New error estimates for balance laws with position-dependent sources and applications

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Partially joint with D. Amadori (L'Aquila) & N. Vauchelet (LJLL, Paris)

Plan of the talk

1. New error estimates by Lyapunov functional
   - Outline of the computation
   - Scalar equation: Lyapunov vs. Kuznetsov
   - A 2-velocity relaxation kinetic model
2. Kinetic models of $(1+1)$-dim. chemotaxis dynamics
   - Hydrodynamic limit of 2-velocity model
   - Continuous velocities $v \in (-1, 1)$
3. A “baby model” of relativistic self-gravitating fluids
   - The (Nordstrom) gravitational field equations
   - Coupled field-matter hydrodynamics
   - Locally inertial “WB” numerical discretization

Systems of Balance Laws in 1d

$$\partial_t u + \partial_x f(u) = k(x)g(u), \qquad x \in \mathbb{R}, t > 0.$$

Different large-time behavior according to $k \geq 0$ in unbounded domain $\mathbb{R}$:

- $k \equiv Cst$, source active (or dissipative) everywhere $\rightarrow$ traveling waves
- $k \in L^p(\mathbb{R})$, scattering state with a stationary soln close to zero.

Aim:

- **Accuracy of "Well-Balanced" approximations**, that preserve stationary solutions (prototype: 1D shallow water with topography)
- Theoretical method to prove rigorous L^1 error estimates.
- Dependence in time of the error: avoid the use of Gronwall Lemma

Outline of the approach

- From $\partial_t u + \partial_x f(u) = k(x) g(u)$ to a smooth augmented system,

$$\partial_t u + \partial_x f(u) - g(u)\partial_x a = 0 \qquad \partial_t a = 0\,. \qquad (*)$$

  Non-resonance ($\lambda(u)$ or $f'(u) \neq 0$) $\Rightarrow$ strict hyperbolicity of $(*)$.

- Numerical scheme for $(*)$: $(u^{\Delta x}, a^{\Delta x})$, $a^{\Delta x}$ piecewise constant, so $\partial_x a^{\Delta x} = \Delta x \sum k(x_j)\delta(x - x_j)$, countable "local scattering centers"

$$(*) \quad \text{for} \quad u^{\Delta x}(t=0,\cdot) = P^{\Delta x} u_0, \quad a^{\Delta x}(t,\cdot) \equiv P^{\Delta x} a\,.$$

  Stationary waves at each discontinuity point of $a^{\Delta x} \to$ Dirac mass.

- Estimate $\|(u,a) - (u^{\Delta x}, a^{\Delta x})\|_{L^1}$ via L^1 stability theory for $(*)$ with $u, a \to$ exact, but $u^{\Delta x}, a^{\Delta x} \to$ approx. [M. Laforest, SIAM Math. Anal. 2004]

Outline of the approach: A Lyapunov-type functional

Let $U = (u, a)$ and $V = (v, b)$ be 2 WFT approximate solutions of (*),

Key tool:

Functional $t \mapsto \Phi(U, V)(t)$ equivalent to L^1 norm and decreasing in time:

$$\begin{aligned}\mathcal{O}(1)\big(\|u(t) - v(t)\|_{L^1} + \|a - b\|_{L^1}\big) &\leq \Phi(U, V)(t) \leq \Phi(U, V)(t = 0) \\ &\leq \mathcal{O}(1)\left(\|u(0) - v(0)\|_{L^1} + \|a - b\|_{L^1}\right)\end{aligned}$$

Here $\mathcal{O}(1)$ are indep. on t. For certain systems they can be made explicit.

Yet, as a special case, fix $v(t = 0, \cdot) = P^{\Delta x} u_0$ piecewise cst:

$$\|u(0) - u^{\Delta x}(0)\|_{L^1} \leq \Delta x \, \mathrm{TV}\{u_0\}.$$

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$$\leq \mathcal{O}(1)\,(\|u(0) - v(0)\|_{L^1} + \|a - b\|_{L^1})$$

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Outline of the approach: A Lyapunov-type functional

Source term contribution reduced to,

$$\boxed{\|a - a^{\Delta x}\|_{L^1} \le \Delta x \, \mathrm{TV}\{a\} = \Delta x \, \|k\|_{L^1}}$$

$$\begin{aligned} \|u(t) - u^{\Delta x}(t)\|_{L^1} &\le \mathcal{O}(1) \left(\|u(0) - u^{\Delta x}(0)\|_{L^1} + \|a - a^{\Delta x}\|_{L^1} \right) \\ &\le \mathcal{O}(\Delta x) (\mathrm{TV}\{u_0\} + \|k\|_{L^1}) \end{aligned}$$

