Well-Balanced Schemes for the Euler Equations with Gravity

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Euler Equations with Gravity

$$\boldsymbol{U}_t + \boldsymbol{f}(\boldsymbol{U})_x + \boldsymbol{g}(\boldsymbol{U})_y = \boldsymbol{S}(\boldsymbol{U})$$

$$\boldsymbol{U} := \begin{pmatrix} \rho \\ \rho u \\ \rho v \\ E \end{pmatrix} \quad \boldsymbol{f}(\boldsymbol{U}) = \begin{pmatrix} \rho u \\ \rho u^2 + p \\ \rho u v \\ u(E+p) \end{pmatrix} \quad \boldsymbol{g}(\boldsymbol{U}) := \begin{pmatrix} \rho v \\ \rho u v \\ \rho v^2 + p \\ v(E+p) \end{pmatrix}$$

$$\boldsymbol{S}(\boldsymbol{U}) = -\begin{pmatrix} 0 \\ \rho \phi_x \\ \rho \phi_y \\ \rho(u\phi_x - v\phi_y) \end{pmatrix}$$

- ρ is the density
- u, v are the x- and y-velocities
- E is the total energy
- p is the pressure; $E = \dfrac{p}{\gamma - 1} + \dfrac{\rho}{2}(u^2 + v^2)$
- ϕ is the time-independent linear gravitational potential; $\phi_x = 0$ and $\phi_y = g$

Euler Equations with Gravity

- Plays an important role in modeling model astrophysical and atmospheric phenomena in many fields including supernova explosions, (solar) climate modeling and weather forecasting;
- This is a system of balance laws: in many physical applications, solutions of the system are small perturbations of the steady states.
- Capturing such solutions numerically is a challenging task since the size of these perturbations may be smaller than the size of the truncation error on a coarse grid;
- To overcome this difficulty, one can use very fine grid, but in many physically relevant situations, this may be unaffordable;
- It is important to design a well-balanced numerical method, that is, the method which is capable of exactly preserving some steady state solutions. Then, perturbations of these solutions will be resolved on a coarse grid in a non-oscillatory way.

A type of solutions of interest are steady state ones

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Steady States

$$\begin{cases} \rho_t + (\rho u)_x + (\rho v)_y = 0 \\ (\rho u)_t + (\rho u^2 + p)_x + (\rho u v)_y = 0 \\ (\rho v)_t + (\rho u v)_x + (\rho v^2 + p)_y = -\rho g \\ E_t + (u(E+p))_x + (v(E+p))_y = -\rho v g \end{cases}$$

Steady state solution:

$$\boxed{u \equiv 0, \;\; v \equiv 0, \;\; w \equiv \text{Const}}$$

Here,

$$w := p + R, \quad R(x,y,t) := g \int^{y} \rho(x,\xi,t)\, d\xi$$

Numerical Challenges : **Well-balanced scheme** should exactly balance the flux and source terms so that the steady states are preserved.

Well-Balanced Methods Some References

- Developed mainly in the context of shallow water models.
- Euler equations with gravitational fields:
  - R. Leveque and D. Bale (1998) – quasi-steady wave-propagation methods for models with a static gravitational field
  - N. Botta, R. Klein, S. Langenberg, and S. Lützenkirchen (2004) – well-balanced finite-volume methods, which preserve a certain class of steady states for nearly hydrostatic flows
  - C. T. Tian, K. Xu, K. L. Chan, and L. C. Deng (2007), K. Xu, J. Luo, and S. Chen (2010), J. Luo, K. Xu, and N. Liu (2011) – gas-kinetic schemes for multi-D gas dynamic equations and well-balanced numerical methods for problems, in which the gravitational potential was modeled by a piecewise step function
  - Y. Xing and C.-W. Shu (2013) – higher order finite-difference methods for the gas dynamics with gravitation
  - M. Zenk, C. Berthon and C. Klingenberg (2014), P. Chandrashekar, C. Klingenberg (2015) – FV methods
  - ...

1-D System – For Simplicity

$$\begin{cases} \rho_t + (\rho v)_y = 0 \\ (\rho v)_t + (\rho v^2 + p)_y = -\rho g \\ E_t + (v(E+p))_y = -\rho v g \end{cases}$$

Steady state solution:

$$\boxed{v \equiv 0, \;\; w \equiv \text{Const}}$$

Here,

$$w := p + R, \quad R(y,t) := g \int^{y} \rho(\xi, t)\, d\xi$$

How to design a well-balanced scheme?

Finite-Volume Methods

$$\boldsymbol{U}_t + \boldsymbol{g}(\boldsymbol{U})_y = \boldsymbol{S}$$

- $\overline{\boldsymbol{U}}_k^n \approx \dfrac{1}{\Delta y} \displaystyle\int\limits_{C_k} \boldsymbol{U}(y, t^n)\, dy$: cell averages over $C_k := (y_{k-\frac{1}{2}}, y_{k+\frac{1}{2}})$

- Semi-discrete FV method:

$$\frac{d}{dt}\overline{\boldsymbol{U}}_k(t) = -\frac{\boldsymbol{\mathcal{G}}_{k+\frac{1}{2}}(t) - \boldsymbol{\mathcal{G}}_{k-\frac{1}{2}}(t)}{\Delta y} + \overline{\boldsymbol{S}}_k$$

$\boldsymbol{\mathcal{G}}_{k+\frac{1}{2}}(t)$: numerical fluxes

$\overline{\boldsymbol{S}}_k$: quadrature approximating the corresponding source terms

- Central-Upwind (CU) Scheme:

[Kurganov, Lin, Noelle, Petrova, Tadmor, et al.; 2000–2007]

$$\{\overline{\boldsymbol{U}}_k(t)\} \to \widetilde{\boldsymbol{U}}(\cdot,t) \to \left\{\boldsymbol{U}_k^{\mathrm{N,S}}(t)\right\} \to \left\{\boldsymbol{\mathcal{G}}_{k+\frac{1}{2}}(t)\right\} \to \{\overline{\boldsymbol{U}}_k(t+\Delta t)\}$$

(Discontinuous) piecewise-linear reconstruction:

$$\widetilde{\boldsymbol{U}}(y,t) := \overline{\boldsymbol{U}}_k(t) + (\boldsymbol{U}_y)_k(y-y_k), \quad y \in C_k$$

It is conservative, second-order accurate, and non-oscillatory provided the slopes, $\{(\boldsymbol{U}_y)_j\}$, are computed by a nonlinear limiter

