Well-balanced and Asymptotic Schemes for Friedrichs systems

B. Després (LJLL-University Paris VI)
C. Buet (CEA)

thanks to
T. Leroy (PhD CEA), X. Valentin (PhD CEA) , C. Enaux (CEA),
P. Lafitte (Centrale Paris) and E. Franck (Inria)

Section 1

Introduction

Discretize $\partial_t U + \nabla \cdot f(U) = S(U, \dots)$ with **FV**-P^0-cell centered schemes

$$\frac{U_j^{n+1} - U_j^n}{\Delta t} + \frac{f_{j+\frac{1}{2}}^n - f_{j-\frac{1}{2}}^n}{\Delta x} = S(U_j^n, \dots)$$

Tentative definition :

WB-FV scheme = exact for stationary solutions
AP (asymptotic preserving) schemes = WB with stiffness.

- Literature on WB : it is a "one case after another" strategy.
- Can we have a global view (classification) on WB methods ?

Some references

- Cargo-LeRoux, A WB for a model of an atmosphere with gravity CRAS I 318 (1994)
- Greenberg-Leroux, A WB scheme for the numerical processing of source terms in hyp. eq., 1996.
- Audusse-Bouchut-Bristeau-Klein-Perthame, A fast and stable well-balanced scheme with hydrostatic reconstruction for shallow water flows, (2004).
- Bouchut. Nonlinear stability of finite volume methods for hyperbolic conservation laws, and well-balanced schemes for sources, Frontiers in Mathematics series, 2004.
- Coquel-Godlewski. Asymptotic preserving scheme for Euler system with large friction, 2011.
- T. Muller and A. Pfeiffer, Well-balanced simulation of geophysical flows via the shallow water equations with bottom topography : consistency and numerical computations, 2014.
- Larsen-Morel-Miller, Asymptotic solution of num. transport solutions in optical thick regimes 1987'.
- L. Gosse and G. Toscani, An AP WB scheme for the hyp. heat equations. CRAS I, 2002.
- S. Jin, AP schemes for multiscale kinetic and hyperbolic equations : a review, 2010.
- Buet-Després-Franck Design of AP schemes for the hyp. heat equation on unstructured meshes, 2012.
- Buet-Després-Franck, AP Schemes on Distorted Meshes for Friedrichs Systems with Stiff Relaxation : Application to Angular Models in Linear Transport, 2014.
- L. Gosse, Computing Qualitatively Correct Approximations of Balance Laws . . ., 2013.

Model problem : Friedrichs systems with relaxation

- A paradigm is the p-system with friction : linearization yields

$$\left\{\begin{array}{l}\partial_t\tau-\partial_x u=0,\\ \partial_t u+\partial_x p(\tau)=g-\sigma u.\end{array}\right. \Longrightarrow \left\{\begin{array}{l}\partial_t\tau_1-\partial_x u_1=0,\\ \partial_t u_1-\partial_x\left(c_0(x)^2\tau_1\right)=-\sigma u_1.\end{array}\right.$$

One gets the linear hyperbolic heat equation

$$\partial_t U+\partial_x(A(x)U)=-R(x)U, \qquad U=(c_0(x)\tau_1,u_1)^t,$$

where $A=-c_0(x)\begin{pmatrix}0&1\\1&0\end{pmatrix}=A^t$ and $R=\begin{pmatrix}0&c_0'(x)\\0&\sigma\end{pmatrix}$.

Notice the compatibility relation $R+R^t+A'(x)\geq 0$

$$\partial_t\frac{|U|^2}{2}+\partial_x\frac{(U,A(x)U)}{2}=-\frac{1}{2}\left(R(x)+R(x)^t+A'(x)U,U\right)\leq 0.$$

S_n and DOM

- Another example is DOM (Discrete Ordinates Method) for transfer

$$\partial_t I + \mu \partial_x I = \sigma \left(< I > - I \right).$$

Approximate

$$I(x,t,\mu) = \sum_{i=1}^{n} w_i f_i(x,t) \delta(\mu - \mu_i) + \sum_{i=1}^{n} w_i g_i(x,t) \delta(\mu + \mu_i).$$

Normalize $U = (\sqrt{\text{diag}(w_i)}\ f, \sqrt{\text{diag}(w_i)}\ g)^t \in \mathbb{R}^{2n}$, with $\mathbf{w} = (\sqrt{w_1}, \ldots, \sqrt{w_n}, \sqrt{w_1}, \ldots, \sqrt{w_n}) \in \mathbb{R}^{2n}$.