Remarks

- Depends on $\|k\|_{L^1}$ and not on $\mathcal{O}(t)\mathrm{TV}\{k\} \rightarrow$ Poincaré ineq. in BV
- For scalar eqns, $f' > 0$ + possibly accretive source: Godunov proj'ns, at $t^n = n\Delta t$, yield an additional term growing linearly in time

Scalar 1D error estimate (cf. [AG, JDE (2013)])

Let $f' > 0$ and $N = \sup_{x,u} k(x)g'(u) > 0$, $\forall t > 0$ and $x_1 < x_2$,

$$\boxed{\int_{x_1}^{x_2} |u^{\Delta t}(t,x) - u(t,x)|dx \leq \min\{E_1, E_2\}}$$

$$\begin{aligned} E_1(\Delta x, t) &= C_1 \Delta x + C_2 t \rightarrow \text{specific WB}, \\ E_2(\Delta x, t) &= \sqrt{\Delta x}\, A(t) + \Delta x\, B(t). \end{aligned}$$

For small t: $A(t) \sim \sqrt{t}$, $B(t) \sim const. > 0$;

For large t: $A(t)$, $B(t) \sim \exp(Nt) \rightarrow$ standard time-split algos.

The estimate $E_2(t)$ results of Kuznetsov's method + Gronwall lemma: it's convenient for very small values of $t\Delta x$. The estimate $E_1(t)$ is good for large ones (C_2 comes from time-steps averaging, $\sum_n P^{\Delta x}\delta(t - t^n)$).

Exponentially amplified soln: $\partial_t u + u\partial_x u = 0.2u$

D. Amadori, L. Gosse / J. Differential Equations 255 (2013) 469–502 493

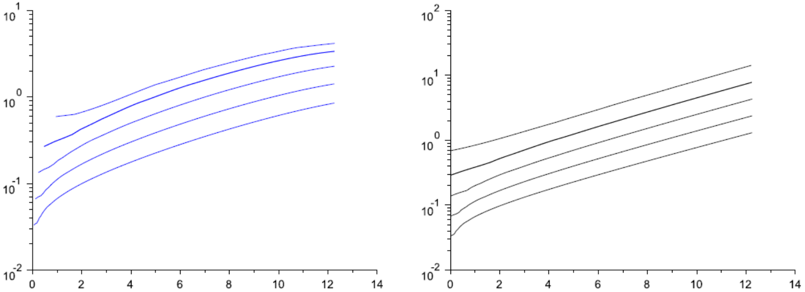
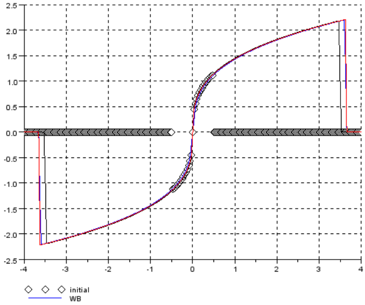
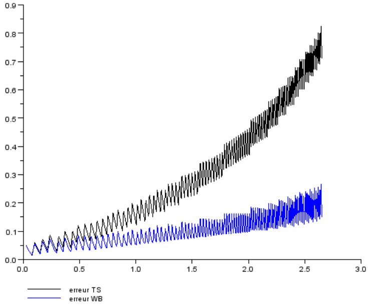


Fig. 3. Time evolution of the measured L^1 error for (5) with $\Delta x = 2^{-n}$, $n = 0, 1, 2, 3, 4$. The WB scheme (left) shows a weaker dependence on the grid compared to the TS one (right) which displays a neat exponential growth.

Decoupling effect between Δx and t.

Amplified N-wave: $\partial_t u + u\partial_x u = u^4/4, \;\rightarrow$ [Kim, JCP]





$$u(t,x) = \mathsf{sgn}(x)(3|x|)^{\frac{1}{3}}\chi(|x| < \exp(3t/4)/2).$$

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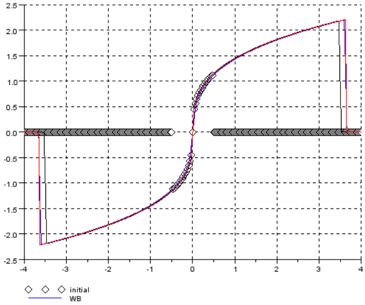
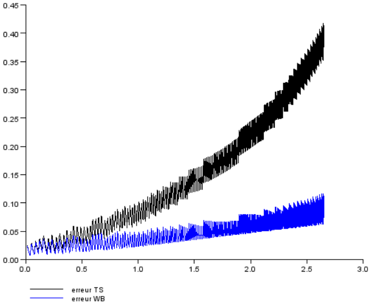




$$u(t,x) = \mathsf{sgn}(x)(3|x|)^{\frac{1}{3}}\chi(|x| < \exp(3t/4)/2).$$

A semilinear 2×2 system with space-dependent source

$$\begin{cases} \partial_t \rho + \partial_x J = 0 \\ \partial_t J + \partial_x \rho = k(x)(A(\rho) - J) \end{cases}$$

- $k \in L^1 \cap BV(\mathbb{R})$, $\quad k(x) \geq 0 \to$ dissipative relaxation structure
- $A \in C^1(\mathbb{R})$, $\quad A(0) = 0$, $\quad Lip(A) < 1$ sub-characteristic cond'n.