Example — Generalized Minmod Limiter

$$(\boldsymbol{U}_y)_k = \mathrm{minmod}\left(\theta\frac{\overline{\boldsymbol{U}}_k - \overline{\boldsymbol{U}}_{k-1}}{\Delta y}, \frac{\overline{\boldsymbol{U}}_{k+1} - \overline{\boldsymbol{U}}_{k-1}}{2\Delta y}, \theta\frac{\overline{\boldsymbol{U}}_{k+1} - \overline{\boldsymbol{U}}_j}{\Delta y}\right)$$

where

$$\mathrm{minmod}(z_1, z_2, ...) := \begin{cases} \min_k\{z_k\}, & \text{if } z_k > 0 \ \forall k, \\ \max_k\{z_k\}, & \text{if } z_k < 0 \ \forall k, \\ 0, & \text{otherwise}, \end{cases}$$

and $\theta \in [1,2]$ is a constant

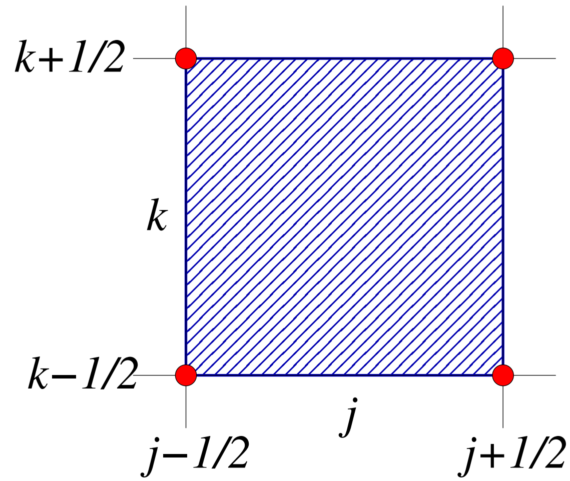
$$\{\overline{\boldsymbol{U}}_k(t)\} \to \widetilde{\boldsymbol{U}}(\cdot,t) \to \left\{\boldsymbol{U}_k^{\mathrm{N,S}}(t)\right\} \to \left\{\boldsymbol{\mathcal{G}}_{k+\frac{1}{2}}(t)\right\} \to \{\overline{\boldsymbol{U}}_k(t+\Delta t)\}$$

$\boldsymbol{U}_k^{\mathrm{N}}$ and $\boldsymbol{U}_k^{\mathrm{S}}$ are the point values at $y_{k+\frac{1}{2}}$ and $y_{k-\frac{1}{2}}$:

$$\widetilde{\boldsymbol{U}}(y,t) = \overline{\boldsymbol{U}}_k + (\boldsymbol{U}_y)_k(y-y_k), \quad y \in C_k$$

$$\boldsymbol{U}_k^{\mathrm{N}} := \overline{\boldsymbol{U}}_k + \frac{\Delta x}{2}(\boldsymbol{U}_y)_k$$

$$\boldsymbol{U}_k^{\mathrm{S}} := \overline{\boldsymbol{U}}_k - \frac{\Delta x}{2}(\boldsymbol{U}_y)_k$$



$$\{\overline{\boldsymbol{U}}_k(t)\} \to \widetilde{\boldsymbol{U}}(\cdot,t) \to \left\{\boldsymbol{U}_k^{\mathrm{N,S}}(t)\right\} \to \left\{\boldsymbol{\mathcal{G}}_{k+\frac{1}{2}}(t)\right\} \to \{\overline{\boldsymbol{U}}_k(t+\Delta t)\}$$

$$\boxed{\frac{d}{dt}\overline{\boldsymbol{U}}_k = -\frac{\boldsymbol{\mathcal{G}}_{k+\frac{1}{2}} - \boldsymbol{\mathcal{G}}_{k-\frac{1}{2}}}{\Delta y} + \overline{\boldsymbol{S}}_k}$$

where

$$\boldsymbol{\mathcal{G}}_{k+\frac{1}{2}} = \frac{b^+_{k+\frac{1}{2}}\boldsymbol{g}(\boldsymbol{U}_k^{\mathrm{N}}) - b^-_{k+\frac{1}{2}}\boldsymbol{g}(\boldsymbol{U}_{k+1}^{\mathrm{S}})}{b^+_{k+\frac{1}{2}} - b^-_{k+\frac{1}{2}}} + \beta_{k+\frac{1}{2}}\left(\boldsymbol{U}_{k+1}^{\mathrm{S}} - \boldsymbol{U}_k^{\mathrm{N}}\right)$$

$$\beta_{k+\frac{1}{2}} = \frac{b^+_{k+\frac{1}{2}} b^-_{k+\frac{1}{2}}}{b^+_{k+\frac{1}{2}} - b^-_{k+\frac{1}{2}}}$$

$$b^+_{k+\frac{1}{2}} = \max\left(v_k^{\mathrm{N}} + c_k^{\mathrm{N}},\, v_{k+1}^{\mathrm{S}} + c_{k+1}^{\mathrm{S}},\, 0\right)$$

$$b^-_{k+\frac{1}{2}} = \min\left(v_k^{\mathrm{N}} - c_k^{\mathrm{N}},\, v_{k+1}^{\mathrm{S}} - c_{k+1}^{\mathrm{S}},\, 0\right), \quad c^2 = \frac{\gamma p}{\rho}$$

and

$$\overline{\boldsymbol{S}}_k = (0, -g\bar{\rho}_k, -g(\overline{\rho v})_k)^T$$

No Well-Balanced Property

For steady-state solution: $v \equiv 0, \quad w \equiv \text{Const}$, we have

$$
\begin{cases}
\dfrac{d\,\bar{\rho}_k}{dt} = -\dfrac{\beta_{k+\frac{1}{2}}(\rho^{\mathrm{S}}_{k+1} - \rho^{\mathrm{N}}_{k}) - \beta_{k-\frac{1}{2}}(\rho^{\mathrm{S}}_{k} - \rho^{\mathrm{N}}_{k-1})}{\Delta y} \\[2ex]
\dfrac{d(\overline{\rho v})_k}{dt} = -\dfrac{(p^{\mathrm{S}}_{k+1} + p^{\mathrm{N}}_{k}) - (p^{\mathrm{S}}_{k} + p^{\mathrm{N}}_{k-1})}{2\Delta y} \\[2ex]
\dfrac{d\,\overline{E}_k}{dt} = -\dfrac{\beta_{k+\frac{1}{2}}(p^{\mathrm{S}}_{k+1} - p^{\mathrm{N}}_{k}) - \beta_{k-\frac{1}{2}}(p^{\mathrm{S}}_{k} - p^{\mathrm{N}}_{k-1})}{(\gamma-1)\Delta y}
\end{cases}
$$

- The RHS does not necessarily vanish and hence the steady state would not be preserved at the discrete level;
- This would also true for the first-order version of the scheme;
- For smooth solutions, the balance error is expected to be of order $(\Delta y)^2$, but a coarse grid solution may contain large spurious waves;
- The lack of balance between the numerical flux and source terms is a fundamental problem of the scheme.

