Well-posedness and numerics for the uncertain Cauchy problem for conservation laws

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Overview

Consider the scalar conservation law

$$\begin{aligned} \partial_t u + \partial_x f(u) &= 0, \qquad x \in \mathbb{R}, t > 0 \\ u(x,0) &= \bar{u}(x). \end{aligned} \tag{1}$$

(1) is well-posed for $\bar{u} \in L^1(\mathbb{R})$:

- There exists an **entropy solution**.
- The entropy solution is unique.
- The entropy solution is stable w.r.t. $\bar{u}$.

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What if the initial data $\bar{u}$ is uncertain?

- What is the right notion of solution?
- Does there exist such a solution?
- What does “uniqueness” mean for uncertain data?
- How can we approximate the “uncertain” solution numerically?
- How to measure convergence of numerical schemes?

Entropy solutions

$$\begin{aligned} \partial_t u + \partial_x f(u) &= 0 \\ u(x,0) &= \bar{u}(x) \end{aligned} \tag{1}$$

Definition

A function $u \in L^1(\mathbb{R} \times \mathbb{R}_+)$ is a **weak solution** of (1) if

$$\int_{\mathbb{R}} \int_{\mathbb{R}_+} u\varphi_t + f(u) \cdot \nabla\varphi \; dxdt + \int_{\mathbb{R}} \bar{u}(x)\varphi(x,0) \; dx = 0 \qquad \forall \; \varphi \in C_c^1(\mathbb{R} \times \mathbb{R}_+).$$

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Entropy conditions are imposed to single out a unique "physical" solution.

Definition

- **Entropy pair:** Functions (η, q), with $\eta(u)$ convex and $q'(u) = \eta'(u) \cdot f'(u)$.
- A weak solution u is an **entropy solution** of (1) if for all entropy pairs (η, q)

$$\partial_t \eta(u) + \partial_x q(u) \leqslant 0 \qquad \text{in } \mathcal{D}'(\mathbb{R} \times \mathbb{R}_+).$$

Total entropy/energy decreases in time:

$$\int_{\mathbb{R}} \eta(u(x,T))\ dx \leqslant \int_{\mathbb{R}} \eta(\bar{u}(x))\ dx.$$

Well-posedness: scalar equations

$$\begin{aligned} \partial_t u + \partial_x f(u) &= 0 \\ u(x,0) &= \bar{u}(x) \end{aligned} \tag{1}$$

Theorem (Kruzkov 1970)

For **scalar** *conservation laws there* **exists a unique entropy solution** *of* (1) *whenever* $\bar{u} \in L^1(\mathbb{R})$*.* *The solutions are stable with respect to initial data:*

$$\int_{\mathbb{R}} |u(x,t) - v(x,t)|\, dx \leqslant \int_{\mathbb{R}} |\bar{u}(x) - \bar{v}(x)|\, dx \qquad \text{for all } t > 0$$

for entropy solutions u *and* v *with initial data* $\bar{u}$ *and* $\bar{v}$*.*

We denote the solution operator for (1) by

$$S_t : L^1(\mathbb{R}) \to L^1(\mathbb{R}), \qquad S_t\bar{u} := u(\cdot, t).$$

Section 1

Uncertain initial data

Uncertainty quantification

$$\begin{aligned} \partial_t u + \partial_x f(u) &= 0 \\ u(x,0) &= \bar{u}(x), \qquad x \in \mathbb{R}. \end{aligned} \tag{1}$$

- Error and uncertainty in the measurement of $\bar{u}$ is inevitable.
- **Uncertainty quantification:** Given uncertainties in $\bar{u}$, what are the statistics of the solution at time $t > 0$?
- The overall aim is **well-posedness** – our statistical predictions are the *only possible predictions.*

Questions

- How to represent uncertain initial data?
- In what sense is (1) satisfied for uncertain data?
- How do we (numerically) approximate this "statistical solution"?