- Diagonal variables $f^{\pm}$ defined by $\quad \rho = f^+ + f^-, J = f^+ - f^-$, $a(x)$: $a'(x) = k(x)$. Apply [BLY] to "semilinear homog." system,

$$\begin{cases} \partial_t f^- - \partial_x f^- - G(f^-, f^+)\partial_x a & = 0 \\ \partial_t f^+ + \partial_x f^+ + G(f^-, f^+)\partial_x a & = 0 \\ \partial_t a & = 0 \end{cases}$$

"New" error estimate *via* L^1 stability (cf. [ANIHP 2015])

For $x_1 < x_2$ and $2t \leq x_2 - x_1$, set the local L^1 global error

$$I(t) = \int_{x_1+t}^{x_2-t} |f^{\pm}_{\Delta x}(t,x) - f^{\pm}(t,x)|dx\,.$$

Theorem: For small $4\,C_1\|k\|_{L^1} < 1$, and possibly "big BV data" $f_0^{\pm}$,

$$I(t) \leq K \cdot I(0) + \Delta x \cdot \|k\|_{L^1} \cdot \mathcal{E}_1\left(C_0, K, \|k\|_{L^1}, \mathrm{TV}\{f_0^{\pm}\}\right)$$

where $C_1 \simeq 1$ and

$$K = \frac{1}{1 - 4\,C_1\|k\|_{L^1}} \geq 1\,, \qquad C_0 = \max_D |G|$$

Here $\|k\|_{L^1} = \|k\|_{L^1(x_1,x_2)}$, $\mathrm{TV}\{f_0^{\pm}\} = \mathrm{TV}\{f_0^{\pm};(x_1,x_2)\}$.

Standard error estimate: *via* Kuznetsov method

Complementary estimate via Kuznetsov method, for $k \in L^1 \cap BV$, with no restriction on $\|k\|_{L^1}$. (able to cover hydro limit $\to \partial_t \rho + \partial_x A(\rho) = 0$).

Theorem: If $k \in L^1 \cap BV$, WB algorithm obeys another error estimate,

$$I(t) \;\leq\; I(0) \;+\; \sqrt{\Delta x} \cdot t \cdot 2\mathcal{E}_2$$

(like more standard time-split algos), where

$$\begin{aligned} \mathcal{E}_2(t, x_1, x_2) \quad &= \quad \sqrt{C_0 \, \|k\|_{L^1} \, A(t)} + \sqrt{\Delta x}\, C_0 \, \|k\|_{L^1} \|k\|_{L^\infty}, \\ A(t) \quad &\simeq \quad \frac{1}{C_0 \, t} \mathrm{TV}\{f_0^\pm\} + \mathrm{TV}\{k\} \;\to\; \text{ dominates for large } t. \end{aligned}$$

Here, $\|\cdot\|_{L^1}$, $\|\cdot\|_{L^\infty}$, $\mathrm{TV}\{\cdot\}$ are referred to (x_1, x_2): it is linear in t and half-order in Δx.

Chemotaxis dynamics $\rightarrow$ hydrodynamic limits

$$\partial_t f^{\pm} \pm \partial_x f^{\pm} = \pm\frac{1}{\varepsilon}\left([\frac{1}{2} + \phi(\partial_x S)]f^- - [\frac{1}{2} - \phi(\partial_x S)]f^+\right), \quad x \in \mathbb{R},$$

initial/decay conditions $f^{\pm}(t=0,\cdot) = f_0^{\pm}$, $\lim_{x\to\pm\infty} f^{\pm}(t,x) = 0$,

$$-\partial_{xx}S + S = \rho, \qquad S(t,x) = \frac{1}{2}\exp(-|x|) * \rho(t,\cdot) \to 0, \quad |x| \to \infty.$$

Macro. var., $\rho = f^+ + f^-, J = f^+ - f^-$, weakly nonlinear relaxation,

$$\partial_t \rho + \partial_x J = 0, \qquad \partial_t J + \partial_x \rho = \frac{1}{\varepsilon}\big(2\phi(\partial_x S)\rho - J\big).$$