Well-Balanced Scheme

1-D system in the y-direction:

$$\begin{pmatrix} \rho \\ \rho v \\ E \end{pmatrix}_t + \boxed{\begin{pmatrix} \rho v \\ \rho v^2 + p \\ v(E+p) \end{pmatrix}_y = \begin{pmatrix} 0 \\ -\rho g \\ -\rho v g \end{pmatrix}}$$

Define

$$R(y) := g \int^y \rho(\xi) d\xi \quad \Rightarrow \quad R_y = \rho g$$

Then

$$(\rho v^2 + \underbrace{p + R}_{w})_y = 0$$

$$w := p + R$$

Well-Balanced Scheme

- Incorporating the source term $-\rho g$ into the flux:

$$w := p + R, \quad R(y,t) := g \int^{y} \rho(\xi,t)\, d\xi$$

$$\begin{pmatrix} \rho \\ \rho v \\ E \end{pmatrix}_t + \begin{pmatrix} \rho v \\ \rho v^2 + w \\ v(E+p) \end{pmatrix}_y = \begin{pmatrix} 0 \\ 0 \\ -\rho v g \end{pmatrix}$$

- Reconstruct equilibrium variables $\mathbf{W} = (\rho, \rho v, w)^T$

- Compute the point values of conservative variables $\boldsymbol{U} = (\rho, \rho v, E)^T$:

$$\{\overline{\boldsymbol{U}}_k(t)\} \to \widetilde{\boldsymbol{U}}(\cdot,t) \to \left\{\mathbf{W}_k^{\mathrm{N,S}}(t)\right\} \to \left\{\boldsymbol{U}_k^{\mathrm{N,S}}(t)\right\} \to \left\{\boldsymbol{\mathcal{G}}_{k+\frac{1}{2}}(t)\right\} \to \{\overline{\boldsymbol{U}}_k(t+\Delta t)\}$$

1-D Reconstruction

- Compute the point values of ρ and ρv at y_k:

$$\widetilde{\rho}(y) = \sum_k \left[\bar{\rho}_k + (\rho_y)_k (y - y_k)\right] \cdot \chi_{C_k}(y) \quad \longrightarrow \quad \rho_k^{\mathrm{N,S}}$$

$$\widetilde{\rho v}(y) = \sum_k \left[(\overline{\rho v})_k + ((\rho v)_y)_k (y - y_k)\right] \cdot \chi_{C_k}(y) \quad \longrightarrow \quad (\rho v)_k^{\mathrm{N,S}}$$

- Integrate $\widetilde{\rho}(y)$ to obtain an approximation of $R = \int g\rho \, dy$:

$$\widetilde{R}(y) = g \int\limits_{y_{k_L - \frac{1}{2}}}^{y} \widetilde{\rho}(\xi)\, d\xi$$

$$= g \sum_k \left[\Delta y \sum_{i=k_L}^{k-1} \bar{\rho}_i + \bar{\rho}_k (y - y_{k-\frac{1}{2}}) + \frac{(\rho_y)_k}{2}(y - y_{k-\frac{1}{2}})(y - y_{k+\frac{1}{2}})\right] \cdot \chi_{C_k}(y)$$

1-D Reconstruction

- The point values of R at the cell interfaces and cell centers are

$$R_{k+\frac{1}{2}} = g\Delta y \sum_{i=k_L}^{k} \bar{\rho}_i \quad \text{and} \quad R_k = g\Delta y \sum_{i=k_L}^{k-1} \bar{\rho}_i + \frac{g\Delta y}{2}\bar{\rho}_k - \frac{g(\Delta y)^2}{8}(\rho_y)_k$$

or recursively

$$R_{k_L-\frac{1}{2}} = 0, \qquad \begin{cases} R_{k+\frac{1}{2}} = R_{k-\frac{1}{2}} + g\Delta y\,\bar{\rho}_k, \\ R_k = R_{k-\frac{1}{2}} + \dfrac{g\Delta y}{2}\bar{\rho}_k - \dfrac{g(\Delta y)^2}{8}(\rho_y)_k, \end{cases} \qquad k = k_L, \ldots, k_R$$

- The values of w at the cell centers are set as

$$w_k = p_k + R_k$$

where

$$p_k = (\gamma - 1)\left(\overline{E}_k - \frac{\bar{\rho}_k}{2}v_k^2\right)$$

and

$$v_k = (\overline{\rho v})_k / \bar{\rho}_k$$

1-D Reconstruction

- Equipped with $w_k = p_k + R_k$, we apply the minmod reconstruction procedure to $\{w_k\}$ and obtain the point values of w at the cell interfaces:

$$w_k^{\mathrm{N}} = w_k + \frac{\Delta y}{2}(w_y)_k, \quad w_{k+1}^{\mathrm{S}} = w_{k+1} - \frac{\Delta y}{2}(w_y)_{k+1}$$

where

$$(w_y)_k = \mathrm{minmod}\left(\theta\,\frac{w_{k+1} - w_k}{\Delta y},\, \frac{w_{k+1} - w_{k-1}}{2\Delta y},\, \theta\,\frac{w_k - w_{k-1}}{\Delta y}\right)$$

- Finally, the point values of p and E are

$$p_k^{\mathrm{N}} = w_k^{\mathrm{N}} - R_{k+\frac{1}{2}}, \quad p_k^{\mathrm{S}} = w_k^{\mathrm{S}} - R_{k-\frac{1}{2}}$$

and

$$E_k^{\mathrm{N}} = \frac{p_k^{\mathrm{N}}}{\gamma - 1} + \frac{\left((\rho v)_k^{\mathrm{N}}\right)^2}{2\rho_k^{\mathrm{N}}}, \quad E_k^{\mathrm{S}} = \frac{p_k^{\mathrm{S}}}{\gamma - 1} + \frac{\left((\rho v)_k^{\mathrm{S}}\right)^2}{2\rho_k^{\mathrm{S}}}$$