Representation of data: random fields

Approach #0

Fix a probability space $(\Omega, \mathcal{X}, P)$, let the initial data $\bar{X}$ be a **random field**

$$\bar{u} = \bar{u}(\omega; x).$$

- **Pathwise solution:** a random field $u = u(\omega; x, t)$ such that *for every* $\omega \in \Omega$,

  $$u(\omega; \cdot, \cdot) \text{ is a solution of (1) with initial data } \bar{u}(\omega, \cdot)$$

  (similar to “strong” or “weak” solutions of SDEs).
- We can study the **law** of u $\big($say, $(u\#P)(A; x, t) := P(\{u(\omega; x, t) \in A\})$ for $A \subset \mathbb{R}\big)$.

Representation of data: random fields

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Problems with Approach #0:

- Inherently non-unique – $\bar{u}$ can always be reparametrized over a different space $(\tilde{\Omega}, \tilde{\mathcal{X}}, \tilde{P})$.
- Distances (metrics) between two solutions u and $\tilde{u}$ depend on the arbitrary parametrization $\omega \in \Omega$.
- Depends completely on the well-posedness of the deterministic problem (1).

We would like to study the law of u directly.

Probability measures on $L^1(\mathbb{R})$

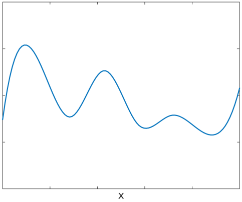
Approach #1

Initial data is a probability measure on solution space: $\bar{\mu} \in Prob(L^1(\mathbb{R}))$.

Examples:

- For some $\bar{u} \in L^1(\mathbb{R})$, let

$$\bar{\mu} = \delta_{\bar{u}} \qquad (\bar{\mu} \text{ is } \textit{atomic})$$



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- For $\bar{u}_1, \bar{u}_2 \in L^1(\mathbb{R})$ and $\alpha_i > 0$, $\alpha_1 + \alpha_2 = 1$, let
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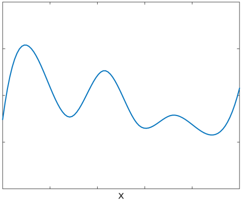

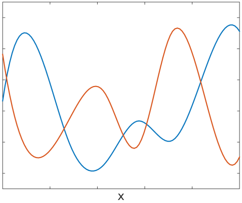

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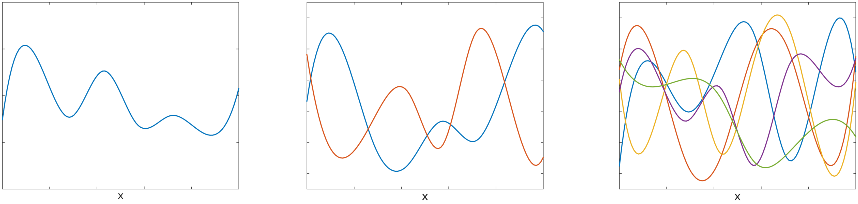
- For $\bar{u}_i \in L^1(\mathbb{R})$ and $\alpha_i > 0$, $\sum_{i=1}^M \alpha_i = 1$, let
$$\bar{\mu} = \sum_{i=1}^M \alpha_i \delta_{\bar{u}_i}.$$



One-point statistics

Approach #2 *(first attempt)*

For each $x \in \mathbb{R}$, assign a probability measure $\bar{\nu}_x \in Prob(\mathbb{R})$, giving the statistics of the value at x.

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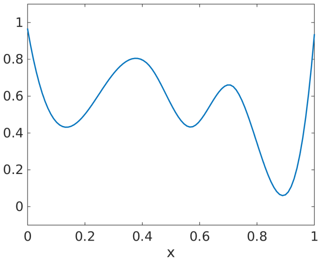

Figure : All mass concentrated at $\bar{u}(x)$.