One gets the $\mathbf{S}_n$ **model**

$$\partial_t U + A \partial_x U = -RU, \qquad A = A^t, \quad R = -\mathbf{w} \otimes \mathbf{w} + I_d,$$

Main idea

• **Generic model problem** is

$$\partial_t U + \partial_x (A(\mathbf{x})U) + \partial_y (B(\mathbf{x})U) = -R(\mathbf{x})U, \quad U(t,\mathbf{x}) \in \mathbb{R}^n,$$

where $\mathbf{x} = (x,y) \in \mathbb{R}^2$, $A(\mathbf{x}) = A(\mathbf{x})^t$, $B(\mathbf{x}) = B(\mathbf{x})^t$.

• The vectorial space of stationary states, needed for WB analysis, is : $\mathcal{U} = \{\mathbf{x} \mapsto U(\mathbf{x});\ \ \partial_x (A(\mathbf{x})U) + \partial_y (B(\mathbf{x})U) = -R(\mathbf{x})U\}$.

Main idea - consider the adjoint equation :

$$\partial_t V + A(\mathbf{x})^t \partial_x V + B(\mathbf{x})^t \partial_y V = R(\mathbf{x})^t V, \quad V(t,\mathbf{x}) \in \mathbb{R}^n.$$

Magic property - this is just duality :

$$\partial_t (U,V) + \partial_x (A(\mathbf{x})U, V) + \partial_y (B(\mathbf{x})U, V) = 0.$$

Strategy

- The vectorial space of **dual stationary states** is

$$\mathcal{V} = \left\{\mathbf{x} \mapsto V(\mathbf{x});\ \ A(\mathbf{x})^t \partial_x V + B(\mathbf{x})^t \partial_y V = R(\mathbf{x})^t U\right\}.$$

- Pick up $V_p \in \mathcal{V}$ and define $\alpha_p = (U, V_p) \in \mathbb{R}$ which is solution of a **conservative equation**

$$\partial_t \alpha_p + \partial_x (A(\mathbf{x})U, V_p) + \partial_y (B(\mathbf{x})U, V_p) = 0$$

- a) In 1D, assemble the system of conservation laws for $\alpha = (\alpha_p) \in \mathbb{R}^{\dim(\mathcal{V})} = \mathbb{R}^{\dim(\mathcal{U})}$.
- b) Discretize the new system of conservation laws.
- c) Write the new scheme for the original variable U.

For simplicity : A is constant with $\det(A) \neq 0$

So $\dim(\mathcal{V}) = \dim(\mathcal{U}) = n$.

$A^t \partial_x V = R^t V \Longleftrightarrow \partial_x V = A^{-t} R^t V \Longleftrightarrow V(x) = e^{A^{-t}R^t x} V(0)$
where $e^{A^{-t}R^t x}$ is a **matrix exponential**.

Proposition : Define the change-of-basis matrix $P(x) = e^{RA^{-1}x}$, and the change of unknown $\alpha = P(x)U \Longleftrightarrow U = P^{-1}(x)\alpha$.

The new conservative system rewrites

$$\partial_t \alpha + \partial_x (Q(x)\alpha) = 0$$

with a matrix defined through a global change of basis

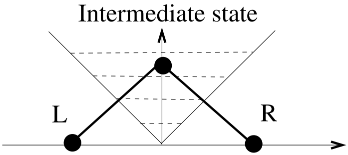
$$Q(x) = P(x) A P^{-1}(x)$$

Note $Q(x)$ is similar to A, so has the same eigenvalues.
But the eigenvectors depend on x.

Discretization

• For additional simplification (once again), A is also **symmetric** : $A = A^t \in \mathbb{R}^{n\times n}$. One has the spectral decomposition $Au_p = \lambda_p u_p$, $\lambda_p \neq 0$

$$\boxed{\begin{cases} \left(u_p, U^*_{j+\frac{1}{2}} - U_j\right) = 0, & \lambda_p > 0, \\ \left(u_p, U^*_{j+\frac{1}{2}} - U_{j+1}\right) = 0, & \lambda_p < 0. \end{cases}}$$



The intermediate state is computed in function of $L = j$ and $R = j + 1$.