Sub-characteristic condition allows passing to limit $\varepsilon \to 0$ with $a = 2\phi$,

$$\|a(\partial_x S(t,x))\|_\infty \leq 1 \qquad \partial_t \rho + \partial_x(a(\partial_x S)\rho) = 0.$$

Local scatt. centers $\rightarrow$ WB scheme with scatt. matrix

Independently of numerical handling of Vol'Pert product $a(\partial_x S) \cdot \rho$,

$$\begin{bmatrix} f^+_{j,n+1} \\ f^-_{j-1,n+1} \end{bmatrix} = (1 - \frac{\Delta t}{\Delta x}) \begin{bmatrix} f^+_{j,n} \\ f^-_{j-1,n} \end{bmatrix} + \frac{\Delta t}{\Delta x} \mathcal{S}^n_{j-\frac{1}{2}} \begin{bmatrix} f^+_{j-1,n} \\ f^-_{j,n} \end{bmatrix}.$$

Stationary eqns. $\rightarrow$ $\mathcal{S}^n_{j-1/2} = \begin{pmatrix} \dfrac{2c_L}{1-c_R+c_L} & \dfrac{1+c_R+c_L}{1-c_R+c_L} \\ \dfrac{1-c_R-c_L}{1-c_R+c_L} & \dfrac{-2c_R}{1-c_R+c_L} \end{pmatrix}$,

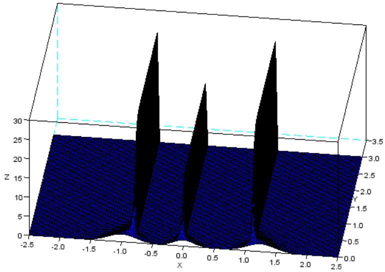
with the signed coefficients ($a^n_{j-\frac{1}{2},L}$ and $a^n_{j-\frac{1}{2},R}$ may be equal) :

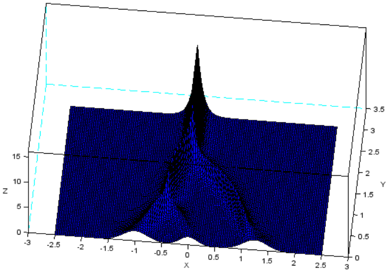
$$c_R = \frac{a^n_{j-\frac{1}{2},R}}{1-\exp(a^n_{j-\frac{1}{2},R}\Delta x/\varepsilon)} \le 0, \quad c_L = \frac{a^n_{j-\frac{1}{2},L}}{1-\exp(-a^n_{j-\frac{1}{2},L}\Delta x/\varepsilon)} \ge 0.$$

If $|a^n_{j-\frac{1}{2},L/R}| \le 1$, $\mathcal{S}^n_{j-\frac{1}{2}}$ is a left-stochastic matrix (positivity + mass).

Delicate handling of Vol'Pert product

Numerical handling of Vol'Pert product $a(\partial_x S) \cdot \rho \to$ Heaviside $\times$ Dirac!





$\to$ other computations in Monika's Ph.D. Thesis (isothermal Euler).

Two different schemes $\rightarrow$ Gosse-Vauchelet [SISC'15]

- Full WB approach "$a^n_{j-\frac{1}{2},L} \neq a^n_{j-\frac{1}{2},R}$" $\rightarrow$ Poisson coupling included in the scattering matrix $\mathcal{S}^n_{j-1/2}$. Upwind emerging in the limit $\varepsilon \rightarrow 0$

$$\boxed{J^n_{j-1/2} \rightarrow \max\left(0, a^n_{j-\frac{1}{2},L}\right)\rho^n_{j-\frac{1}{2},L} + \min\left(0, a^n_{j-\frac{1}{2},R}\right)\rho^n_{j-\frac{1}{2},R}}$$

instead of Lx-F (cf. [Ja-Va]) in $\rho^{n+1}_j = \rho^n_j - \frac{\Delta t}{\Delta x}\left(J^n_{j+1/2} - J^n_{j-1/2}\right)$