Well-Balanced Evolution

$$\frac{d}{dt}\overline{\boldsymbol{U}}_k = -\frac{\boldsymbol{\mathcal{G}}_{k+\frac{1}{2}} - \boldsymbol{\mathcal{G}}_{k-\frac{1}{2}}}{\Delta y} + \overline{\boldsymbol{S}}_k.$$

Here, the second and third components of the numerical fluxes $\boldsymbol{\mathcal{G}}$ are computed as before

$$G^{(2)}_{k+\frac{1}{2}} = \frac{b^+_{k+\frac{1}{2}}\left(\rho^{\mathrm{N}}_k (v^{\mathrm{N}}_k)^2 + w^{\mathrm{N}}_k\right) - b^-_{k+\frac{1}{2}}\left(\rho^{\mathrm{S}}_{k+1}(v^{\mathrm{S}}_{k+1})^2 + w^{\mathrm{S}}_{k+1}\right)}{b^+_{k+\frac{1}{2}} - b^-_{k+\frac{1}{2}}}$$
$$+ \beta_{k+\frac{1}{2}}\left((\rho v)^{\mathrm{S}}_{k+1} - (\rho v)^{\mathrm{N}}_k\right)$$

$$G^{(3)}_{k+\frac{1}{2}} = \frac{b^+_{k+\frac{1}{2}} v^{\mathrm{N}}_k (E^{\mathrm{N}}_k + p^{\mathrm{N}}_k) - b^-_{k+\frac{1}{2}} v^{\mathrm{S}}_{k+1}(E^{\mathrm{S}}_{k+1} + p^{\mathrm{S}}_{k+1})}{b^+_{k+\frac{1}{2}} - b^-_{k+\frac{1}{2}}} + \beta_{k+\frac{1}{2}}\left(E^{\mathrm{S}}_{k+1} - E^{\mathrm{N}}_k\right)$$

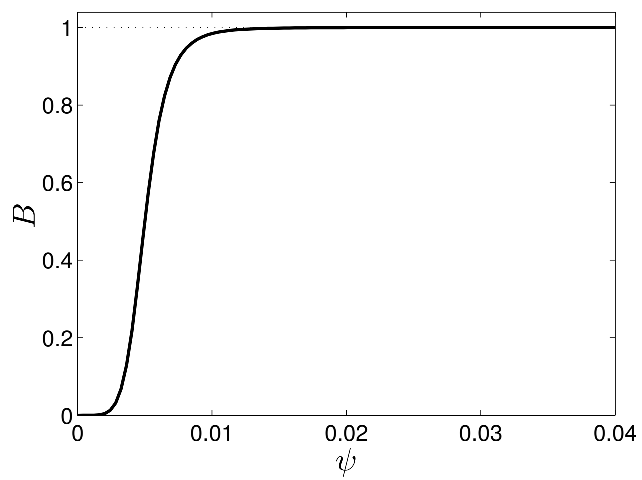
Well-Balanced Evolution

However, to ensure a well-balanced property of the scheme, the first component of the flux and the source term have to be modified:

$$G^{(1)}_{k+\frac{1}{2}} = \frac{b^+_{k+\frac{1}{2}}(\rho v)^{\mathrm{N}}_k - b^-_{k+\frac{1}{2}}(\rho v)^{\mathrm{S}}_{k+1}}{b^+_{k+\frac{1}{2}} - b^-_{k+\frac{1}{2}}}$$

$$+ \beta_{k+\frac{1}{2}} B\left(\frac{|w_{k+1} - w_k|}{\Delta y} \cdot \frac{y_{k_R+\frac{1}{2}} - y_{k_L-\frac{1}{2}}}{\max_k\{w_k\}}\right)\left(\rho^{\mathrm{S}}_{k+1} - \rho^{\mathrm{N}}_k\right)$$

$$\overline{\boldsymbol{S}}_k = \left(0, 0, -g(\overline{\rho v})_k\right)^T$$



Proof of the Well-Balanced Property

Theorem. *The semi-discrete scheme coupled with the well-balanced reconstruction and evolution is well-balanced in the sense that it preserves the steady state* $v \equiv 0,\ w \equiv \mathrm{Const}.$

Proof: Assume that at certain time level, we have

$$v_k^{\mathrm{N}} \equiv v_k \equiv v_k^{\mathrm{S}} \equiv 0 \quad \text{and} \quad w_k^{\mathrm{N}} \equiv w_k \equiv w_k^{\mathrm{S}} \equiv \widehat{w} \equiv \mathrm{Const}$$

To show that the proposed scheme is well-balanced, we need to show that

$$\frac{d}{dt}\overline{\boldsymbol{U}}_k = -\frac{\boldsymbol{\mathcal{G}}_{k+\frac{1}{2}} - \boldsymbol{\mathcal{G}}_{k-\frac{1}{2}}}{\Delta y} + \overline{\boldsymbol{S}}_k \equiv \mathbf{0}$$

$$\overline{\boldsymbol{S}}_k = (0, 0, -g v_k \rho_k)^T = \mathbf{0}$$

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$$G^{(1)}_{k+\frac{1}{2}} = \frac{b^+_{k+\frac{1}{2}}(\rho v)^{\mathrm{N}}_k - b^-_{k+\frac{1}{2}}(\rho v)^{\mathrm{S}}_{k+1}}{b^+_{k+\frac{1}{2}} - b^-_{k+\frac{1}{2}}} + \beta_{k+\frac{1}{2}} B\left(\frac{|w_{k+1} - w_k|}{\Delta y} \cdot \frac{y_{k_R+\frac{1}{2}} - y_{k_L-\frac{1}{2}}}{\max_k\{w_k\}}\right)\left(\rho^{\mathrm{S}}_{k+1} - \rho^{\mathrm{N}}_k\right) = 0$$

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$$G^{(2)}_{k+\frac{1}{2}} = \frac{b^+_{k+\frac{1}{2}}\left(\rho_k^{\mathrm{N}}(v_k^{\mathrm{N}})^2 + w_k^{\mathrm{N}}\right) - b^-_{k+\frac{1}{2}}\left(\rho_{k+1}^{\mathrm{S}}(v_{k+1}^{\mathrm{S}})^2 + w_{k+1}^{\mathrm{S}}\right)}{b^+_{k+\frac{1}{2}} - b^-_{k+\frac{1}{2}}} = 0$$

