mathbf{w}'_t + \nabla \cdot \mathcal{F}_{NL}(\mathbf{w}', \bar{\mathbf{w}}) = 0, \quad \mathcal{F}_{NL} = \begin{bmatrix} 0 \\ \frac{\mathbf{q}\mathbf{q}^T}{\bar{\rho}+\rho'} \\ \mathbf{q}^T \frac{(\rho\theta)'}{\bar{\rho}+\rho'} \end{bmatrix} \tag{6}$$

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- eigenvalues:

$$\lambda_1 = 0, \quad \lambda_2 = \lambda_3 = \lambda_4 = \mathbf{u}\cdot\mathbf{n}, \quad \lambda_5 = 2\mathbf{u}\cdot\mathbf{n}$$

$\Longrightarrow$ **non-stiff nonlinear subsystem**

Semi-implicit time discretization

- first order IMEX scheme

$$
\begin{aligned}
\mathbf{w}^{n+1} &= \mathbf{w}^n + \Delta t \mathcal{L}(\mathbf{w}^{n+1}) + \Delta t \mathcal{N}(\mathbf{w}^n) \\
&= \mathbf{w}^n - \Delta t \nabla \cdot \mathcal{F}_L(\mathbf{w}^{n+1}) + \Delta t S(\mathbf{w}^{n+1}) - \Delta t \nabla \cdot \mathcal{F}_{NL}(\mathbf{w}^n)
\end{aligned}
$$

Higher time discretization

- **RK2CN scheme:** (midpoint for the nonlinear & trap rule for the linear part)

$$\mathbf{w}^{n+\frac{1}{2}} = \mathbf{w}^n - \frac{\Delta t}{2}\nabla \cdot \mathcal{F}_{NL}(\mathbf{w}^n) - \frac{\Delta t}{2}\nabla \cdot \mathcal{F}_L(\mathbf{w}^{n+\frac{1}{2}}) + \frac{\Delta t}{2} S(\mathbf{w}^{n+\frac{1}{2}})$$

$$\mathbf{w}^{n+1} = \mathbf{w}^n - \Delta t \nabla \cdot \mathcal{F}_{NL}(\mathbf{w}^{n+\frac{1}{2}}) - \frac{\Delta t}{2}\nabla \cdot \left[\mathcal{F}_L(\mathbf{w}^n) + \mathcal{F}_L(\mathbf{w}^{n+1})\right] + \frac{\Delta t}{2}\left[S(\mathbf{w}^n) + S(\mathbf{w}^{n+1})\right],$$

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$$\mathbf{w}^{n+1} = \mathbf{w}^n - \Delta t\nabla\cdot\mathcal{F}_{NL}(\mathbf{w}^{n+\frac{1}{2}}) - \frac{\Delta t}{2}\nabla\cdot\left[\mathcal{F}_{L}(\mathbf{w}^n) + \mathcal{F}_{L}(\mathbf{w}^{n+1})\right] + \frac{\Delta t}{2}\left[S(\mathbf{w}^n) + S(\mathbf{w}^{n+1})\right],$$

- **ARS (2,2,2) scheme:** (Boscarino, Pareschi, Russo)

$$\mathbf{w}_1 = \mathbf{w}^n$$

$$\mathbf{w}_2 + \Delta t\gamma[\nabla\cdot\mathcal{F}_L(\mathbf{w}_2) + S(\mathbf{w}_2)] = \Delta t\gamma\nabla\cdot\mathcal{F}_{NL}(\mathbf{w}^n)$$

$$\mathbf{w}^{n+1} + \Delta t\gamma[\nabla\cdot\mathcal{F}_L(\mathbf{w}^{n+1}) + S(\mathbf{w}^{n+1})] = \mathbf{w}^n - \Delta t\gamma[\delta\nabla\cdot\mathcal{F}_{NL}(\mathbf{w}^n) + (1-\delta)\nabla\cdot\mathcal{F}_{NL}(\mathbf{w}_2)] - \Delta t\left[(1-\gamma)\left(\nabla\cdot\mathcal{F}_L(\mathbf{w}_2) + S(\mathbf{w}_2)\right)\right],$$