- Hybridization TS-WB, where $\mathcal{S}^n_{j-1/2}$ contains just "$a^n_{j-\frac{1}{2}}$" (easier):

$$\partial_t f^\pm \mp \frac{\mathcal{L}(x; f^\pm)}{\varepsilon_{TS}} = \mp\partial_x f^\pm \pm \frac{\mathcal{L}(x; f^\pm)}{\varepsilon_{WB}}, \qquad \frac{1}{\varepsilon} = \frac{1}{\varepsilon_{WB}} + \frac{1}{\varepsilon_{TS}},$$

$$\begin{pmatrix} \tilde{f}^+_{j-1} \\ \tilde{f}^-_j \end{pmatrix} = O^{\Delta t/\varepsilon_{TS}}_{j-\frac{1}{2}} \begin{pmatrix} f^+_{j-1} \\ f^-_j \end{pmatrix}, \qquad \text{TS pre-processing step,}$$

$$O^{\Delta t}_{j-\frac{1}{2}} = \frac{1}{2}\begin{pmatrix} 1 + a_{j-\frac{1}{2}} + (1 - a_{j-\frac{1}{2}})e^{-\Delta t} & (1 + a_{j-\frac{1}{2}})(1 - e^{-\Delta t}) \\ (1 - a_{j-\frac{1}{2}})(1 - e^{-\Delta t}) & 1 - a_{j-\frac{1}{2}} + (1 + a_{j-\frac{1}{2}})e^{-\Delta t} \end{pmatrix}.$$

Continuous velocities → "Caseology" elementary solns

$$\partial_t f + v\partial_x f = \frac{\int_{-1}^{1} h(v'\partial_x S)dv'}{\varepsilon}\left(\frac{h(v\partial_x S)}{\int_{-1}^{1} h(v'\partial_x S)dv'}\int_{-1}^{1} f(t,x,v')dv' - f\right)$$

Asymptotically, as $\varepsilon \to 0$, $\rho(t,x) = \int_{-1}^{1} f(v)dv$ satisfies continuity eqn.

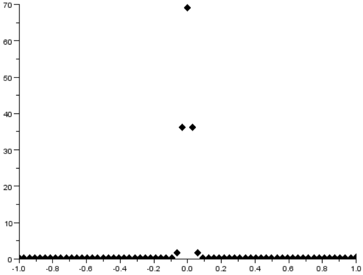
$$\partial_t \rho + \partial_x(a(\partial_x S)\rho) = 0, \qquad a(\partial_x S) = \frac{\int_{-1}^{1} v.h(v\partial_x S)dv}{\int_{-1}^{1} h(v\partial_x S)dv}.$$

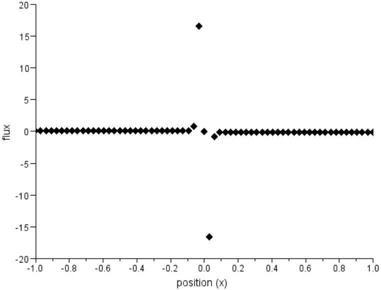
Hybrid num. method based on [Paveri-Fontana et al., J. Stat. Phys. 1989] $\to \mathcal{S}^n_{j-\frac{1}{2}}$

$$\begin{bmatrix} f_j^{n+1}(|\mathbf{v}|) \\ f_{j-1}^{n+1}(-|\mathbf{v}|) \end{bmatrix} = (1-\frac{\Delta t}{\Delta x}|\mathbf{v}|)\begin{bmatrix} f_j^{n}(|\mathbf{v}|) \\ f_{j-1}^{n}(-|\mathbf{v}|) \end{bmatrix} + \frac{\Delta t}{\Delta x}|\mathbf{v}|\mathcal{S}^n_{j-\frac{1}{2}}\begin{bmatrix} f_{j-1}^{n}(|\mathbf{v}|) \\ f_{j}^{n}(-|\mathbf{v}|) \end{bmatrix},$$

Positivity + mass $\to \mathcal{S}^n_{j-\frac{1}{2}}$ diagonally similar to a left-stochastic matrix.

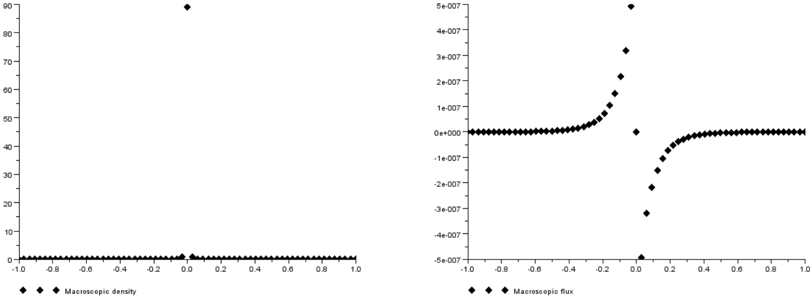
Asympt.-Pres. time-split: James-Vauchelet [SINUM'15]





$$\rho(T,\cdot) = \int_{-1}^{1} f(T,\cdot,v)dv, \qquad J(T,\cdot) = \int_{-1}^{1} v.f(T,\cdot,v)dv.$$

