Modelling of shallow dispersive water waves

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Acknowledgements

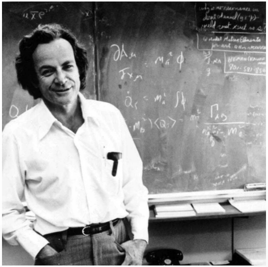
to my precious collaborators

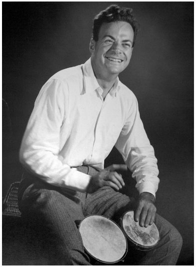
Collaborators:

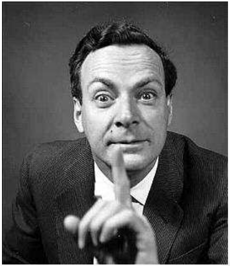
- Didier CLAMOND: Professor (LJAD) Université de Nice Sophia Antipolis, France
- Dimitrios MITSOTAKIS: Victoria University of Wellington, New Zealand

Water wave problem - I

Prof. Richard FEYNMAN *(1918 – 1988)*







The Feynman Lectures on Physics (Vol. I)

> *[. . .] the next waves of interest, that are easily seen by everyone and which are usually used as an example of waves in elementary courses, are water waves. As we shall soon see, they are the worst possible example, because they are in no respects like sound and light; they have all the complications that waves can have [. . .]*

- Continuity equation (incompressibility + irrotationality)

$$\nabla^2_{\boldsymbol{x},y}\phi = 0, \quad (\boldsymbol{x}, y) \in \times[-d, \eta],$$

- Kinematic bottom condition:

$$\frac{\partial \phi}{\partial y} + \nabla\phi \cdot \nabla d = 0, \quad y = -d,$$

- Kinematic free surface condition:

$$\frac{\partial \eta}{\partial t} + \nabla\phi \cdot \nabla\eta = \frac{\partial \phi}{\partial y}, \quad y = \eta(\boldsymbol{x}, t),$$

- Dynamic free surface condition (Cauchy–Lagrange integral):

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}|\nabla_{\boldsymbol{x},y}\phi|^2 + g\eta = 0, \quad y = \eta(\boldsymbol{x}, t).$$

- Continuity equation (incompressibility + irrotationality)

$$\nabla^2_{\boldsymbol{x},y}\phi \;=\; 0, \quad (\boldsymbol{x}, y) \in \times[-d, 0],$$

- Kinematic bottom condition:

$$\frac{\partial\phi}{\partial y} \;=\; 0, \quad y \;=\; -d,$$

- Kinematic free surface condition:

$$\frac{\partial\eta}{\partial t} \;=\; \frac{\partial\phi}{\partial y}, \quad y \;=\; 0,$$

- Dynamic free surface condition (Cauchy–Lagrange integral):

$$\frac{\partial\phi}{\partial t} + g\eta \;=\; 0, \quad y \;=\; 0.$$

Dispersion relation analysis - II

Infinitesimal periodic waves on flat bottom

Plane wave solutions:

$$\eta(\boldsymbol{x}, t) = a\,\mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t)}, \qquad \phi(\boldsymbol{x}, y, t) = b\,\varphi(y)\,\mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t)}.$$

Dispersion relation for gravity waves:

$$\omega^2(k) = gk\tanh(kd), \qquad k := |\boldsymbol{k}| \in \mathbb{R}^+$$

Definition:

The wave propagation is:

- Dispersive: $c_p(k) = \frac{\omega(k)}{k} \neq \mathrm{const}$
- Non-dispersive: $c_p(k) = \mathrm{const}$ (*e.g.* sound, light, elastic waves)

Unfortunately, the water waves are dispersive, since

$$c_p(k) = \frac{\omega}{k} = \sqrt{gd\,\frac{\tanh(kd)}{kd}}$$

This analysis is fundamentally linear!

- Linearized Saint-Venant equations:

$$\eta_t + d\,\nabla \cdot \boldsymbol{u} = 0,$$

$$\boldsymbol{u}_t + g\,\nabla \eta = 0.$$

- Plane wave solutions:

$$\eta(\boldsymbol{x}, t) = a\,\mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t)}, \quad u(\boldsymbol{x}, t) = b\,\mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t)}$$

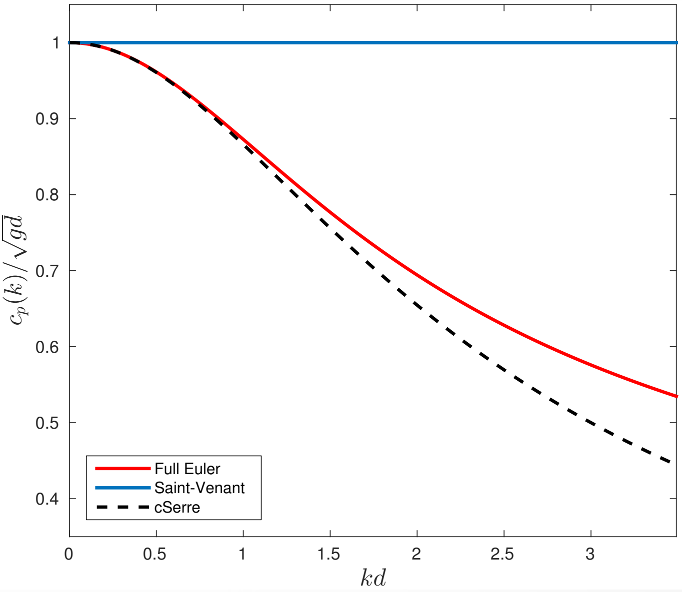
The dispersion relation is:

$$c_{\mathrm{SV}} = \frac{\omega}{k} = \sqrt{gd} \equiv \lim_{kd \to 0} c_{\mathrm{Euler}}(k)$$

The Saint-Venant equations are dispersionless!