• The FV discretization of $\partial_t \alpha + \partial_x \beta = 0$ with $\beta = Q(x)\alpha$ writes

$$\frac{\alpha_j^{n+1} - \alpha_j^n}{\Delta t} + \frac{\beta^n_{j+\frac{1}{2}} - \beta^n_{j-\frac{1}{2}}}{\Delta x_j} = 0$$

where $\beta^n_{j+\frac{1}{2}}$ is the flux at time step $t_n = n\Delta t$.

Diagonalisation of Q

The **right** and **left** spectral decompositions of $Q^* = Q(x^*) = P(x^*)AP(x^*)^{-1} \neq Q^*$ are

$$\left\{ \begin{array}{ll} Q(x^*)r_p^* = \lambda_p r_p^*, & r_p^* = P(x^*)u_p, \\ Q^t(x^*)s_p^* = \lambda_p s_p^*, & s_p^* = P(x^*)^{-t}u_p. \end{array} \right.$$

The solution at interface of the system $\partial_t \left(s_p^*, \beta\right) + \lambda_p \partial_x \left(s_p^*, \beta\right) = 0$ yields a family of Riemann solvers.

One-state solvers : defined by the well-posed linear system

$$\boxed{\left\{ \begin{array}{ll} \left(s_p^*, \beta^* - \beta_L\right) = 0, & \lambda_p > 0, \\ \left(s_p^*, \beta^* - \beta_R\right) = 0, & \lambda_p < 0. \end{array} \right.}$$

It defines a first Riemann solver $\varphi : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$

$$\varphi(\beta_L, \beta_R, x^*) = \beta^*.$$

The WB scheme with the one-state solver

- First formulation with the global change of basis

$$\frac{\alpha_j^{n+1}-\alpha_j^n}{\Delta t} + \frac{\varphi(\beta_j^n,\beta_{j+1}^n,x_{j+\frac12})-\varphi(\beta_{j-1}^n,\beta_j^n,x_{j-\frac12})}{\Delta x_j} = 0$$

- Second formulation $\frac{U_j^{n+1}-U_j^n}{\Delta t} + P(x_j)^{-1}\frac{\varphi(\beta_j^n,\beta_{j+1}^n,x_{j+\frac12})-\varphi(\beta_{j-1}^n,\beta_j^n,x_{j-\frac12})}{\Delta x_j} = 0$
- Third purely local formulation, the one for implementation,

$$\frac{U_j^{n+1}-U_j^n}{\Delta t} + A\frac{U^*_{j+\frac12}-U^*_{j-\frac12}}{\Delta x_j}$$

$$+\frac{P(x_j)^{-1}P(x_{j+\frac12})-I}{\Delta x_j}AU^*_{j+\frac12} + \frac{I-P(x_j)^{-1}P(x_{j-\frac12})}{\Delta x_j}AU^*_{j-\frac12} = 0.$$

where the flux $U^*_{j+\frac12}$ is solution of the linear system (with $\Delta x_j^{\pm} = x_{j\pm\frac12}-x_j$)

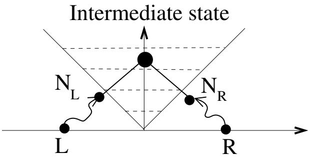
$$\begin{cases} \left(u_p, U^*_{j+\frac12} - e^{-A^{-1}R\Delta x_j^+}U_j\right) = 0, & \lambda_p > 0, \\ \left(u_p, U^*_{j+\frac12} - e^{-A^{-1}R\Delta x_{j+1}^-}U_{j+1}\right) = 0, & \lambda_p < 0. \end{cases}$$

Property : the scheme is WB.

Proof : If $\beta_j = \beta$ for all j, the solution is stationary in the first formulation.

Schematic

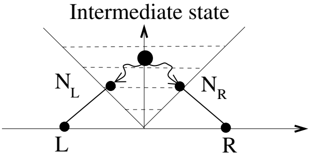
- Structure of the one-state solver.