Asympt.-Pres. hybrid WB-TS: Gosse-Vauchelet



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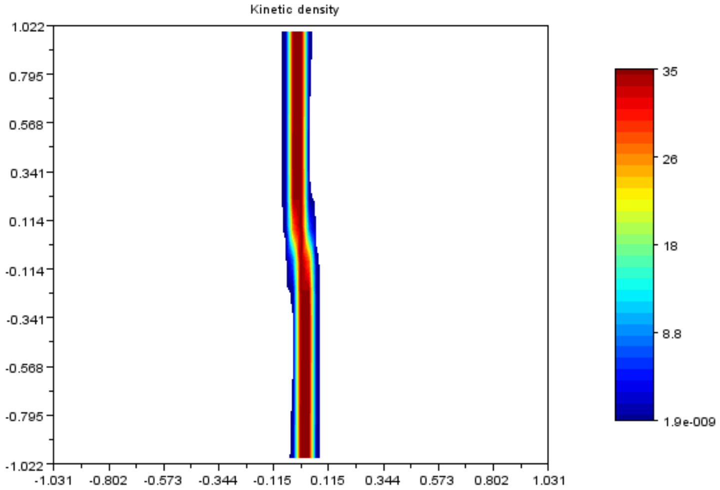
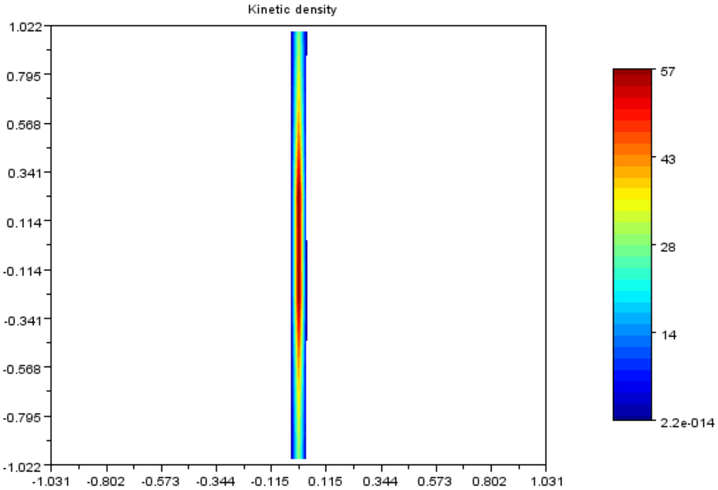
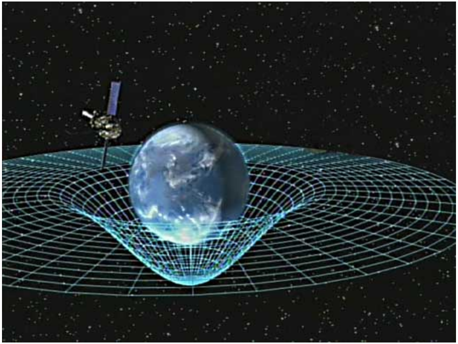


Asympt.-Pres. hybrid WB-TS: Gosse-Vauchelet



Application to "R=T" 1+1 GR model (cf. [SIAP 2015])

General relativity is a geometric theory of gravity which stipulates that it's not a force but the deformations of space-time itself (photons with zero mass deviate close to massive stars): $G = T$, G the Einstein tensor.



J. Wheeler: "Matter tells space-time how to curve, and curvature tells matter how to move". Nonlinearly coupled matter-field eqns. in low dim.

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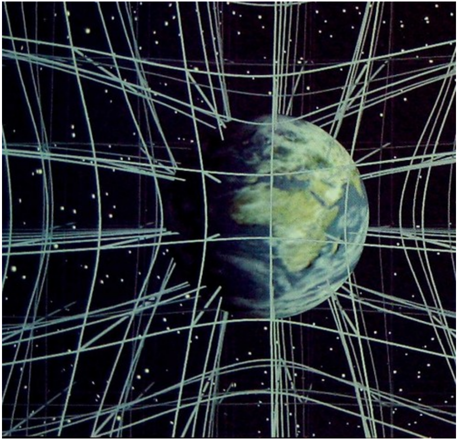


Analogue of $(1+3)$-dim. GR in 1 space dimension

P. Collas [1977] $\rightarrow$ GR is trivial in 1 space dim. $(0=0)$. So R. Mann and colleagues developed a scalar "baby model" in the 90's based on "dilaton gravity", called "R=T". The scalar field eqn. relates spacetime curvature (expressed by a metric) to the mass-energy density at each point t, x.