Comparison of dispersion relations - I

for models considered so far...



- Courtesy of V. I. Bukreev *et al.* [2]:

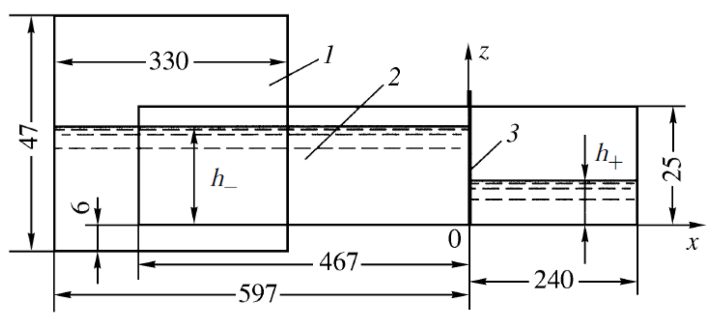


Figure: *Experimental set-up.*

- Courtesy of V. I. BUKREEV *et al.* [2]:

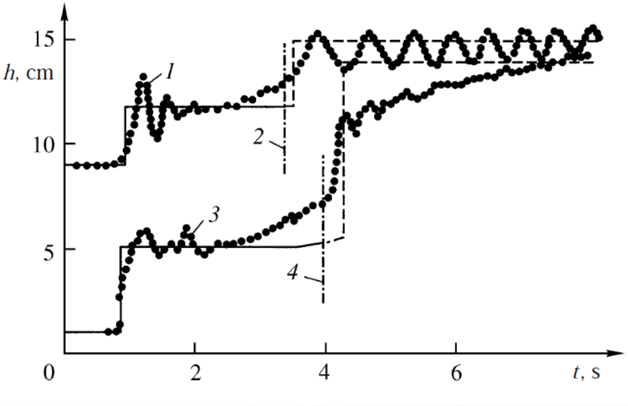


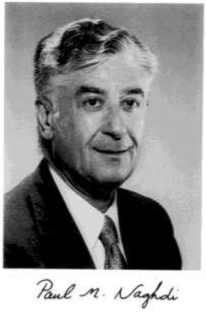
Figure: Wave height records.

Main constitutive assumptions:

- Free surface is a graph: $y = \eta(\boldsymbol{x}, t)$
- Long wave: $\lambda \gg d$
- Nonlinearity is finite: $\varepsilon \equiv a/d \propto \mathcal{O}(1)$

Credits:

- Lord RAYLEIGH (1876) [3] (only *steady* version)
- François SERRE (1953) [4]
- C. SU & C. GARDNER (1969) [5]
- A. GREEN & P. NAGHDI (1976) [6]
- E. PELINOVSKY & ZHELEZNYAK (1985) [7]
- ...
- Nora AISSIOUENE. (15 June 2015) ☺



- Total water depth: $h(x,t) \;:=\; d \;+\; \eta(x,t)$

Assumption:

- Depth-averaged profile:

$$u(x,y,t) \;\approx\; \bar{u}(x,t) \;\equiv\; \frac{1}{h}\int_{-d}^{\eta} u(x,y,t)\,\mathrm{d}y$$

Incompressibility yields:

$$u_x \;+\; v_y \;=\; 0 \Longrightarrow \quad v(x,y,t) \;\approx\; -(y+d)\bar{u}_x$$

The energy can be computed:

Kinetic: $\frac{\mathscr{K}}{\rho} \;=\; \frac{1}{2}\int_{-d}^{\eta}(u^2+v^2)\,\mathrm{d}y \;\approx\; \frac{1}{2}h\,\bar{u}^2 \;+\; \frac{1}{6}h^3\,\bar{u}_x^2$

Potential: $\frac{\mathscr{V}}{\rho} \;=\; \int_{-d}^{\eta} g\,(y+d)\,\mathrm{d}y \;=\; \frac{1}{2}g\,h^2$

- The action is:

$$\frac{\mathcal{S}}{\rho} = \int_{t_1}^{t_2}\int_{x_1}^{x_2}\Big[\frac{h\,\bar{u}^2}{2} + \frac{h^3\,\bar{u}_x^2}{6} - \frac{g\,h^2}{2} + \{\,h_t + [\,h\,\bar{u}\,]_x\,\}\,\tilde{\phi}\Big]\,\mathrm{d}x\,\mathrm{d}t$$

- Euler–Lagrange equations:

$$\begin{aligned}
\delta\tilde{\phi} : 0 &= h_t + [\,h\,\bar{u}\,]_x\,,\\
\delta\bar{u} : 0 &= \tilde{\phi}\,h_x - [\,h\,\tilde{\phi}\,]_x - \tfrac{1}{3}[\,h^3\,\bar{u}_x\,]_x + h\,\bar{u},\\
\delta h : 0 &= \tfrac{1}{2}\,\bar{u}^2 - g\,h + \tfrac{1}{2}\,h^2\,\bar{u}_x^2 - \tilde{\phi}_t + \tilde{\phi}\,\bar{u}_x - [\,\bar{u}\,\tilde{\phi}\,]_x.
\end{aligned}$$