- Two-states solver. Same principle but with upwinding of the eigenvectors

$$\begin{cases} \left(s_p^L, \beta^{**} - \beta_L\right) = 0, & \lambda_p > 0, \\ \left(s_p^R, \beta^{**} - \beta_R\right) = 0, & \lambda_p < 0, \end{cases}$$

It defines $\psi : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}^n : \psi(\beta_L, \beta_R, x_L, x_R) = \beta^{**}$.



Th. : If $R + R^t \geq 0$, then $\left\{s_p^L\right\}_{\lambda_p>0} \cup \left\{s_p^R\right\}_{\lambda_p<0}$ is linearly independent.

Interpretation for : $\partial_t p + \partial_x u = 0, \quad \partial_t u + \partial_x p = -\sigma u$

$A^{-1}R = \begin{pmatrix} 0 & \sigma \\ 0 & 0 \end{pmatrix}$ is nilpotent. So $e^{A^{-1}Rx} = I + A^{-1}Rx$.

• One-state solver = Steady state approximation of Huang-Liu 1986, or Jin-Levermore 1996

$$\begin{cases} p^*_{j+\frac{1}{2}} & = \dfrac{p_j + p_{j+1}}{2} & + \dfrac{(1-\sigma\Delta x_j^+)u_j - (1+\sigma\Delta x_{j+1}^-)u_{j+1}}{2}, \\ u^*_{j+\frac{1}{2}} & = \dfrac{p_j - p_{j+1}}{2} & + \dfrac{(1-\sigma\Delta x_j^+)u_j + (1+\sigma\Delta x_{j+1}^-)u_{j+1}}{2}. \end{cases}$$

• Two-state solver= Equal to the Gosse-Toscani scheme 2002.

$$\begin{cases} p^{**}_{j+\frac{1}{2}} = \frac{1-\sigma\Delta x_{j+1}^-}{2+\sigma\Delta x_j^+ - \sigma\Delta x_{j+1}^-}(p_j - u_j) + \frac{1+\sigma\Delta x_j^+}{2+\sigma\Delta x_j^+ - \sigma\Delta x_{j+1}^-}(p_{j+1} - u_{j+1}), \\ u^{**}_{j+\frac{1}{2}} = \frac{1}{2+\sigma\Delta x_j^+ - \sigma\Delta x_{j+1}^-}(u_j + u_{j+1} + p_j - p_{j+1}). \end{cases}$$

- Immediate interest for **non linear** fluid solvers with σ the friction parameter.

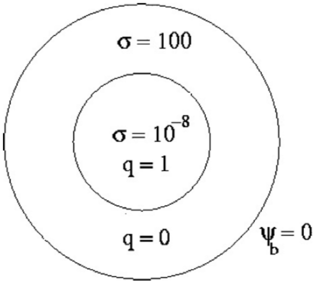
Section 3

Applications

#1 : P_n method, Valentin (PhD-CEA)+Enaux+Laffitte

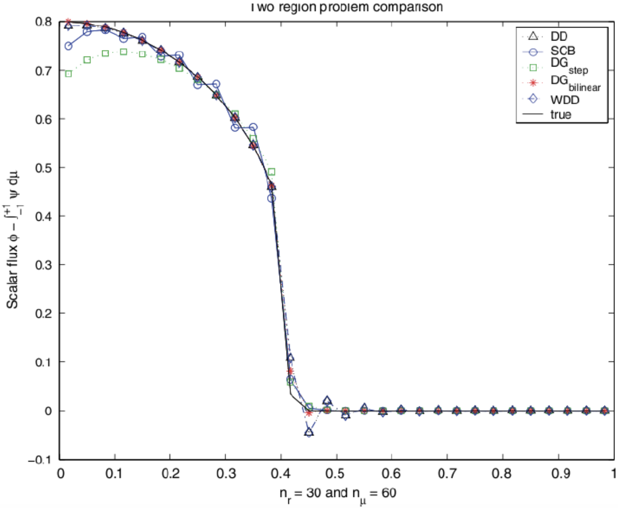
This is a challenging problem with a vast literature in neutron propagation.

Machorro 2007 : consider $\mu\partial_r u + \frac{1-\mu^2}{r} u = -\sigma u + q$ with $u(r,\mu) = \sum_{i=0}^{n} u_i(r) P_i(\mu)$ in the domain



Solve $\partial_t U + A\partial_r U + \frac{1}{r} GU = -\sigma U + b$.
Valentin-Enaux-Lafitte propose to use Magnus series to compute the local matrix exponentials.