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Proof: Assume that at certain time level, we have

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$$\frac{d}{dt}\overline{\boldsymbol{U}}_k = -\frac{\boldsymbol{\mathcal{G}}_{k+\frac{1}{2}} - \boldsymbol{\mathcal{G}}_{k-\frac{1}{2}}}{\Delta y} + \overline{\boldsymbol{S}}_k \equiv \boldsymbol{0}$$

$$\begin{aligned} G^{(3)}_{k+\frac{1}{2}} &= \beta_{k+\frac{1}{2}}\left(E^{\mathrm{S}}_{k+1} - E^{\mathrm{N}}_k\right) = \frac{\beta_{k+\frac{1}{2}}}{\gamma-1}\left(p^{\mathrm{S}}_{k+1} - p^{\mathrm{N}}_k\right) \\ &= \frac{\beta_{k+\frac{1}{2}}}{\gamma-1}\left[\left(w^{\mathrm{S}}_{k+1} - R_{k+\frac{1}{2}}\right) - \left(w^{\mathrm{N}}_k - R_{k+\frac{1}{2}}\right)\right] = 0 \end{aligned}$$

2-D Well-Balanced Scheme

$$\begin{pmatrix} \rho \\ \rho u \\ \rho v \\ E \end{pmatrix}_t + \boxed{\begin{pmatrix} \rho u \\ \rho u^2 + p \\ \rho u v \\ u(E+p) \end{pmatrix}_x + \begin{pmatrix} \rho v \\ \rho u v \\ \rho v^2 + p \\ v(E+p) \end{pmatrix}_y = - \begin{pmatrix} 0 \\ 0 \\ g\rho \\ g\rho v \end{pmatrix}}$$

- Incorporating the source term $-\rho g$ into the flux:

$$w := p + R, \quad R(x,y,t) := g \int^{y} \rho(x,\xi,t)\, d\xi$$

$$\begin{pmatrix} \rho \\ \rho u \\ \rho v \\ E \end{pmatrix}_t + \boxed{\begin{pmatrix} \rho u \\ \rho u^2 + p \\ \rho u v \\ u(E+p) \end{pmatrix}_x + \begin{pmatrix} \rho v \\ \rho u v \\ \rho v^2 + w \\ v(E+p) \end{pmatrix}_y = - \begin{pmatrix} 0 \\ 0 \\ 0 \\ g\rho v \end{pmatrix}}$$

2-D Reconstruction

In the x-direction: Reconstruct conservative variables

$$\{\overline{\boldsymbol{U}}_{j,k}(t)\} \to \widetilde{\boldsymbol{U}}(\cdot,\cdot,t) \to \left\{\boldsymbol{U}^{\mathrm{E,W}}_{j,k}\right\} \to \left\{\boldsymbol{\mathcal{F}}_{j+\frac{1}{2},k}\right\} \to \{\overline{\boldsymbol{U}}_{j,k}(t+\Delta t)\}$$

- In the reconstruction step, we compute

$$\begin{array}{ccccc} \rho_{j,k} & \to & \widetilde{\rho}(\cdot,\cdot,t) & \to & \left\{\rho^{\mathrm{E,W}}_{j,k}(t)\right\} \\ (\rho u)_{j,k} & \to & \widetilde{\rho u}(\cdot,\cdot,t) & \to & \left\{(\rho u)^{\mathrm{E,W}}_{j,k}(t)\right\} \\ (\rho v)_{j,k} & \to & \widetilde{\rho v}(\cdot,\cdot,t) & \to & \left\{(\rho v)^{\mathrm{E,W}}_{j,k}(t)\right\} \end{array}$$

- From the EOS:

$$p_{j,k} = (\gamma-1)\left(\overline{E}_{j,k} - \frac{(\overline{\rho u})^2_{j,k} + (\overline{\rho v})^2_{j,k}}{2\,\overline{\rho}_{j,k}}\right) \quad \to \quad \widetilde{p}(\cdot,\cdot,t) \;\to\; \left\{p^{\mathrm{E,W}}_{j,k}(t)\right\}$$

$$\left\{p^{\mathrm{E,W}}_{j,k}(t)\right\} \;\to\; \left\{E^{\mathrm{E,W}}_{j,k}(t)\right\}$$

2-D Reconstruction

In the y-direction: Reconstruct equilibrium variables $\mathbf{W} = (\rho, \rho u, \rho v, w)^T$:

$$\{\overline{\boldsymbol{U}}_{j,k}(t)\} \to \widetilde{\boldsymbol{U}}(\cdot,\cdot,t) \to \left\{\mathbf{W}^{\mathrm{N,S}}_{j,k}\right\} \to \left\{\boldsymbol{U}^{\mathrm{N,S}}_{j,k}\right\} \to \left\{\boldsymbol{\mathcal{G}}_{j,k+\frac{1}{2}}\right\} \to \{\overline{\boldsymbol{U}}_{j,k}(t+\Delta t)\}$$

- In the reconstruction step, we compute

$$\begin{array}{lcll} \rho_{j,k} & \to & \widetilde{\rho}(\cdot,\cdot,t) & \to \left\{\rho^{\mathrm{N,S}}_{j,k}(t)\right\} \\ (\rho u)_{j,k} & \to & \widetilde{\rho u}(\cdot,\cdot,t) & \to \left\{(\rho u)^{\mathrm{N,S}}_{j,k}(t)\right\} \\ (\rho v)_{j,k} & \to & \widetilde{\rho v}(\cdot,\cdot,t) & \to \left\{(\rho v)^{\mathrm{N,S}}_{j,k}(t)\right\} \end{array}$$

- Compute $w_{j,k} = p_{j,k} + R_{j,k}$, $p_{j,k} = (\gamma - 1)\left(\overline{E}_{j,k} - \frac{(\overline{\rho u})^2_{j,k} + (\overline{\rho v})^2_{j,k}}{2\overline{\rho}_{j,k}}\right)$

$$R_{j,k_L-\frac{1}{2}} = 0, \qquad \begin{cases} R_{j,k+\frac{1}{2}} = R_{j,k-\frac{1}{2}} + g\Delta y\,\overline{\rho}_{j,k}, \\ R_{j,k} = R_{j,k-\frac{1}{2}} + \dfrac{g\Delta y}{2}\overline{\rho}_{j,k} - \dfrac{g(\Delta y)^2}{8}(\rho_y)_{j,k} \end{cases}$$