- After some simple algebra:

$$\begin{aligned}
\tilde{\phi}_x &= \bar{u} - \tfrac{1}{3}\,h^{-1}\,[\,h^3\,\bar{u}_x\,]_x,\\
\tilde{\phi}_t &= \tfrac{1}{2}\,h^2\,\bar{u}_x^2 - \tfrac{1}{2}\,\bar{u}^2 - g\,h + \tfrac{1}{3}\,\bar{u}\,h^{-1}\,[\,h^3\,\bar{u}_x\,]_x.
\end{aligned}$$

$$\Longrightarrow \quad \tilde{\phi}_{tx} = \tilde{\phi}_{xt}\ !$$

- Mass conservation:

$$h_t \;+\; [\,h\,\bar{u}\,]_x \;=\; 0$$

- Momentum conservation:

$$[\,h\,\bar{u}\,]_t \;+\; \Bigl[\,h\,\bar{u}^{\,2} \;+\; \tfrac{1}{2}\,g\,h^2 \;+\; \tfrac{1}{3}\,h^2\,\tilde{\gamma}\,\Bigr]_x \;=\; 0$$

$$\tilde{\gamma} \;\equiv\; h\Bigl[\,\bar{u}_x^2 \;-\; \bar{u}_{xt} \;-\; \bar{u}\,\bar{u}_{xx}\,\Bigr] \;=\; 2\,h\,\bar{u}_x^2 \;-\; h\,\partial_x\,[\,\bar{u}_t \;+\; \bar{u}\,\bar{u}_x\,]$$

- Surface tangential momentum:

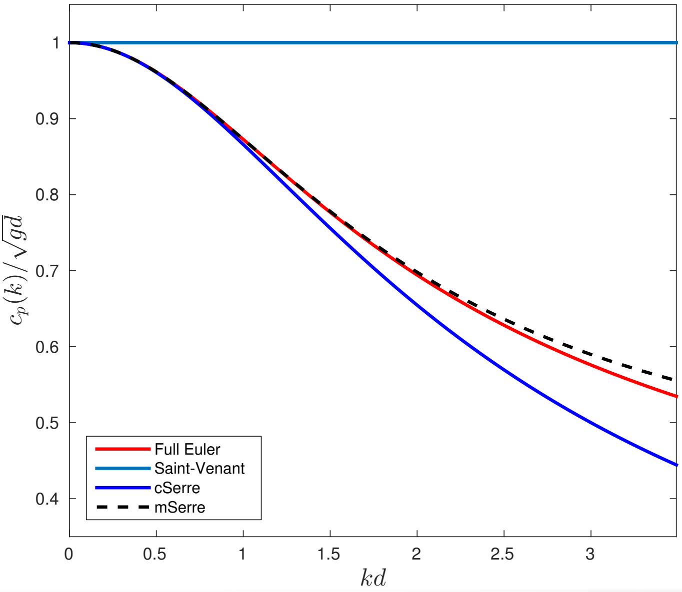
$$\Bigl[\,\bar{u} \;-\; \tfrac{1}{3}\,h^{-1}(h^3\bar{u}_x)_x\,\Bigr]_t + \Bigl[\,\tfrac{1}{2}\,\bar{u}^2 + g\,h - \tfrac{1}{2}\,h^2\,\bar{u}_x^2 - \tfrac{1}{3}\,\bar{u}\,h^{-1}(h^3\bar{u}_x)_x\,\Bigr]_x = 0$$

- Energy conservation:

$$\Bigl[\,\tfrac{1}{2}\,h\,\bar{u}^2 \;+\; \tfrac{1}{6}\,h^3\bar{u}_x^2 \;+\; \tfrac{1}{2}\,g\,h^2\,\Bigr]_t \;+\; \Bigl[\,(\tfrac{1}{2}\,\bar{u}^2 \;+\; \tfrac{1}{6}\,h^2\,\bar{u}_x^2 + \;g\,h \;+\; \tfrac{1}{3}\,h\,\gamma\,)\,h\,\bar{u}\,\Bigr]_x \;=\; 0$$

Comparison of dispersion relations - II

look at the black dotted line



- Introduction of scaled variables ($\partial_{x,t} = \mathcal{O}(\sigma)$):

$$x^\star = \sigma\, x, \quad t^\star = \sigma\, t, \qquad \tilde{\gamma}^\star = \sigma^{-2}\,\tilde{\gamma}$$

- Derived above vertical acceleration at free surface:

$$\tilde{\gamma}_{\mathrm{old}} = 2\sigma^2\, h\,\bar{u}_{x^\star}^2 - \sigma^2 h\,\partial_{x^\star}\left[\,\bar{u}_{t^\star} + \bar{u}\,\bar{u}_{x^\star}\,\right]$$

- Non-conservative form of momentum equation:

$$\bar{u}_{t^\star} + \bar{u}\,\bar{u}_{x^\star} = -\,g\,h_{x^\star} - \sigma^2\,\tfrac{1}{3}\,h^{-1}\,\partial_{x^\star}\Big[\,h^2\,\tilde{\gamma}^\star\,\Big],$$

- New *asymptotically equivalent* acceleration:

$$\tilde{\gamma}_{\mathrm{new}} = \sigma^2\,2\,h\,\bar{u}_{x^\star}^2 + \sigma^2\,g\,h\,h_{x^\star x^\star} + \mathcal{O}(\sigma^4),$$

- Bona–Smith [8] & Nwogu [9] trick ($0 \leq \alpha \leq 1$):

$$\tilde{\gamma}_\alpha := \alpha\tilde{\gamma}_{\mathrm{old}} + (1-\alpha)\tilde{\gamma}_{\mathrm{new}} = 2\,\sigma^2\,h\,\bar{u}_{x^\star}^2 + (1-\alpha)\,\sigma^2\,g\,h\,h_{x^\star x^\star} - \alpha\,\sigma^2\,h\,\partial_{x^\star}[\,\bar{u}_{t^\star} + \bar{u}\,\bar{u}_{x^\star}\,] + \mathcal{O}(\sigma^4)$$