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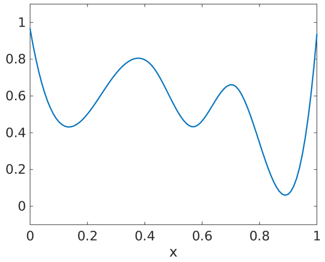


Figure : All mass concentrated at $\bar{u}(x)$.

- For functions $\bar{u}(x)$, $\bar{\sigma}(x)$, let

$$\bar{\nu}_x = N\big(\bar{u}(x), \bar{\sigma}^2(x)\big)$$

(normal distribution with mean value $\bar{u}(x)$).

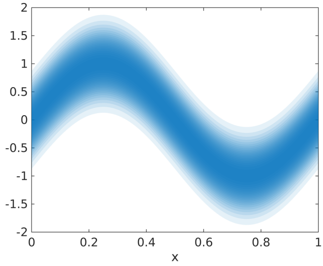
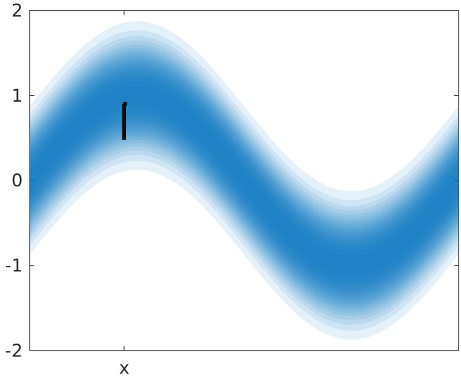


Figure : Mass normally distributed around $\sin(2\pi x)$.

One-point statistics

If $x \in \mathbb{R}$ and $A \subset \mathbb{R}$ then

$$\nu_x(A) = \text{probability of } u(x) \in A.$$



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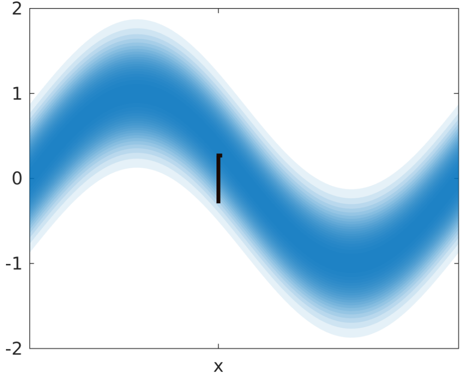

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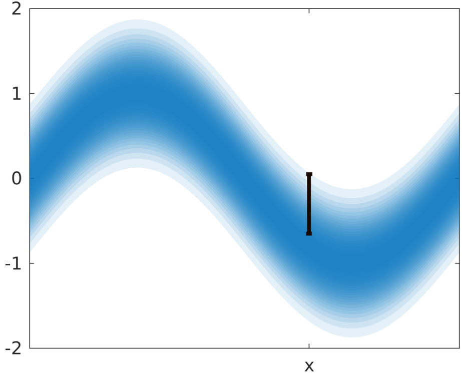

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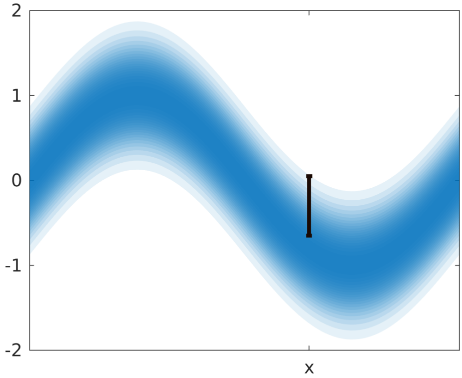

However, one-point statistics do **not** adequately describe the uncertain data: There are several different *realizations* (random fields) corresponding to the same $\bar{\nu}_x$.