P_n results : Machorro 2007



Discontinuity of coefficients. flux dip, oscillations, higher cost with DG.

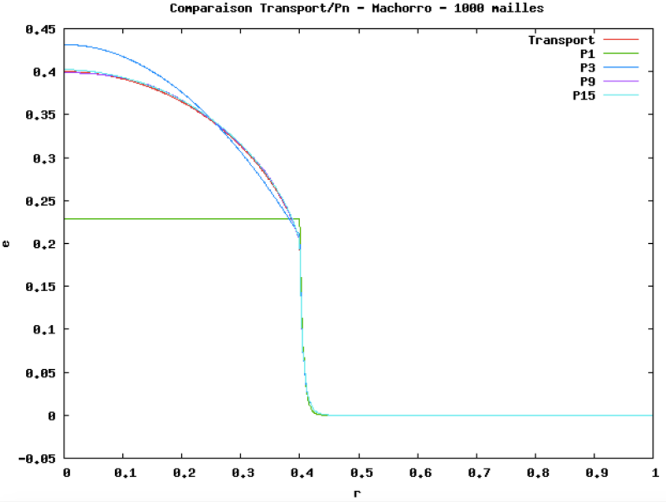
P_n results : Valentin 2015

Introduction

Detail of 1D Riemann solvers

Applications

2D P_1



No flux dip, no oscillations, reduced cost.

#2 : Transport-Doppler effect, Leroy (PhD-CEA)+Buet+D.

Solve : $\partial_t \rho = \frac{\kappa}{3}\nu \partial_\nu \rho + \sigma(\nu)(B(\nu)-\rho), \quad v, t > 0$ with $\kappa = \partial_x u \in \mathbb{R}$.

The SWB (Spectrally Well Balanced) scheme writes

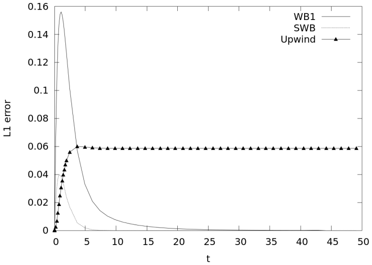
$$\partial_t \rho_j = \frac{\sigma(\nu_j)}{1 - M(\nu_{j+1}, \nu_j)} \frac{\rho(\nu_j \mid \nu_{j+1}, \rho_{j+1}) - \rho_j}{\Delta \nu_j}$$

where

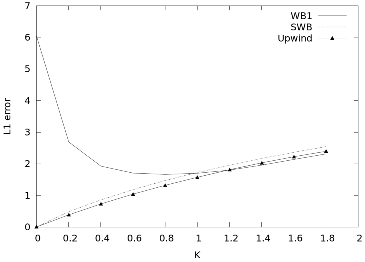
$$\rho(\nu \mid \nu_*, \rho_*) = \rho_* e^{-\frac{3}{\kappa}\int_\nu^{\nu_*} \frac{\sigma(s)}{s} ds} + \frac{3}{\kappa}\int_\nu^{\nu_*} \frac{B(s)\sigma(s)}{s} \int_\tau^s \frac{\sigma(\tau)}{\tau} d\tau ds$$

is the stationary solution at ν_j and the additional term $M(\nu_{j+1}, \nu) = e^{-\frac{3}{\kappa}\int_{\nu_j}^{\nu_{j+1}} \frac{\sigma(s)}{s} ds}$ gives the correct limit $\kappa \to 0$.

Upwing/WB/SWB



WB and SWB correct



WB non correct for $\kappa = 0$

#3 : System with a zero eigenvalue

This is a serious issue, related to "resonance" in hyperbolic theory.