- Modified Serre–(Green–Naghdi) equations:

$$h_t + [h\bar{u}]_x = 0,$$

$$[h\bar{u}]_t + \Big[h\bar{u}^2 + \tfrac{1}{2}gh^2 + \tfrac{1}{3}h^2\tilde{\gamma}_\alpha\Big]_x = 0.$$

$$\tilde{\gamma}_\alpha = 2h\bar{u}_x^2 + (1-\alpha)ghh_{xx} - \alpha h\partial_x[\bar{u}_t + \bar{u}\bar{u}_x]$$

Dispersion relation analysis:

- mSerre system:

$$\frac{c^2}{gd} = \frac{3 + (\alpha-1)(kd)^2}{3+\alpha(kd)^2} = 1 - \tfrac{1}{3}(kd)^2 + \tfrac{1}{9}\alpha(kd)^4 - \tfrac{1}{27}\alpha^2(kd)^6 + \cdots$$

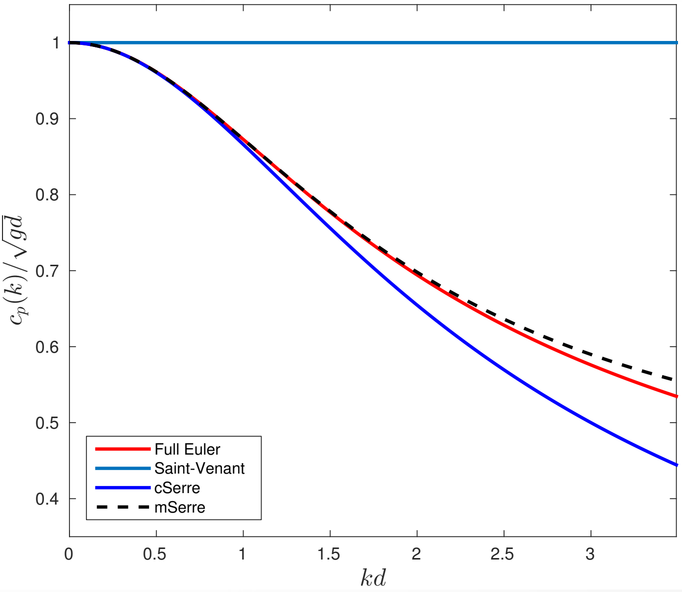
- Full Euler:

$$\frac{c^2}{gd} = \frac{\tanh(kd)}{kd} = 1 - \tfrac{1}{3}(kd)^2 + \tfrac{2}{15}(kd)^4 - \tfrac{17}{315}(kd)^6 + \cdots$$

An *optimal* value: $\alpha = 6/5 = 1.2$!

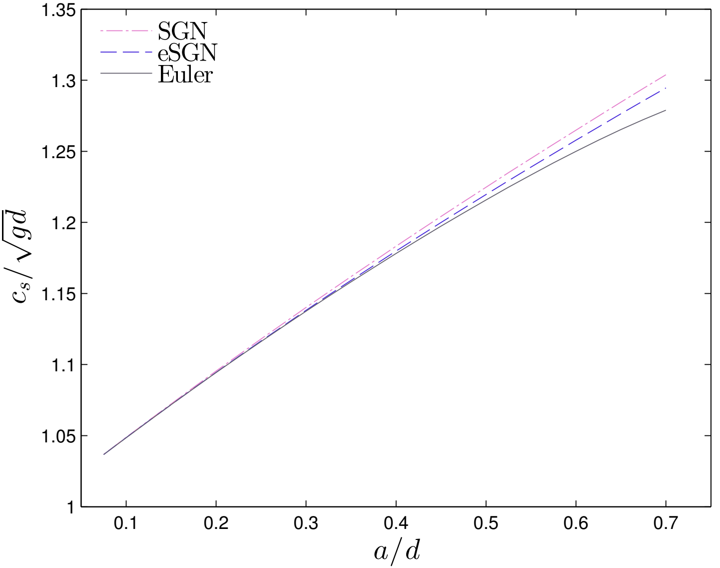
Comparison of dispersion relations - II

look at the black dotted line



Performance of the modified Serre model - I

Speed-amplitude relation [10]



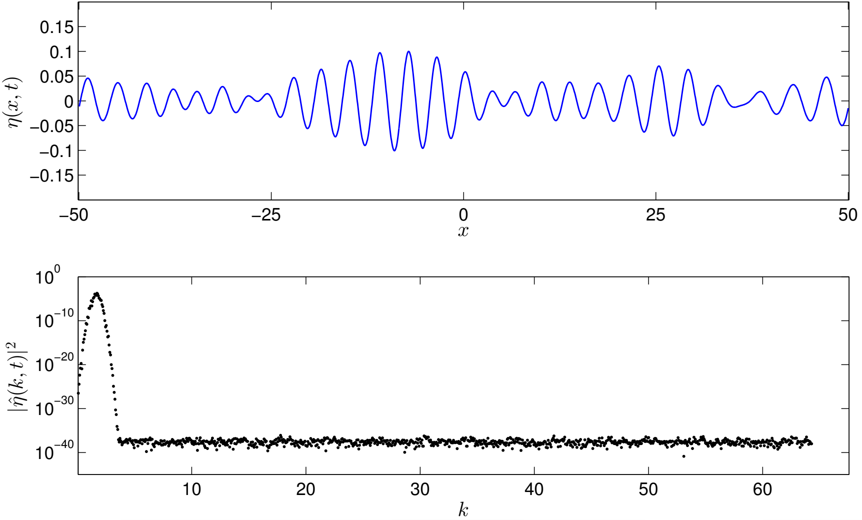


Figure: *Initial condition (up) along with its Fourier power spectrum (down).*

Performance of the modified Serre model - II

Comparison of the classical, modified Serre and the full Euler equations [10]