Pointwise statistics: correlations

We can *add information* in the form of **correlations** ("joint probability distributions"):

Two-point correlations Correlation between values at $x = x_1$ and $x = x_2$:

$$\nu^2_{x_1,x_2} \in Prob(\mathbb{R}^2).$$

k-point correlations Correlation between values at $x = x_1, \ldots, x_k$:

$$\nu^k_{x_1,\ldots,x_k} \in Prob(\mathbb{R}^k).$$

We call these *k-point correlation marginals.*

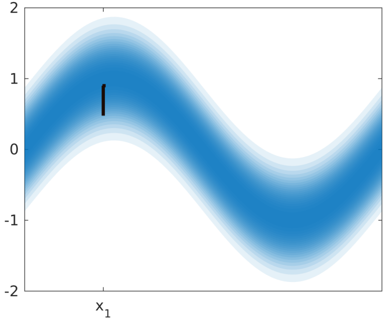

Figure : $\nu^1_{x_1}(A)$

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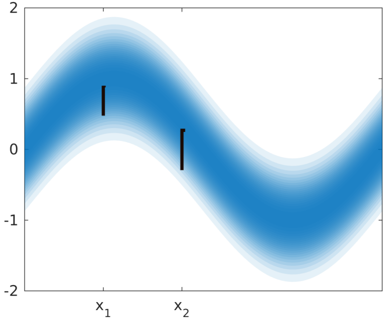

Figure : $\nu^2_{x_1,x_2}(A \times B)$

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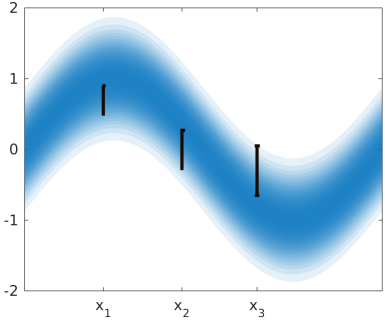


Figure : $\nu^3_{x_1,x_2,x_3}(A \times B \times C)$

Correlation measures: definition

Definition

Fix $p \in [1, \infty)$. A **correlation measure** is a collection $\nu = (\nu^1, \nu^2, \dots)$ satisfying:

(i) Weak* measurability: Each map $\nu^k : \mathbb{R}^k \to Prob(\mathbb{R}^k)$ is weak*-measurable

(ii) L^p-boundedness: ν^k is bounded in $\mathcal{F}$: there exists an $R > 0$ such that

$$|\nu^k|_{p,k} := \left(\int_{\mathbb{R}^k} \langle \nu^k_x, |\xi_1|^p \cdots |\xi_k|^p \rangle \, dx \right)^{1/p} \leqslant R^k \qquad \forall\ k \in \mathbb{N}.$$

(iii) Symmetry: If σ is a permutation of $\{1, \dots, k\}$ and $f \in C_0(\mathbb{R}^k)$ then

$$\langle \nu^k_{\sigma(x)}, f(\sigma(\xi)) \rangle = \langle \nu^k_x, f(\xi) \rangle \qquad \text{for a.e. } x \in \mathbb{R}^k.$$

(iv) Consistency: If $f \in C_0(\mathbb{R}^{k-1})$ then

$$\langle \nu^k_{x_1,\dots,x_k}, f(\xi_1, \dots, \xi_{k-1}) \rangle = \langle \nu^{k-1}_{x_1,\dots,x_{k-1}}, f(\xi_1, \dots, \xi_{k-1}) \rangle.$$

Each element ν^k is a **correlation marginal**. Denote by $\mathcal{L}^p = \mathcal{L}^p(\mathbb{R}, \mathbb{R})$ the set of all correlation measures from $\mathbb{R}$ to $\mathbb{R}$.

(Here, $\langle \lambda, f \rangle = \int_{\mathbb{R}^k} f(\xi) d\lambda(\xi)$, the expected value of f w.r.t. $\lambda \in Prob(\mathbb{R}^k)$.)