Consider P^1 model coupled a linear temperature equation

$$\begin{cases} \partial_t p & +\partial_x u & = \tau(T-p), \\ \partial_t u & +\partial_x p & = -\sigma u, \\ \partial_t T & & = \tau(p-T), \end{cases}$$

$$A = A^t = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } R = R^t = \begin{pmatrix} \tau & 0 & -\tau \\ 0 & \sigma & 0 \\ -\tau & 0 & \tau \end{pmatrix}.$$

Assume $\sigma, \tau > 0$: the solutions of the adjoint stationary equation satisfy $\widehat{T} = \widehat{p}$, so $\partial_x \widehat{u} = 0$ and $\partial_x \widehat{p} = \sigma \widehat{u}$. Therefore

$$\dim(\mathcal{V}) = \dim(\mathcal{U}) = 2 < 3 \text{ with basis}$$

$$V_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \text{ and } V_2 = \begin{pmatrix} \sigma x \\ 0 \\ \sigma x \end{pmatrix}.$$

One abstract solution= shift the spectrum

Consider progressive solutions of the adjoint equation

$$\begin{cases} \partial_t V + A\partial_x V = RV, \\ \partial_t V + \xi\partial_x V = 0, \qquad \xi \in \mathbb{R}, \end{cases}$$

where $\xi \in \mathbb{R}$ is arbitrary. It yields the shifted problem

$$A_\xi \partial_x V = RV, \qquad A_\xi = A - \xi I.$$

So $\det(A_\xi) \neq 0$.

Probably non correct in the regime $\xi \to 0$, so useless.

Ad-hoc integration in time

Set $\alpha_1 = (U, V_1) = p + T$, $\alpha_2 = (U, V_2) = \sigma x(p + T) + u$ and $\alpha_3 = T$

$$\begin{cases} \partial_t \alpha_1 + \partial_x(-\sigma x \alpha_1 + \alpha_2) = 0, \\ \partial_t \alpha_2 + \partial_x \left((1 - \sigma^2 x^2)\alpha_1 + \sigma x \alpha_2 - \alpha_3 \right) = 0, \\ \partial_t \alpha_3 = \tau(\alpha_1 - 2\alpha_3). \end{cases}$$

The last equation is more an ODE $(e^{2\tau t}\alpha_3)' = \tau e^{2\tau t}\alpha_1$

$$\alpha_3(x, t) = e^{-2\tau t} \int_0^t \tau e^{2\tau s} \alpha_1(x, s) ds + e^{-2\tau t} \alpha_3(x, 0).$$

Explicit Euler approximation over Δt reads

$$\alpha_3(x, \Delta t) \approx \frac{1}{2} \left(1 - e^{-2\tau \Delta t} \right) \alpha_1(x, 0) + e^{-2\tau \Delta t} \alpha_3(x, 0).$$

It yields

$$\begin{cases} \partial_t \alpha_1 + \partial_x(-\sigma x \alpha_1 + \alpha_2) = 0, \\ \partial_t \alpha_2 + \partial_x \left((1 - \sigma^2 x^2)\alpha_1 + \sigma x \alpha_2 - \frac{1}{2} \left(1 - e^{-2\tau \Delta t} \right) \alpha_1 - e^{-2\tau \Delta t} \alpha_3(0) \right) = 0. \end{cases}$$

Section 4

2D P_1

- Assume $\sigma > 0$ is constant

$$\left\{\begin{array}{llll} \partial_t p & +\partial_x u & +\partial_y v & = 0, \\ \partial_t u & +\partial_x p & & = -\sigma u, \\ \partial_t v & & +\partial_y p & = -\sigma v, \end{array}\right.$$

Here $\det(A) = 0$, $\det(B) = 0$ and $AB \neq BA$.

- The adjoint stationary states $(\widehat{p}, \widehat{u}, \widehat{v})$ are

$$\left\{\begin{array}{lll} \partial_x \widehat{u} & +\partial_y \widehat{v} & = 0, \\ \partial_x \widehat{p} & & = \sigma \widehat{u}, \\ & \partial_y \widehat{p} & = \sigma \widehat{v}. \end{array}\right.$$

So $(\widehat{u}, \widehat{v}) = \frac{1}{\sigma} \nabla \widehat{p}$ and $\Delta \widehat{p} = 0$. Therefore $\dim(\mathcal{V}) = \dim(\mathcal{U}) = \infty$

$$\mathcal{V} = \left\{ (\widehat{p}, \widehat{u}, \widehat{v});\ \widehat{p} \text{ is harmonic and } (\widehat{u}, \widehat{v}) = \frac{1}{\sigma} \nabla \widehat{p} \right\}.$$

New formulation

- Set $\mathcal{V}_n = \mathcal{V} \cap \{\widehat{p} \text{ is an harmonic polynomial of degree } \leq n\}$.
$p \in \mathcal{V}_1$ is equivalent to $\widehat{p} = a + bx + cy$.