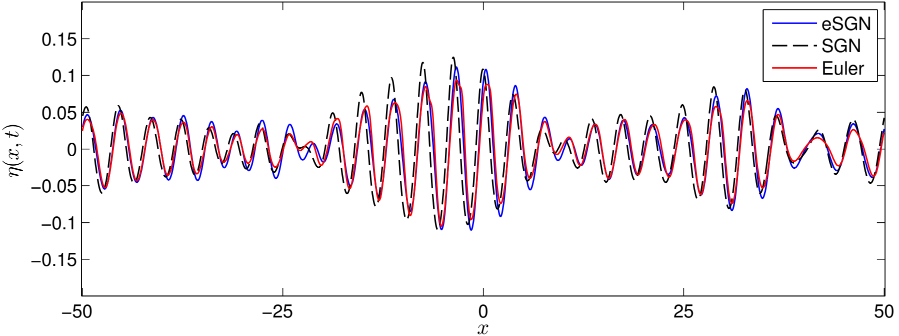


Figure: Free surface elevations at $t = 10\sqrt{d/g}$.

Performance of the modified Serre model - II

Comparison of the classical, modified Serre and the full Euler equations [10]

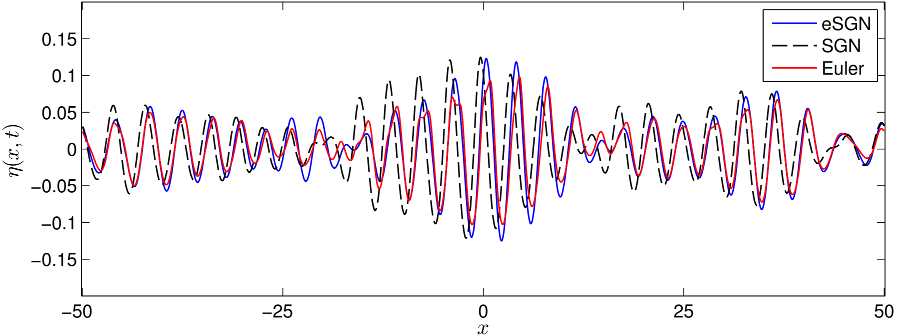


Figure: Free surface elevations at $t = 20\sqrt{d/g}$.

Performance of the modified Serre model - II

Comparison of the classical, modified Serre and the full Euler equations [10]

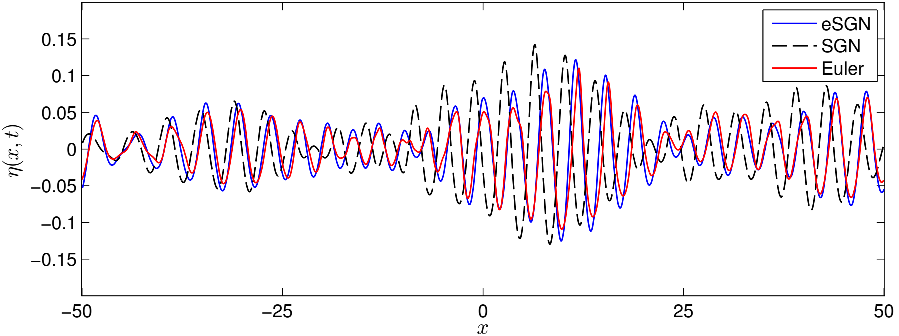


Figure: Free surface elevations at $t = 40\sqrt{d/g}$.

Performance of the modified Serre model - II

Comparison of the classical, modified Serre and the full Euler equations [10]

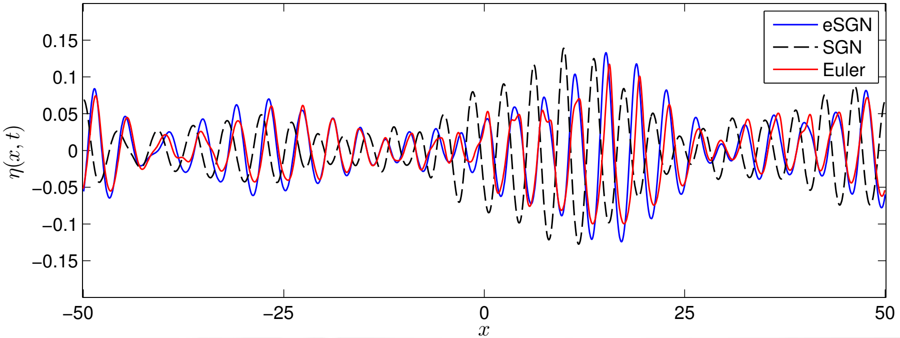


Figure: Free surface elevations at $t = 50\sqrt{d/g}$.