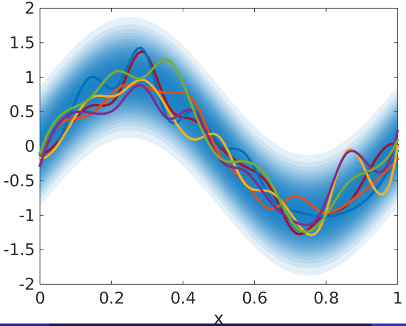
Equivalence between $Prob(L^p(\mathbb{R}))$ and $\mathcal{L}^p(\mathbb{R})$

Theorem (USF, Lanthaler, Mishra 2015)

- *Fix $p \in [1,\infty)$. For every correlation measure $\nu \in \mathcal{L}^p(\mathbb{R})$ there exists a unique probability measure $\mu \in Prob(L^p(\mathbb{R}))$ with bounded support such that*

$$\int_{\mathbb{R}^k}\int_{\mathbb{R}^k} g(x,\xi)\ d\nu_x^k(\xi)dx = \int_{L^p(\mathbb{R})}\int_{\mathbb{R}^k} g(x,u(x))\ dxd\mu(u) \qquad \forall\ g \in \mathcal{C}^k. \tag{2}$$

- *Conversely, for every probability measure $\mu \in Prob(L^p(\mathbb{R}))$ with bounded support there exists a unique correlation measure $\nu \in \mathcal{L}^p(\mathbb{R})$ such that (2) holds.*



Section 2

Statistical solutions

Statistical solutions – overview

- By the equivalence theorem, we can view every

$$\mu \in \mathit{Prob}(L^1(\mathbb{R}))$$

as a correlation measure

$$\nu = (\nu^1, \nu^2, \dots) \in \mathcal{L}^1(\mathbb{R}),$$

and vice versa.

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and vice versa.

- We consider **initial data** given by

$$\bar{\mu} \in Prob(L^1(\mathbb{R}))$$

(or, equivalently, $\bar{\nu} \in \mathcal{L}^1(\mathbb{R})$).

- We **propagate** the initial data

$$\bar{\mu} \mapsto \mu_t \in Prob(L^1(\mathbb{R})), \qquad t > 0$$

(or $\bar{\nu} \mapsto \nu_t = (\nu_t^1, \nu_t^2, \dots)$), obtaining a **statistical solution.**

The canonical statistical solution

Question:

How do we *expect* the solution to look like?

- If $\bar{\mu} = \delta_{\bar{u}} \in Prob(L^1(\mathbb{R}))$ for some $\bar{u} \in L^1(\mathbb{R})$ then

$$\mu_t = \delta_{S_t \bar{u}} = S_t \# \bar{\mu}$$

should be the *only* solution.

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- If $\bar{\mu} = \sum_{i=1}^{M} \alpha_i \delta_{\bar{u}_i}$ then

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Answer:

For general $\bar{\mu} \in Prob(L^1(\mathbb{R}))$, we want

$$\mu_t = S_t \# \bar{\mu}.$$

This is the **canonical solution.**

Evolution equation for statistical solutions

Question:

What equations does the canonical solution satisfy?

- We want an evolution equation for the statistical solution $\mu_t \leftrightarrow \nu_t$.
- Correlation measures $\nu = (\nu^1, \nu^2, \dots) \in \mathcal{L}^1(\mathbb{R})$ are uniquely determined by its **moments**

$$\langle \nu^1_{x_1}, \xi_1 \rangle, \quad \langle \nu^2_{x_1,x_2}, \xi_1\xi_2 \rangle, \quad \dots, \quad \langle \nu^k_{x_1,\dots,x_k}, \xi_1 \cdots \xi_k \rangle, \quad \dots$$

  (recall: e.g. $\langle \nu^2_{x_1,x_2}, \xi_1\xi_2 \rangle = \int_{\mathbb{R}^2} \xi_1\xi_2 \; d\nu^2_{x_1,x_2}(\xi_1,\xi_2)$).
- **We write down evolution equations for these (infinite number of) moments.**

Evolution equation for statistical solutions (motivation)

If μ_t is atomic (i.e. $\mu_t = \delta_{u(t)}$) then its k-th moment is

$$\langle \nu^k_{x_1,\dots,x_k}, \xi_1\xi_2\cdots\xi_k\rangle = u(x_1,t)u(x_2,t)\cdots u(x_k,t).$$

Let now $u = u(x,t)$ be a (classical) solution of (1).