2-D Reconstruction

In the y-direction: Reconstruct equilibrium variables $\mathbf{W} = (\rho, \rho u, \rho v, w)^T$:

$$\{\overline{\boldsymbol{U}}_{j,k}(t)\} \to \widetilde{\boldsymbol{U}}(\cdot,\cdot,t) \to \left\{\mathbf{W}_{j,k}^{\mathrm{N,S}}\right\} \to \left\{\boldsymbol{U}_{j,k}^{\mathrm{N,S}}\right\} \to \left\{\boldsymbol{\mathcal{G}}_{j,k+\frac{1}{2}}\right\} \to \{\overline{\boldsymbol{U}}_{j,k}(t+\Delta t)\}$$

- Reconstruct w

$$w_{j,k} \quad \to \quad \widetilde{w}(\cdot,\cdot,t) \;\to\; \left\{w_{j,k}^{\mathrm{N,S}}(t)\right\}$$

- Finally, obtain the point values:

$$p_{j,k}^{\mathrm{N}} = w_{j,k}^{\mathrm{N}} - R_{j,k+\frac{1}{2}}, \quad p_{j,k}^{\mathrm{S}} = w_{j,k}^{\mathrm{S}} - R_{j,k-\frac{1}{2}}$$

and from the EOS:

$$E_{j,k}^{\mathrm{N}} = \frac{p_{j,k}^{\mathrm{N}}}{\gamma-1} + \frac{\left((\rho u)_{j,k}^{\mathrm{N}}\right)^2 + \left((\rho v)_{j,k}^{\mathrm{N}}\right)^2}{2\rho_{j,k}^{\mathrm{N}}}, \; E_{j,k}^{\mathrm{S}} = \frac{p_{j,k}^{\mathrm{S}}}{\gamma-1} + \frac{\left((\rho u)_{j,k}^{\mathrm{S}}\right)^2 + \left((\rho v)_{j,k}^{\mathrm{S}}\right)^2}{2\rho_{j,k}^{\mathrm{S}}}$$

Well-Balanced Evolution

$$\frac{d}{dt}\overline{\boldsymbol{U}}_{j,k} = -\frac{\boldsymbol{\mathcal{F}}_{j+\frac{1}{2},k} - \boldsymbol{\mathcal{F}}_{j-\frac{1}{2},k}}{\Delta x} - \frac{\boldsymbol{\mathcal{G}}_{j,k+\frac{1}{2}} - \boldsymbol{\mathcal{G}}_{j,k-\frac{1}{2}}}{\Delta y} + \overline{\boldsymbol{S}}_{j,k}$$

Similar to the 1-D case ...

Theorem. *The 2-D semi-discrete central-upwind scheme scheme described with the described well-balanced reconstruction and evolution is well-balanced in the sense that it preserves the steady state exactly.*

Example — 1-D Isothermal Equilibrium Solution

[Xing, Shu; 2013]

- The ideal gas with $\gamma = 1.4$; domain $[0, 1]$
- The gravitational force is $\phi_y = \mathrm{g} = 1$
- The steady-state initial conditions are

$$\rho(y,0) = e^{-y}, \quad p(y,0) = e^{-y}, \quad v(y,0) = 0$$

- A zero-order extrapolation at the boundaries is used

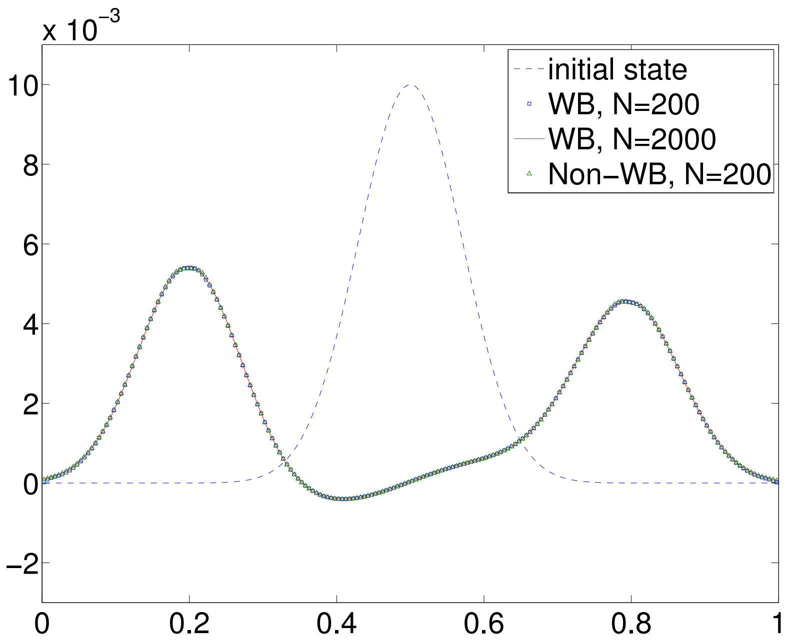
Well-Balanced Test – L^1-errors at $T = 2$:

N	ρ	ρv	E
40	2.33E-016	1.15E-016	1.27E-015
100	4.33E-016	9.64E-016	5.46E-016
200	7.58E-016	1.48E-015	1.23E-015
400	1.14E-015	3.97E-016	1.24E-015

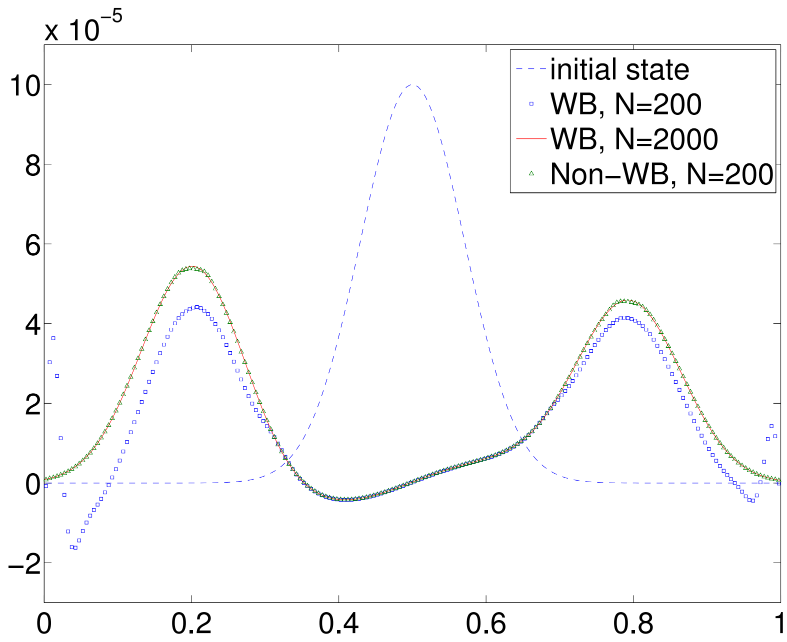
Perturbation

A small initial pressure perturbation:

$$\rho(y,0) = e^{-y}, \quad v(y,0) \equiv 0, \quad p(y,0) = ge^{-y} + \eta e^{-100(y-0.5)^2},$$