Performance of the modified Serre model - II

Comparison of the classical, modified Serre and the full Euler equations [10]

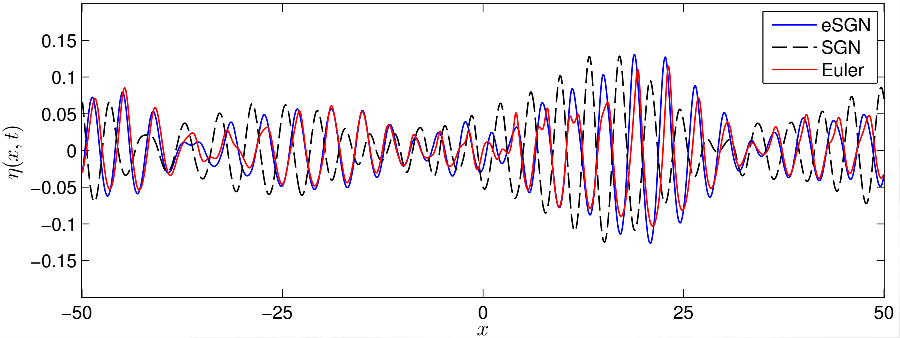


Figure: *Free surface elevations at $t = 60\sqrt{d/g}$.*

Performance of the modified Serre model - II

Comparison of the classical, modified Serre and the full Euler equations [10]

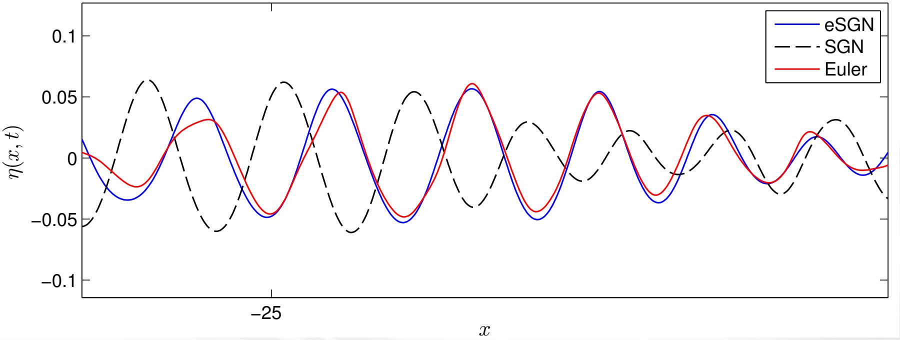


Figure: Zoom on free surface elevations at $t = 60\sqrt{d/g}$.

The mSerre system is not perfect (as well as Nwogu's system)!

There is at least one important issue

- Galilean invariance: OK ☺
- Energy conservation: presumably NO ☹

- The Lagrangian density is:

$$\mathscr{L} = \frac{h\,\bar{u}^2}{2} + \frac{h^3\,\bar{u}_x^2}{6} - \frac{g\,h^2}{2} + \{\, h_t + [\,h\,\bar{u}\,]_x \,\}\,\tilde{\phi}$$

- Introduce explicitly the vertical acceleration:

$$\mathscr{L} = \frac{h\,\bar{u}^2}{2} + \frac{h^2\,\tilde{\gamma}}{12} + \frac{h^3}{12}\,[\,\bar{u}_t + \bar{u}\,\bar{u}_x\,]_x - \frac{g\,h^2}{2} + \{\, h_t + [\,h\,\bar{u}\,]_x \,\}\,\tilde{\phi}$$

- Substitute the relaxed version $\tilde{\gamma}_\alpha$ and simplify ($\beta \equiv 1 - \alpha$):

$$\mathscr{L}' = \frac{h\,\bar{u}^2}{2} + \frac{(2+3\beta)\,h^3\,\bar{u}_x^2}{12} - \frac{g\,h^2}{2} - \frac{\beta\,g\,h^2\,h_x^2}{4} + \{\, h_t + [\,h\,\bar{u}\,]_x \,\}\,\tilde{\phi}$$

- New expressions for the kinetic and potential energies:

$$\frac{\mathscr{K}'}{\rho} = \frac{h\,\bar{u}^2}{2} + \frac{(2+3\beta)\,h^3\,\bar{u}_x^2}{12}, \qquad \frac{\mathscr{V}'}{\rho} = \frac{g\,h^2}{2}\left(1 + \frac{\beta\,h_x^2}{2}\right)$$

- The Euler–Lagrange equations:

$$\begin{aligned}
\delta\tilde{\phi} : 0 &= h_t + [\,h\,\bar{u}\,]_x\,, \\
\delta\bar{u} : 0 &= h\,\bar{u} + \tilde{\phi}\,h_x - \Big[\,h\,\tilde{\phi}\,\Big]_x - \big(\tfrac{1}{3} + \tfrac{1}{2}\beta\big)\Big[\,h^3\,\bar{u}_x\,\Big]_x\,, \\
\delta h : 0 &= \tfrac{1}{2}\,\bar{u}^2 - g\,h - \tilde{\phi}_t + \tilde{\phi}\,\bar{u}_x - [\,\bar{u}\,\tilde{\phi}\,]_x \\
&\quad + \big(\tfrac{1}{2} + \tfrac{3}{4}\beta\big)\,h^2\,\bar{u}_x^2 - \tfrac{1}{2}\,\beta\,g\,h\,h_x^2 + \tfrac{1}{2}\,\beta\,g\,[\,h^2\,h_x\,]_x
\end{aligned}$$

- After eliminating the velocity potential:

$$\begin{aligned}
\tilde{\phi}_x &= \bar{u} - \big(\tfrac{1}{3} + \tfrac{1}{2}\beta\big)\,h^{-1}\Big[\,h^3\,\bar{u}_x\,\Big]_x\,, \\
\tilde{\phi}_t &= \tfrac{1}{2}\,\bar{u}^2 - \bar{u}\,\tilde{\phi}_x - g\,h + \big(\tfrac{1}{2} + \tfrac{3}{4}\beta\big)\,h^2\,\bar{u}_x^2 + \tfrac{1}{2}\,\beta\,g\,h\,[\,h\,h_x\,]_x
\end{aligned}$$