$$g = \begin{pmatrix} E & 0 \\ 0 & G \end{pmatrix}.$$

Gaussian curvature in these ortho. coordinates reads: ($R \rightarrow$ Ricci scalar)

$$K = -\frac{1}{2\sqrt{|EG|}}\left[\partial_t\left(\frac{\partial_t G}{\sqrt{|EG|}}\right) + \partial_x\left(\frac{\partial_x E}{\sqrt{|EG|}}\right)\right] = \frac{R}{2}$$

Mass-energy tensor in 1+1 dim. with $c=1 \rightarrow$ a 2×2 matrix,

$$T^{\alpha\beta} = (\rho+p)u^\alpha u^\beta - p\, g^{\alpha\beta}, \qquad u^\alpha u_\alpha = (g^{\alpha\beta}\, u_\beta)u_\alpha = -1$$

2 main expressions of field equation

Any 2-D surface is locally "conformally flat", so the metric can be

$$g = \exp(2\phi) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} := \exp(2\phi)\eta, \quad \eta \text{ the Minkowski metric,}$$

$\phi(t,x)$ the conformal factor. Gravitational field eqn. reads accordingly,

$$R = 2K = -\exp(-2\phi)\left(\frac{\partial^2(2\phi)}{\partial t^2} - \frac{\partial^2(2\phi)}{\partial x^2}\right) = T,$$

being T the trace of $T^{\alpha\beta}$ $\rightarrow$ Liouville wave eqn. In Schwarzschild coord.,

$$g = \begin{pmatrix} -\alpha & 0 \\ 0 & \frac{1}{\alpha} \end{pmatrix}, \qquad R = \frac{\partial^2}{\partial t^2}\left(\frac{1}{\alpha}\right) - \frac{\partial^2}{\partial x^2}(\alpha) = T.$$

The field eqn. develops shocks in finite time (Nishida), "gauge shocks".

Inclusion of matter and resulting dynamics

Define a (local) Lorentz factor γ and a scalar 1D velocity v:

$$\gamma = \frac{1}{\sqrt{1-v^2}}, \qquad u := (1, v)\exp(-\phi)\gamma.$$

Covariant divergence: $T^{\alpha\beta}{}_{;\beta} = \nabla_\beta T^{\alpha\beta} = \dfrac{\partial_k\left(\sqrt{-\det g}\; T^{\alpha k}\right)}{\sqrt{-\det g}} + \Gamma^{\alpha}_{lm} T^{lm} = 0$

$$\left\{ \begin{array}{rcccl} \tau & = & \exp(2\phi)T^{tt} & = & (\rho+p)\gamma^2 - p, \\ S & = & \exp(2\phi)T^{tx} & = & (\rho+p)\gamma^2 v, \\ \Sigma(\tau, S) & = & \exp(2\phi)T^{xx} & = & (\rho+p)(\gamma v)^2 + p. \end{array} \right.$$

Within those notations, field-matter eqns: $(\tau - \Sigma = \rho - p)$

$$\boxed{\left\{ \begin{array}{rcl} \partial_t\tau + \partial_x S + 2S\;\partial_x\phi + (\tau+\Sigma)\;\partial_t\phi & = & 0, \\ \partial_t S + \partial_x\Sigma + (\tau+\Sigma)\;\partial_x\phi + 2S\;\partial_t\phi & = & 0, \end{array} \right.}$$

$$\boxed{\frac{\partial^2(2\phi)}{\partial t^2} - \frac{\partial^2(2\phi)}{\partial x^2} + (\tau - \Sigma)\exp(2\phi) = 0.}$$

Specificities of these conformal coordinates

In Schwarzschild coords. (more massive obj.), more pathological system,

$$\begin{cases} \partial_t(\alpha T^{tt}) + \partial_x(\alpha T^{tx}) = V \cdot \partial_t(\log\sqrt{\alpha}), \\ \partial_t(T^{tx}/\alpha) + \partial_x(T^{xx}/\alpha) = V \cdot \partial_x \log(1/\sqrt{\alpha}), \\ V = (\rho+p)\gamma^2(1+v^2),\ T = p - \rho, \end{cases} \qquad \partial_{tt}\left(\frac{1}{\alpha}\right) - \partial_{xx}\alpha = T$$

with the tensor components,

$$\begin{cases} T^{tt} & = & (\rho+p)u^t u^t + p\ g^{tt} & = & (\rho+p)\frac{\gamma^2}{\alpha} - \frac{p}{\alpha}, \\ T^{tx} & = & (\rho+p)u^t u^x + p\ g^{tx} & = & (\rho+p)\ \gamma^2 v, \\ T^{xx} & = & (\rho+p)u^x u^x + p\ g^{xx} & = & \alpha(\rho+p)(\gamma v)^2 + p\alpha, \end{cases}$$