- $\mathcal{V}_1$ yields three test functions

$$V_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad V_2 = \begin{pmatrix} \sigma x \\ 1 \\ 0 \end{pmatrix}, \quad V_3 = \begin{pmatrix} \sigma y \\ 0 \\ 1 \end{pmatrix}$$

Set $\alpha_i = (U, V_i)$ for $i = 1, 2, 3$. The new system is

$$\partial_t \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} + \partial_x \begin{pmatrix} m_1 \\ \alpha_1 + \sigma x m_1 \\ \sigma y m_1 \end{pmatrix} + \partial_y \begin{pmatrix} m_2 \\ \sigma x m_2 \\ 1 + \sigma y m_2 \end{pmatrix} = 0$$

with $m_1 = -\sigma x \alpha_1 + \alpha_2$ and $m_2 = -\sigma y \alpha_1 + \alpha_3$.

The original variable is recovered as $U = P(\mathbf{x})^{-1}\alpha$ where

$$P(\mathbf{x}) = \begin{pmatrix} 1 & 0 & 0 \\ \sigma x & 1 & 0 \\ \sigma y & 0 & 1 \end{pmatrix}$$ is non singular, and $P(\mathbf{x})P(\mathbf{y}) = P(\mathbf{y})P(\mathbf{x})$.

Convergence

- Performed on the problem with an additional small parameter

$$\begin{cases} \partial_t p + \frac{1}{\varepsilon}\nabla \cdot \mathbf{u} & = 0, \\ \partial_t \mathbf{u} + \frac{1}{\varepsilon}\nabla p & = -\frac{\sigma}{\varepsilon^2}\mathbf{u}, \end{cases}$$

where $\sigma > 0$ is given and $\varepsilon > 0$ is a small parameter.

- In the limit $\varepsilon \approx 0^+$, one gets asymptotically the parabolic equation $\partial_t p - \frac{1}{\sigma}\Delta p = 0$.

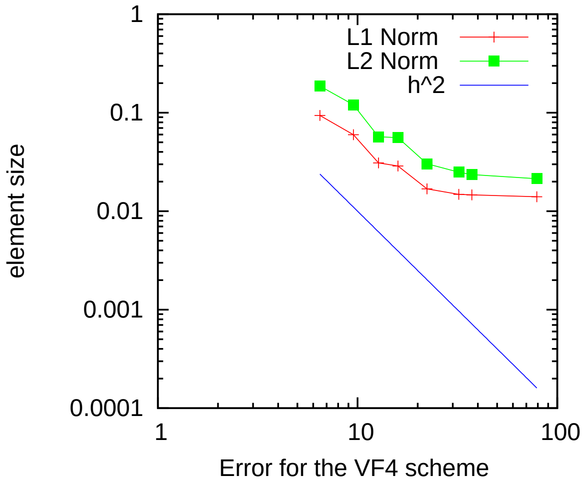
- **Theorem (2D <u>on general grids</u>) :** The implicit edge-based-FV WB scheme ($=$ Gosse-Toscani if 1D on uniform mesh) is convergent

$$\left\| p^{\varepsilon}_{h,\Delta t} - p^{\varepsilon}, \mathbf{u}^{\varepsilon}_{h,\Delta t} - \mathbf{u}^{\varepsilon} \right\|_{L^2([0,T]\times\Omega)} \leq C(h^{\frac{1}{4}} + \Delta t^{\frac{1}{2}}),$$

<u>uniformly</u> with respect to small parameter $\varepsilon \in (0,1]$ (numerics show convergence order between 1 and 2).

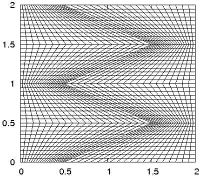
Sketch of the proof : linear estimates but of hyperbolic-parabolic type, balance between dissipativity and source terms, general grid, control of the constants, ...