- These equations have the full set of conservation laws:

$$\begin{aligned}
&\partial_t\Big[\,\tfrac{1}{2}\,h\,\bar{u}^2 + \tfrac{1}{12}\,(2+3\beta)\,h^3\,\bar{u}_x^2 + \tfrac{1}{2}\,g\,h^2 + \tfrac{1}{4}\,\beta\,g\,h^2\,h_x^2\,\Big] \\
&\qquad + \partial_x\Big[\,\big(\tfrac{1}{2}\,\bar{u}^2 + \tfrac{1}{12}\,(2+3\beta)\,h^2\,\bar{u}_x^2 + g\,h + \tfrac{1}{3}\,h\,\Gamma\,\big)\,h\,\bar{u}\,\Big] = 0
\end{aligned}$$

- The mass conservation yields:

$$\bar{u} = -c\,d\,/\,h$$

- After substituting it in momentum conservation and some algebra:

$$\left(\frac{\mathrm{d}\,h}{\mathrm{d}x}\right)^2 = \frac{\mathcal{F} - (1 + \mathcal{C}_2 + 2\mathcal{F})\,(h/d) + (2 + 2\mathcal{C}_1 + \mathcal{F})\,(h/d)^2 - (h/d)^3}{\left(\frac{1}{3} + \frac{1}{2}\beta\right)\mathcal{F} - \frac{1}{2}\,\beta\,(h/d)^3}$$

Solvable analytically with Jacobi elliptic functions!

- For solitary waves $\mathcal{C}_1 = \mathcal{C}_2 = 0$:

$$\left(\frac{\mathrm{d}\,\eta}{\mathrm{d}x}\right)^2 = \frac{(\mathcal{F} - 1)\,(\eta/d)^2 - (\eta/d)^3}{\left(\frac{1}{3} + \frac{1}{2}\beta\right)\mathcal{F} - \frac{1}{2}\,\beta\,(1 + \eta/d)^3}$$

Solvable in *elementary functions* in parametric form!

- Introduce a new independent variable:

$$x(\xi) \;=\; \int_0^{\xi} \left| \frac{(\beta + 2/3)\mathcal{F} \;-\; \beta\, h^3(\xi')/d^3}{(\beta + 2/3)\,\mathcal{F} \;-\; \beta} \right|^{1/2} \mathrm{d}\xi'$$

- The analytical solution:

$$\frac{\eta(\xi)}{d} \;=\; (\mathcal{F} - 1)\,\mathrm{sech}^2\left(\frac{\kappa\,\xi}{2}\right), \qquad (\kappa d)^2 \;=\; \frac{6\,(\mathcal{F} - 1)}{(2 + 3\beta)\,\mathcal{F} \;-\; 3\,\beta}$$

- Wave-amplitude relation:

$$a \;\equiv\; \eta(0) \;=\; (\mathcal{F} - 1)d$$

How to choose the free parameter β?

- The same *linear dispersion relation* analysis applies here!
- ...new ideas?

- By following the classical work of McCowan (1891) [11]:

$$\frac{c^2}{gd} = \frac{\tan(\kappa d)}{\kappa d} = 1 + \frac{(\kappa d)^2}{3} + \frac{2\,(\kappa d)^4}{15} + \frac{17\,(\kappa d)^6}{315} + \frac{62\,(\kappa d)^8}{2835} - \cdots$$

- The same result for the mSerre system:

$$\frac{c^2}{gd} = \frac{2 - \beta\,(\kappa d)^2}{2 - (\frac{2}{3} + \beta)\,(\kappa d)^2} \approx 1 + \frac{(\kappa d)^2}{3} + \left(\frac{1}{3} + \frac{\beta}{2}\right)\frac{(\kappa d)^4}{3} + \cdots \quad (\star)$$

Meromorphic interpolation scheme:

- $\tan(z)$ is meromorphic with single poles at $\kappa d = \pm\pi/2, \pm 3\pi/2, \cdots$
- $(\star)$ has a single pole at $\kappa d = \pm\sqrt{6/(2+3\beta)}$
- The first poles at $\kappa d = \pm\pi/2$ coincide if

$$\beta = \tfrac{2}{3}\left(12\,\pi^{-2} - 1\right) \approx 0.1439$$

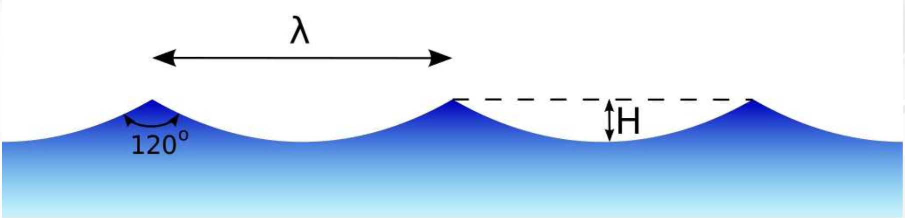
Highest wave
Sir George Gabriel Stokes (1819 – 1903)

In 1880 Stokes showed [12]:

- Existence of a limiting wave
- The angle at the crest is 120°

Reference (see Appendix B)):

Stokes, G.G. (1880). *Supplement to a paper on the theory of oscillatory waves*. Math. Phys. Pap., **1**, 314–326.