$\gamma = 1 - 1/\sqrt{2}$; $\delta = 1 - 1/(2\gamma)$

Higher order time discretization

- BDF2 scheme:

$$\begin{aligned}\mathbf{w}^{n+1} &= \alpha_0 \mathbf{w}^n + \alpha_1 \mathbf{w}^{n-1} \\ &+ \nabla \cdot \Big\{ \beta \mathcal{F}_L(\mathbf{w}^{n+1}) + \beta_0 \mathcal{F}_{NL}(\mathbf{w}^n) + \beta_1 \mathcal{F}_{NL}(\mathbf{w}^{n-1}) \Big\} \\ &- \beta S(\mathbf{w}^{n+1})\end{aligned}$$

$$\alpha_0 = \frac{4}{3};\ \alpha_1 = -\frac{1}{3};\ \beta = -\frac{2}{3}\Delta t;\ \beta_0 = -\frac{4}{3}\Delta t;\ \beta_1 = \frac{2}{3}\Delta t$$

Spatial discretization

Finite Volume update

$$\mathcal{L}(\mathbf{w}^\ell) = -\sum_{k=1}^{2} \frac{1}{\Delta x_k} \delta_{x_k} \mathcal{F}_{L_k}(\mathbf{w}^\ell) + S(\mathbf{w}^\ell), \quad \ell = n, n+1$$

$$\mathcal{N}(\mathbf{w}^\ell) = -\sum_{k=1}^{2} \frac{1}{\Delta x_k} \delta_{x_k} \mathcal{F}_{NL_k}(\mathbf{w}^\ell), \qquad \ell = n, n-1, n+1/2$$

$$\delta_x f_i \equiv f_{i+1/2} - f_{i-1/2}$$

$\mathcal{F}_L$:

• Multi-d evolution operator in [Arun, M.L., Kraft, Prasad (2009)] ... multi-d Riemann solver, see [Yelash, Müller, M.L., Giraldo, Wirth (2014)]

• Central finite-difference scheme (CFD)

$\mathcal{F}_{NL}$:

• evolution along the characteristics

• approximate Riemann solver (e.g. Rusanov, van Leer, etc.)

• comparisons for the shallow water eqs. in [Bispen, Arun, M.L, Noelle (2014)]

AP analysis of the IMEX finite volume scheme

- Consider: CFD-implicitly and Rusanov-flux explicitly, first order IMEX, periodic BC

$$(\mathbb{1} + \Delta t A)\mathbf{W}^{n+1} = RHS(\hat{\mathbf{W}}), \text{ e.g.} \hat{\mathbf{W}} = \mathbf{W}^n$$

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AP analysis of the IMEX finite volume scheme

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- Gauss elimination (Schur complement)

$$(\mathbb{1} + \frac{\Delta t^2}{\varepsilon^2}B)\begin{pmatrix} \rho' \\ (\rho\theta)' \end{pmatrix} = \begin{pmatrix} \hat{\rho}' - \Delta t \nabla_h \cdot \hat{\mathbf{Q}} \\ (\hat{\rho\theta})' - \Delta t \nabla_h \cdot (\bar{\Theta}\hat{\mathbf{Q}}) \end{pmatrix}$$

$$B = \begin{bmatrix} -\mathfrak{D}_{x_3} & -\Delta_h \bar{C}^2\bar{\Theta}^{-1} \\ -\mathfrak{D}_{x_3}\bar{\Theta} & -\nabla_h \cdot ((\mathbf{1}_3 \otimes \bar{\Theta}) * \nabla_h \bar{C}^2\bar{\Theta}^{-1}) \end{bmatrix}$$

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$$E := -(\Delta_h \bar{C}^2 + \mathfrak{D}_{x_3})$$

AP analysis of the IMEX finite volume scheme

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3 $E^2\mathbf{x} = 0 \ \Rightarrow E\mathbf{x} = 0$ thus algebraic and geometric multiplicity of 0 equals $\Rightarrow$

$$\left(\mathbb{1} + \frac{\Delta t^2}{\varepsilon^2} E\right)^{-1} = \pi_{\mathcal{N}(E)} + \mathcal{O}(\varepsilon^2)$$