Non AP feature of the standard edge based VF scheme

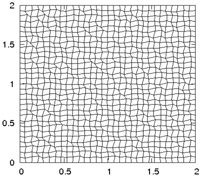


Meshes and P^1 (E. Franck)

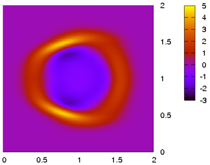
Initial data is Dirac mass at center



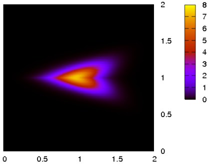
Kershaw mesh



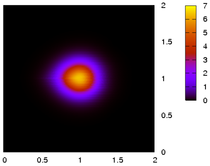
Random mesh



non WB-AP



WB-FV edge, non AP



WB-AP corner

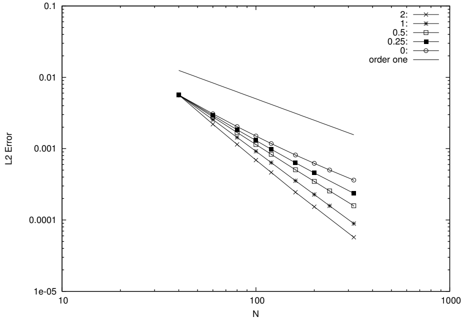
Mesh imprint vastly reduced with WB-AP corner-based-flux FV scheme.

Convergence order

Analytical solution in the torus : $p = f + \frac{\varepsilon^2}{\sigma}\partial_t f$ and $\mathbf{u} = -\frac{\varepsilon}{\sigma}\nabla f$ with

$$\partial_t f + \frac{\varepsilon^2}{\sigma}\partial_t^2 f - \frac{1}{\sigma}\Delta f = 0, \qquad f(t,x,y) = \alpha(t)\cos(L\pi x))\cos(L\pi y).$$

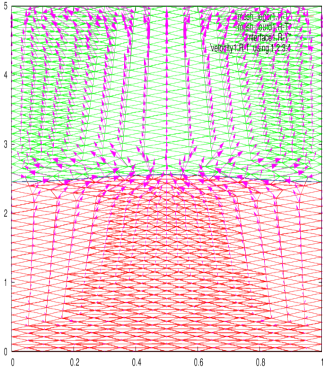
$\alpha'(t) + \frac{\varepsilon^2}{\sigma}\alpha''(t) + \frac{2L^2\pi^2}{\sigma}\alpha(t) = 0$, $\alpha(t) = \frac{\lambda_2}{\lambda_2-\lambda_1}e^{\lambda_1 t} - \frac{\lambda_1}{\lambda_2-\lambda_1}e^{\lambda_2 t}$.



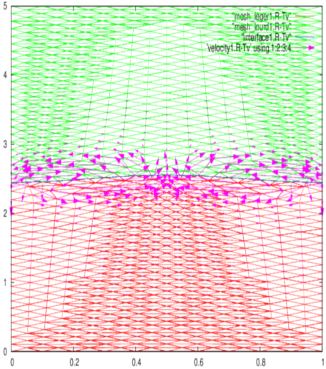
In tests : $\varepsilon = 0.01(40h)^\tau$ for $\tau \in \{0, \frac{1}{4}, \frac{1}{2}, 1, 2\}$.
$1 \leq$ experimental convergence ≤ 2.

#5 : Lagrangian fluid dynamics, comparison standard FV versus WB-FV

Initial stage of Rayleigh-Taylor instability :
source term = gravity.



standard FV
spurious velocities



WB-FV
vortex is "correct"

Conclusions

- Unification of **WB** based on the dual equation
  essentially 2 ideas : duality and linear Riemann solvers,
  essentially 2 families of WB solvers.
- General principle : any **WB-FV** is a standard **FV**.
  AMC 2015, Buet-D : and references therein.
- Validated for Friedrichs systems of large size inspired by radiation/neutron inspired problems.
- Any Riemann solver can be used : high order, even FEM is possible, or central schemes, . . .
- The change-of-basis matrix $P(\mathbf{x})$ seems the good object to manipulate, particularly in multiD.

- Ongoing : convergence estimates, zero eigenvalue problem.
- Open problems :
  - Comparison with other methods for transport equations.
  - Convergence for AP with this method : $e^{A^{-1}R\frac{\Delta x}{\varepsilon}}$
  - Extension to the full non linear case.