The Lorentz factor allows to induces a slightly different scalar velocity v,

$$\gamma = \frac{1}{\sqrt{1-v^2}}, \qquad u := \gamma\left(\frac{1}{\sqrt{\alpha}}, \sqrt{\alpha}v\right).$$

Locally inertial numerical approximation

Work on uniform Cartesian grid with control cells centered around t^n, x_j. Computational cells are such that $\phi \equiv \phi_j^n \simeq \phi(t^n, x_j)$, *i.e.* ϕ is a constant in each cell, meaning that it is a local inertial reference frame (the metric is flat inside each cell): conformal factor jumps only at each interface. Time-splitting + WB strategy, alternating between a first "WB" system,

$$\left.\begin{array}{rcl} \partial_t \tau + \partial_x S + 2S\partial_x \phi & = & 0, \\ \partial_t S + \partial_x \Sigma + (\tau + \Sigma)\partial_x \phi & = & 0, \\ \partial_t \phi & = & 0, \end{array}\right\}$$

and the second one, just for the time-derivatives of the conformal factor,

$$\partial_t \tau(t) = -(\tau + \Sigma)\partial_t \phi \simeq -(\tau + \Sigma^n)\partial_t \phi, \qquad \partial_t S(t) + 2S\partial_t \phi = 0,$$

where Σ^n stands for a "frozen" value of $\Sigma(t)$ at the time $t^n = n\Delta t$.

Corresponding Godunov (subsonic!) scheme for system 1 reads,

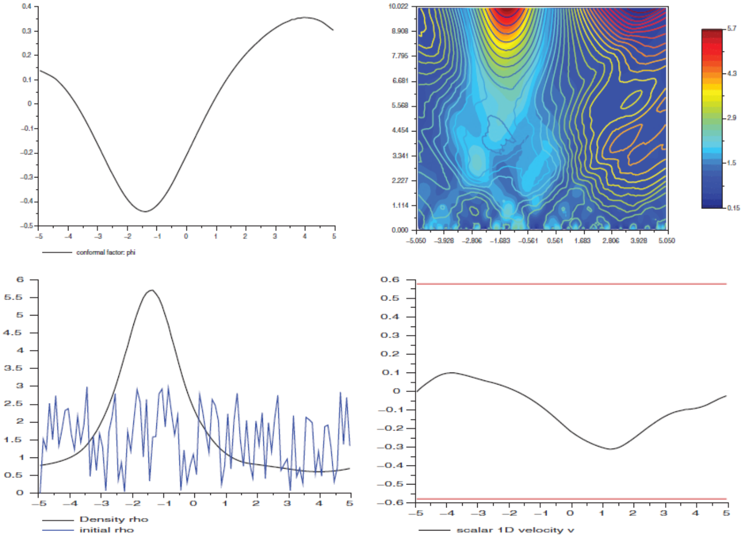
$$\left.\begin{aligned}
\tau_j^{n+1} &= \tau_j^n - \Delta t \overbrace{\left(\frac{S^n_{j+\frac{1}{2},-} - S^n_{j-\frac{1}{2},+}}{\Delta x}\right)}^{numer.\ intrinsic\ space-derivative} \\
S_j^{n+1} &= S_j^n - \Delta t \underbrace{\left(\frac{\Sigma^n_{j+\frac{1}{2},-} - \Sigma^n_{j-\frac{1}{2},+}}{\Delta x}\right)}_{numer.\ intrinsic\ space-derivative}
\end{aligned}\right\}$$

where now $S^n_{j+\frac{1}{2},\mp}, \Sigma^n_{j+\frac{1}{2},\mp}$ are the left/right states in the Riemann fan which are separated by the static discontinuity induced by the jump $\phi^n_{j+1} - \phi^n_j$ of the metric's conformal factor located at the interface $x_{j+\frac{1}{2}}$. Left/right states are values $\rho_\pm, v_\pm$ appearing in the Riemann problem:

$$\begin{pmatrix} \rho_L \\ v_L \end{pmatrix} \overset{\mathcal{W}_+,\phi\equiv C}{\longrightarrow} \begin{pmatrix} \rho_- \\ v_- \end{pmatrix} \overset{S\,\exp(2\phi),\ \Sigma[\dots]\ \equiv C}{\longrightarrow} \begin{pmatrix} \rho_+ \\ v_+ \end{pmatrix} \overset{\mathcal{W}_-,\phi\equiv C}{\longrightarrow} \begin{pmatrix} \rho_- \\ v_- \end{pmatrix},$$

where quantities constant across each simple wave are over each arrow.

Self-gravitating relativistic isothermal gas cloud



Random perturbation of a $1 + 1$ polytrope