4 $\mathcal{R}(E) = \mathcal{N}(E)^{\perp} = \mathcal{N}(\nabla_h)^{\perp} = \mathcal{R}(\nabla_h^T) = \mathcal{R}(\nabla_h \cdot) \Rightarrow$

$$((\rho\theta)')^{n+1} - \hat{\rho}\hat{\theta}', (\rho')^{n+1} - \hat{\rho}', \nabla_h \cdot \mathbf{Q}^{n+1} = \mathcal{O}(\varepsilon^2)$$

AP ✓

Well-Balancing

- Consider: Lineal/Non-lineal splitting, IMEX with CFD (implicit) and Rusanov (expl.)
- Assume we start from the hydrostatic equilibrium, i.e. $\rho' = 0 = (\rho\theta)'$, $\mathbf{u} = 0$.
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$\Rightarrow$ it follows from the construction:

Well-balancing cont....

$$(\mathbb{1} + \Delta t A)\mathbf{W}^{n+1} = RHS(\mathbf{W}^n) \equiv \mathbf{0},$$

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$\mathbb{1} + \Delta t A$ is non-singular $\Rightarrow \mathbf{W}^{n+1} = \mathbf{0}$
it preserves this hydrostatic equilibrium WB ✓

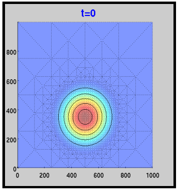
Numerical experiments

Test 1: rising warm air bubble

- bubble with a cosine profile in $\theta = \overline{\theta} + \theta'$:

$$\theta' = \begin{cases} 0 & r > r_C, \; r = \|\mathbf{x} - \mathbf{x}_C\| \\ 0.25[1 + \cos(\pi_c r / r_C)] & r \le r_C \end{cases}$$

$$\mathbf{x}_C = (500, 350), \; r_C = 250m, \; \overline{\theta} = 300K,$$
$$\mathbf{x} \in [0, 1000]^2, \; t \in [0, 700]$$



AP Property for the Euler eqs.: $\|\nabla_h \cdot \mathbf{Q}^\varepsilon\| = \mathcal{O}(\varepsilon^2)$

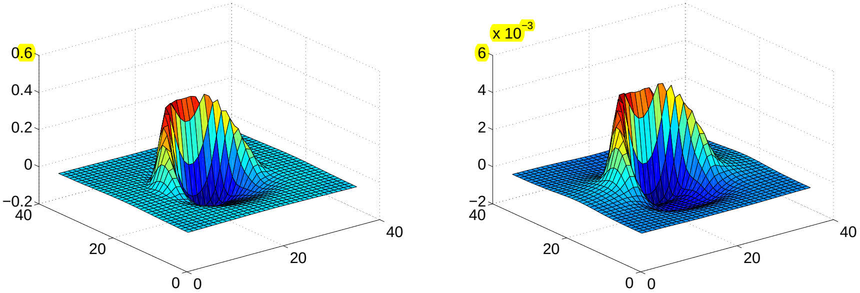


Figure: Warm bubble test; $\epsilon = 0.01,\ 0.001$, 40×40 grid.

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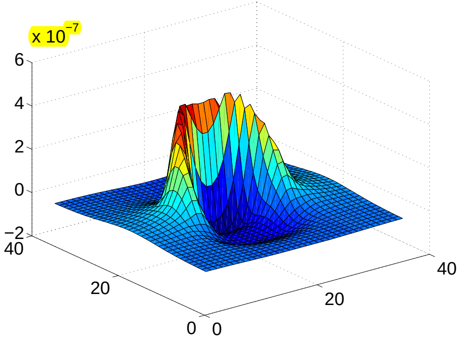

Figure: Warm bubble test; $\epsilon = 10^{-5}$, 40×40 grid.