$\eta = 10^{-2}$



$\eta = 10^{-4}$

Example — 2-D Isothermal Equilibrium Solution

[Xing, Shu; 2013]

- The ideal gas with $\gamma = 1.4$; domain $[0,1] \times [0,1]$
- The gravitational force is $\phi_y = \mathrm{g} = 1$
- The steady-state initial conditions are

$$\rho(x,y,0) = 1.21e^{-1.21y}, \quad p(x,y,0) = e^{-1.21y}, \quad u(x,y,0) \equiv v(x,y,0) \equiv 0$$

- Solid wall boundary conditions imposed at the edges of the unit square

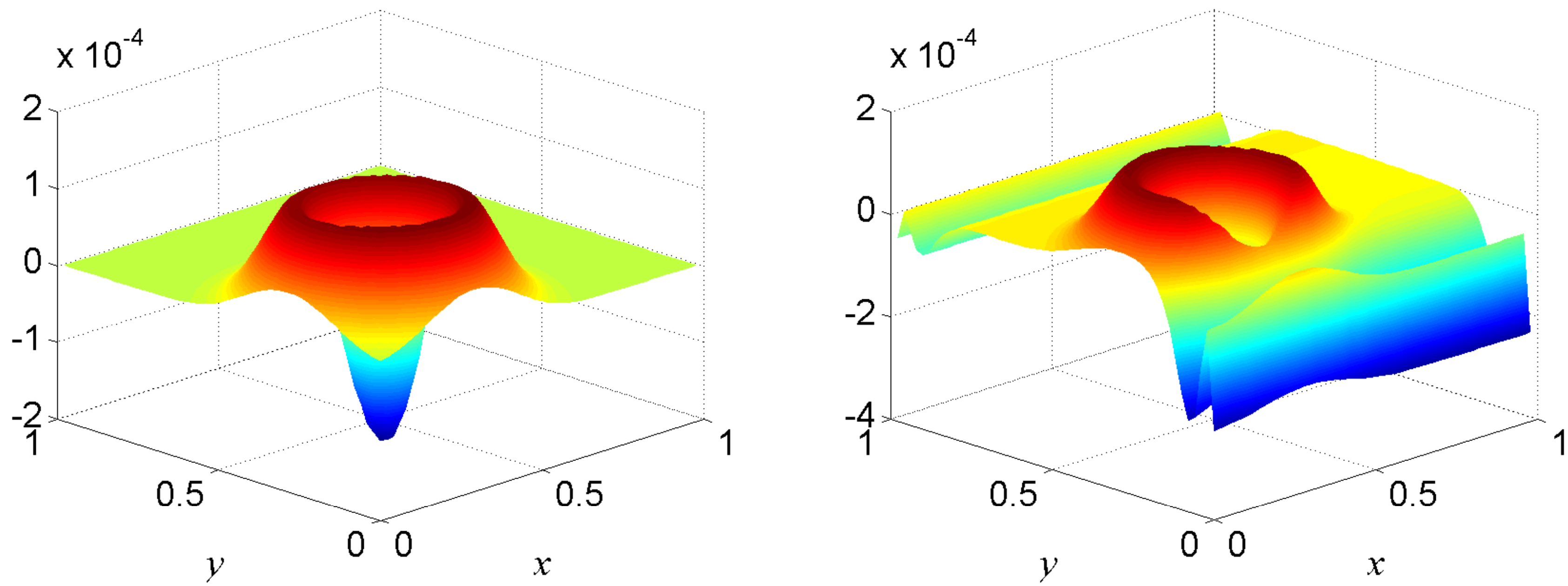
50 $\times$ 50	1.70E-016	0.00E+00	2.43E-016	5.97E-016
100 $\times$ 100	5.88E-017	0.00E+00	3.42E-016	5.31E-016
200 $\times$ 200	1.60E-016	0.00E+00	2.85E-016	5.33E-016

$N \times N$	ρ	ρu	ρv	E
50 $\times$ 50	1.05E-03	0.00E+00	5.72E-05	9.61E-05
100 $\times$ 100	4.02E-04	0.00E+00	2.07E-05	4.10E-05
200 $\times$ 200	1.63E-04	0.00E+00	7.11E-06	1.57E-05