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Contribution à l'étude des ondes longues irrotationnelles

Contribution to the study of long irrotational waves

PAR F. SERRE
INGÉNIEUR DOCTEUR
INGÉNIEUR A L'OMNIUM FRANÇAIS D'ÉTUDES ET DE RECHERCHES

Essentially unknown F. Serre's result

Determination of the highest solitary wave [13]

Le nombre de Froude F_0, correspondant à l'onde solitaire limite, est donné par :

$$F_0^2 = \frac{\operatorname{tg} 2\sigma_0}{2\sigma_0} \ ;$$

on a :

$$1{,}76558 < F_0^2 < 1{,}76707$$

d'où $\boxed{F_0^2 = 1{,}766}$ avec trois décimales exactes; la profondeur d'eau limite à la crête s'en déduit par :

$$\frac{y_c}{y_0} = \zeta_c = 1 + \frac{F_0^2}{2} = \underline{\underline{1{,}883}}.$$

Remarks:

- Result obtained without any computers
- All digits are correct!

The world record in this domain belongs to…

Dmitri V. Maklakov (Kazan State University, Russia) [14]

Euro. Jnl of Applied Mathematics (2002), *vol.* 13, *pp.* 67–93. 67
DOI: 10.1017/S0956792501004739 Printed in the United Kingdom

Almost-highest gravity waves on water of finite depth

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The world record in this domain belongs to...

Dmitri V. Maklakov (Kazan State University, Russia) [14]

Table 8. *Properties of the highest solitary waves*

	Current results	Williams (1981)	Evans & Ford (1996)	Zhitnikov (2000)
$\bar{c}^*_{sol}$	1.2908904558	1.290889	1.29089053	1.290890455863
$\bar{H}^*_{sol}$	0.8331990844	0.833197	0.833199179	0.833199084520
$\bar{M}^*_{sol}$	1.970320658	1.970319	1.97032019	1.970320660132
$\bar{I}^*_{sol}$	2.543468132	2.543463	2.54346767	2.543468135155
$\bar{C}^*_{sol}$	1.714569239	1.714569	1.71456873	1.714569240534
$\bar{T}^*_{sol}$	0.535008835	0.535005	0.535008913	0.535008835971
$\bar{V}^*_{sol}$	0.437672693	0.437670	0.437672702	0.437672693444
$\bar{V}_{sol} + \bar{T}^*_{sol}$	0.972681528	0.972675	0.972681614	0.972681529415

- Taylor expansions around the crest yield

$$x = \left[\frac{(2+3\beta)\,\mathcal{F} \;-\; 3\,\beta\,\mathcal{F}^3}{(2+3\beta)\,\mathcal{F} \;-\; 3\,\beta_2}\right]^{\frac{1}{2}} \xi + \mathcal{O}(\xi^3), \quad \eta = a\left[1 \;-\; \tfrac{1}{4}\,(\kappa\xi)^2\right] + \mathcal{O}(\xi^4)$$

- Peaked crests occur when $\mathrm{d}x/\mathrm{d}\xi = 0$ at $\xi = 0$. It happens if

$$\mathcal{F}^2 \;=\; 1 \;+\; \frac{2}{3\,\beta}$$

- The slopes at the crest are:

$$\left.\frac{\mathrm{d}\,\eta}{\mathrm{d}x}\right|_{x=0^\pm} = \left.\frac{\mathrm{d}\,\eta}{\mathrm{d}\xi}\,\middle/\,\frac{\mathrm{d}\,x}{\mathrm{d}\xi}\right|_{\xi=0^\pm} = \mp\,\frac{\kappa\,a\sqrt{\mathcal{F}^2 + \mathcal{F} + 1}}{\sqrt{3}\,\mathcal{F}} = \mp\,\frac{\sqrt{2\,/\,3\,\beta}\,a}{a + d}$$

We can compare with the full Euler prediction!

Comparison of various choices of the free parameter β

Multiple choices are possible

Full Euler prediction:

$$\mathcal{F} \approx 1.290, \quad a/d \approx 0.8331, \quad \theta = 120^\circ$$

- Linear dispersion relation ($\beta = 2/15$, Taylor correct to $\mathcal{O}((kd)^4)$):
$$\mathcal{F} = \sqrt{6} \approx 2.45, \quad a/d = \sqrt{6} - 1 \approx 1.45, \quad \theta \approx 28.4^\circ$$
- Meromorphic interpolation scheme ($\beta = \frac{2}{3}(12\pi^{-2} - 1)$):
$$\mathcal{F} \approx 2.37, \quad a/d = \sqrt{6} - 1 \approx 1.37, \quad \theta \approx 37.3^\circ$$
- Inner angle interpolation scheme:
$$\beta = \frac{4}{3}\,\frac{\beta + \sqrt{2\,\beta} - 1}{\beta - \sqrt{8\,\beta} + 2} \quad \Longrightarrow \quad \beta \approx 0.3456$$
$$\mathcal{F} \approx 1.711, \quad a/d \approx 0.711, \quad \theta \approx 120^\circ$$

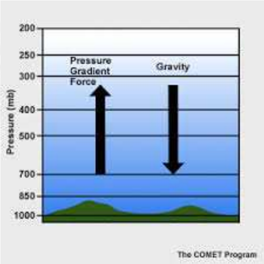
$<$ 15% relative error in amplitude!

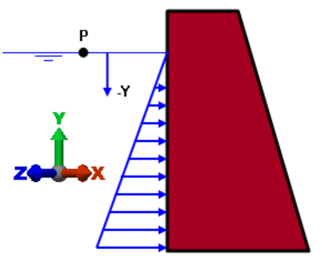
Attention, these are not synonyms:

Dispersive $\neq$ Non-hydrostatic

Counter-examples can be constructed [15]:

There are non-hydrostatic models, which are non-dispersive!





Thank you for your attention!

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