Experimental order of convergence for the IMEX ARS(2,2,2)

$\varepsilon = 10^{-3}$

N	L^1-error ρ'	EOC ρ'	L^1-error m	EOC m	L^1-error $(\rho\theta)'$	EOC $(\rho\theta)'$
20	1.756e-07		5.287e-03		1.214e-07	
40	5.783e-08	1.6021	1.548e-03	1.7725	2.812e-08	2.1097
80	1.802e-08	1.6823	4.602e-04	1.7496	1.177e-08	1.2570
160	9.745e-09	0.8867	1.384e-04	1.7339	7.039e-09	0.7415

$\varepsilon = 10^{-5}$

N	L^1-error ρ'	EOC ρ'	L^1-error m	EOC m	L^1-error $(\rho\theta)'$	EOC $(\rho\theta)'$
20	5.261e-07		2.676e-02		5.261e-07	
40	9.550e-11	12.4275	1.547e-03	4.1126	9.548e-11	12.4278
80	1.860e-12	5.6823	4.600e-04	1.7494	1.102e-12	6.4373
160	1.287e-12	0.5309	1.374e-04	1.7428	1.131e-12	-0.0381

Second order BDF scheme

SWE travelling vortex, $\varepsilon \in \{0.8, 10^{-3}, 10^{-5}\}, CFL_u = 0.3, T = 0.1$

N	L^1-error in z	EOC z	L^1-error in m_1	EOC m_1	L^1-error in m_2	EOC m_2
20	5.035e-02	1.9389	0.10805	1.6333	0.22831	1.5043
40	1.257e-02	2.0020	0.02783	1.9570	0.05553	2.0398
80	3.244e-03	1.9542	0.00722	1.9462	0.01411	1.9765
160	8.148e-04	1.9932	0.00186	1.9561	0.00371	1.9281
N	L^1-error in z	EOC z	L^1-error in m_1	EOC m_1	L^1-error in m_2	EOC m_2
20	9.355e-08	2.1659	0.11412	1.6103	0.23531	1.5518
40	1.643e-08	2.5098	0.02777	2.0388	0.05563	2.0805
80	3.109e-09	2.4016	0.00720	1.9485	0.01435	1.9550
160	7.819e-10	1.9911	0.00187	1.9449	0.00379	1.9200
N	L^1-error in z	EOC z	L^1-error in m_1	EOC m_1	L^1-error in m_2	EOC m_2
20	9.355e-12	2.1661	0.11412	1.6103	0.23531	1.5518
40	1.642e-12	2.5099	0.02777	2.0388	0.05563	2.0805
80	3.105e-13	2.4030	0.00720	1.9485	0.01435	1.9550
160	7.811e-14	1.9912	0.00187	1.9449	0.00379	1.9200

Table: EOC of the second order BDF CFD/Rusanov scheme (elliptic approach)

Second order ARS(2,2,2) scheme

SWE travelling vortex, $\varepsilon \in \{0.8, 10^{-3}, 10^{-5}\}, CFL_u = 0.45, T = 0.1$

N	L^1-error in z	EOC z	L^1-error in m_1	EOC m_1	L^1-error in m_2	EOC m_2
20	4.885e-02	1.9812	0.10303	1.7016	0.21995	1.5502
40	1.243e-02	1.9743	0.02671	1.9476	0.05282	2.0580
80	3.231e-03	1.9442	0.00688	1.9567	0.01307	2.0152
160	8.153e-04	1.9866	0.00177	1.9588	0.00340	1.9429

N	L^1-error in z	EOC z	L^1-error in m_1	EOC m_1	L^1-error in m_2	EOC m_2
20	1.120e-07	2.1192	0.10957	1.6468	0.22691	1.5922
40	2.223e-08	2.3330	0.02658	2.0436	0.05295	2.0995
80	6.223e-09	1.8369	0.00685	1.9560	0.01332	1.9911
160	2.506e-09	1.3123	0.00178	1.9480	0.00349	1.9340

N	L^1-error in z	EOC z	L^1-error in m_1	EOC m_1	L^1-error in m_2	EOC m_2
20	1.126e-11	2.1139	0.10957	1.6467	0.22691	1.5922
40	1.785e-12	2.6572	0.02658	2.0435	0.05295	2.0995
80	5.225e-13	1.7721	0.00685	1.9560	0.01332	1.9911
160	3.159e-13	0.7258	0.00178	1.9479	0.00349	1.9341

Table: EOC of the second order ARS (2,2,2,) CFD/Rusanov scheme (elliptic approach)