- **$k = 1$:**

$$\partial_t u(x_1,t) + \partial_{x_1} f(u(x_1,t)) = 0$$

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- $k = 2$:

$$\begin{aligned}\partial_t \big[u(x_1,t)u(x_2,t)\big] &= (\partial_t u(x_1,t))u(x_2,t) + u(x_1,t)(\partial_t u(x_2,t)) \\ &= -\partial_{x_1} f(u(x_1,t))u(x_2,t) - \partial_{x_2} u(x_1,t) f(u(x_2,t))\end{aligned}$$

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- General $k \in \mathbb{N}$:
$$\partial_t\big[u(x_1,t)\cdots u(x_k,t)\big] + \sum_{i=1}^k \partial_{x_i}\Big[u(x_1,t)\cdots f(u(x_i,t))\cdots u(x_k,t)\Big] = 0,$$

Evolution equation for statistical solutions

Denote $\nabla = (\partial_{x_1}, \dots, \partial_{x_k})$ and $F(\xi_1, \dots, \xi_k) = (\xi_1 \cdots f(\xi_i) \cdots \xi_k)_{i=1}^k$.

Definition

A family $\mu_t \in Prob(L^1(\mathbb{R}))$ (for $t \in [0, \infty)$) with corresponding correlation measures ν_t is a **statistical solution** of (1) if *for every* $k \in \mathbb{N}$,

$$\partial_t \langle \nu^k_{t,x_1,\dots,x_k}, \xi_1 \cdots \xi_k \rangle + \sum_{i=1}^k \partial_{x_i} \langle \nu^k_{t,x_1,\dots,x_k}, \xi_1 \cdots f(\xi_i) \cdots \xi_k \rangle = 0$$

in the sense of distributions on $\mathbb{R}^k \times [0, \infty)$.

Note: The equation for $k = 1$ gives the definition of a **measure-valued solution** (DiPerna, 1985).

Section 3

Entropy conditions – stability and uniqueness

Uniqueness for atomics

Entropy condition, version 1

Entropy condition for one-point correlations:

$$\partial_t \langle \nu^1, |\xi - \zeta| \rangle + \partial_x \langle \nu^1, q(\xi;\zeta) \rangle \leqslant 0.$$

Theorem (USF, Käppeli, Mishra, Tadmor 2015)

Let v be an entropy solution with initial data $\bar{v}$, and let μ be a statistical solution with initial data $\bar{\mu}$. Then for all $t > 0$,

$$\int_{L^1(\mathbb{R})} \int_{\mathbb{R}} |u(x) - v(x,t)| dx d\mu_t(u) \leqslant \int_{L^1(\mathbb{R})} \int_{\mathbb{R}} |u(x) - \bar{v}(x)|\ dx d\bar{\mu}(u) \qquad \forall\ t > 0.$$

i.e.,

$$\int_{\mathbb{R}} W_1\left(\nu^1_{t,x}, \delta_{v(x,t)}\right)\ dx \leqslant \int_{\mathbb{R}} W_1\left(\bar{\nu}^1_x, \delta_{\bar{v}(x)}\right)\ dx.$$

Here, $W_1(\rho, \lambda)$ is the **Wasserstein metric** between $\rho, \lambda \in Prob(\mathbb{R})$.

Uniqueness for convex combinations of atomics

Entropy condition, version 2

For all $k \in \mathbb{N}$ and all $\zeta_1, \ldots, \zeta_k \in \mathbb{R}$,

$$\partial_t \langle \nu^k, |\xi_1 - \zeta_1| \cdots |\xi_k - \zeta_k| \rangle + \nabla_x \cdot \langle \nu^k, Q(\xi; \zeta) \rangle \leqslant 0 \qquad \text{in } \mathcal{D}'(\mathbb{R}^k \times \mathbb{R}_+).$$

Theorem (USF, Lanthaler, Mishra 2015)

If

$$\bar{\mu} = \sum_{i=1}^{M} \alpha_i \delta_{\bar{u}_i}$$

(initial data is convex combination of atomics), then the canonical statistical solution $\mu_t = S_t \# \bar{\mu}$ *is the* **only** *solution.*

Uniqueness of statistical solutions

Recall that the **canonical statistical solution** is

$$\mu_t := S_t \# \bar{\mu}$$

(i.e. S_t applied to initial data $\bar{\mu}$).

Conjecture

The canonical statistical solution is the *only* statistical solution.

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Conjecture

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Corollary (USF, Mishra 2015)

Statistical solutions are stable in the Wasserstein metric on $Prob(L^1(\mathbb{R}))$:

$$W_1(\mu_t, \rho_t) \leqslant W_1(\bar{\mu}, \bar{\rho}).$$

Section 4

Numerical approximation

Monte Carlo method

Recall that the **canonical statistical solution** is

$$\mu_t := S_t \# \bar{\mu}$$

(i.e. S_t applied to initial data $\bar{\mu}$).

Monte Carlo algorithm

1. Pick only finitely many points $\bar{u} \in \mathrm{supp}(\bar{\mu})$
2. Apply numerical scheme $S_t^{\Delta x}$ instead of S_t
3. Put together all computed solutions in an *approximate statistical solution* μ_t

Monte Carlo method

Algorithm

Let initial data $\bar{\mu} \in Prob(L^1(\mathbb{R}))$ be given. Let $M \in \mathbb{N}$.

1. Randomly choose M initial data $\bar{u}_1, \dots, \bar{u}_M$ according to the distribution $\bar{\mu}$.
2. Evolve each data $\bar{u}_i$ numerically
$$u_i(t) = S_t^{\Delta x} \bar{u}_i.$$
3. Compose the statistical solution:
$$\mu_t^{M,\Delta x} := \frac{1}{M} \sum_{i=1}^{M} \delta_{u_i(t)}.$$

Theorem (USF, Mishra 2015)

The above Monte-Carlo method converges in the W_1 metric to a statistical solution:

$$W_1(\mu_t^{M,\Delta x}, \mu_t) \to 0 \qquad \forall\ t > 0 \text{ as } M \to \infty, \Delta x \to 0.$$

Summary and outlook

Summary

- We evolve the **law** of the solution over time. Two equivalent ways to define the law:

  Local: Correlation measures $\nu = (\nu^1, \nu^2, \dots) \in \mathcal{L}^p(\mathbb{R})$
  Global: Probability measures $\mu \in Prob(L^p(\mathbb{R}))$
- A **statistical solution** evolves **moments**

$$\int_{\mathbb{R}^k} \xi_1 \cdots \xi_k \; d\nu^k_{t,x_1,\dots,x_k} = \int_{L^1(\mathbb{R})} u(x_1) \cdots u(x_k) \; d\mu_t(u)$$

  over time.
- Monte-Carlo type numerical schemes converge to the canonical statistical solution.
- The right metric is W_1, the Wasserstein metric on $Prob(L^1(\mathbb{R}))$.

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  over time.

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To do list

- Well-posedness for arbitrary initial data $\bar{\mu} \in Prob(L^1(\mathbb{R}))$
- Study more sophisticated numerical methods (multi-level/quasi Monte Carlo, stochastic Galerkin, ...)
- Extend framework to other equations (linear PDE = easy; nonlinear PDE = hard).

Thank you for your attention!

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