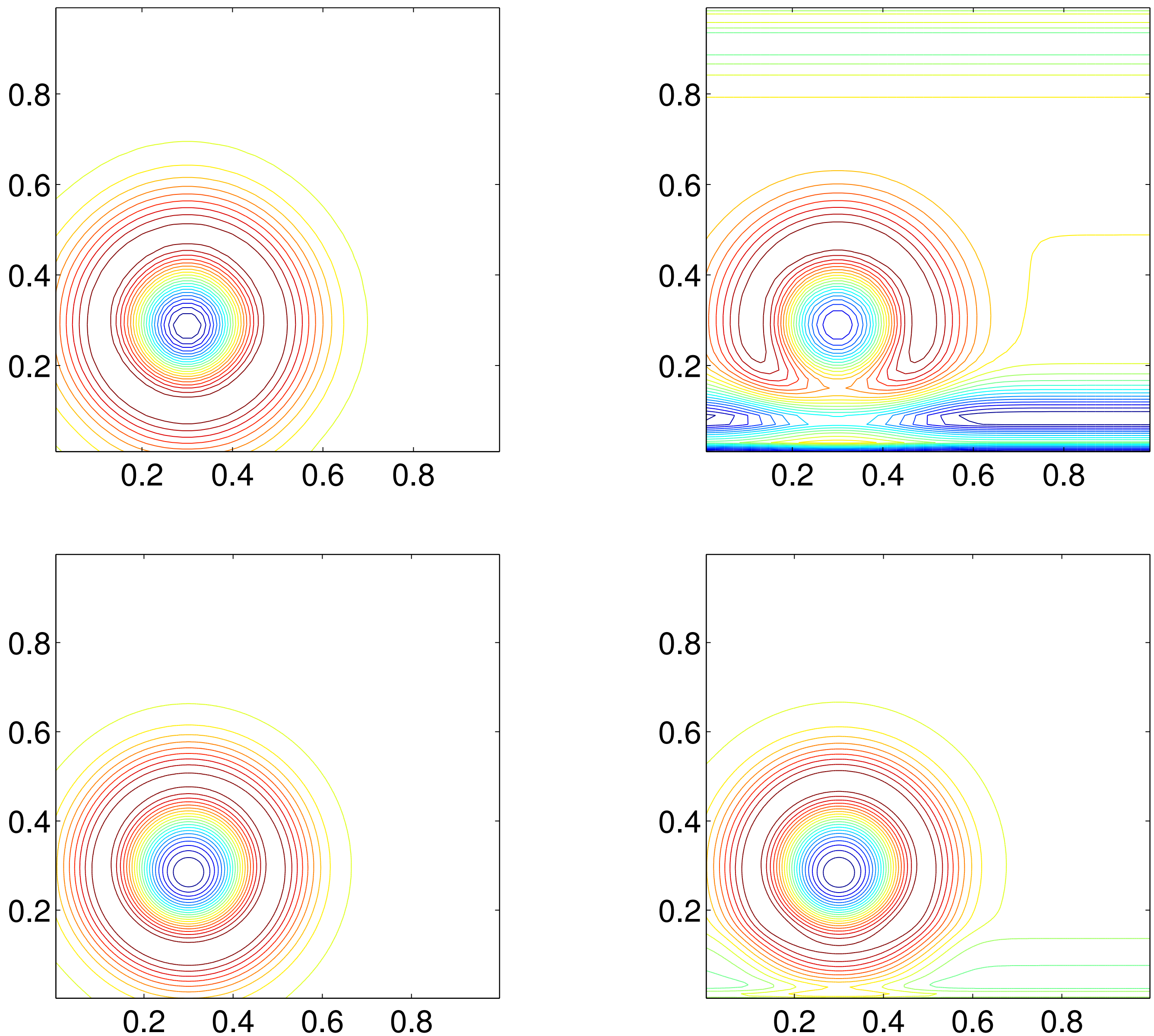
Perturbation

A small initial pressure perturbation:

$$p(x,y,0) = e^{-1.21y} + \eta e^{-121\left((x-0.3)^2+(y-0.3)^2\right)}, \quad \eta = 10^{-3}$$



$$50 \times 50$$



$WB : 50 \times 50,\ 200 \times 200$ $\qquad$ $NWB : 50 \times 50,\ 200 \times 200$

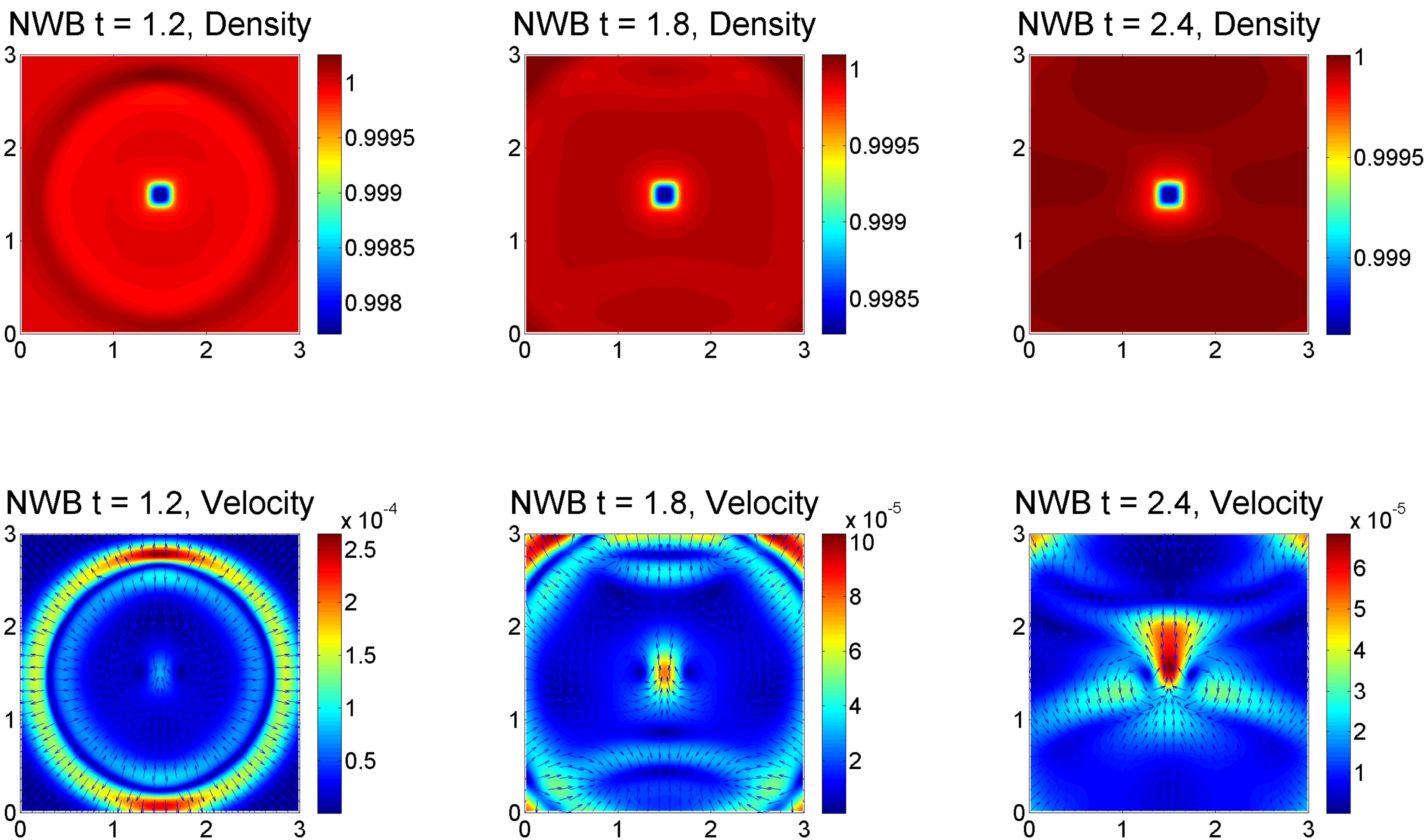
Example — 2-D Explosion

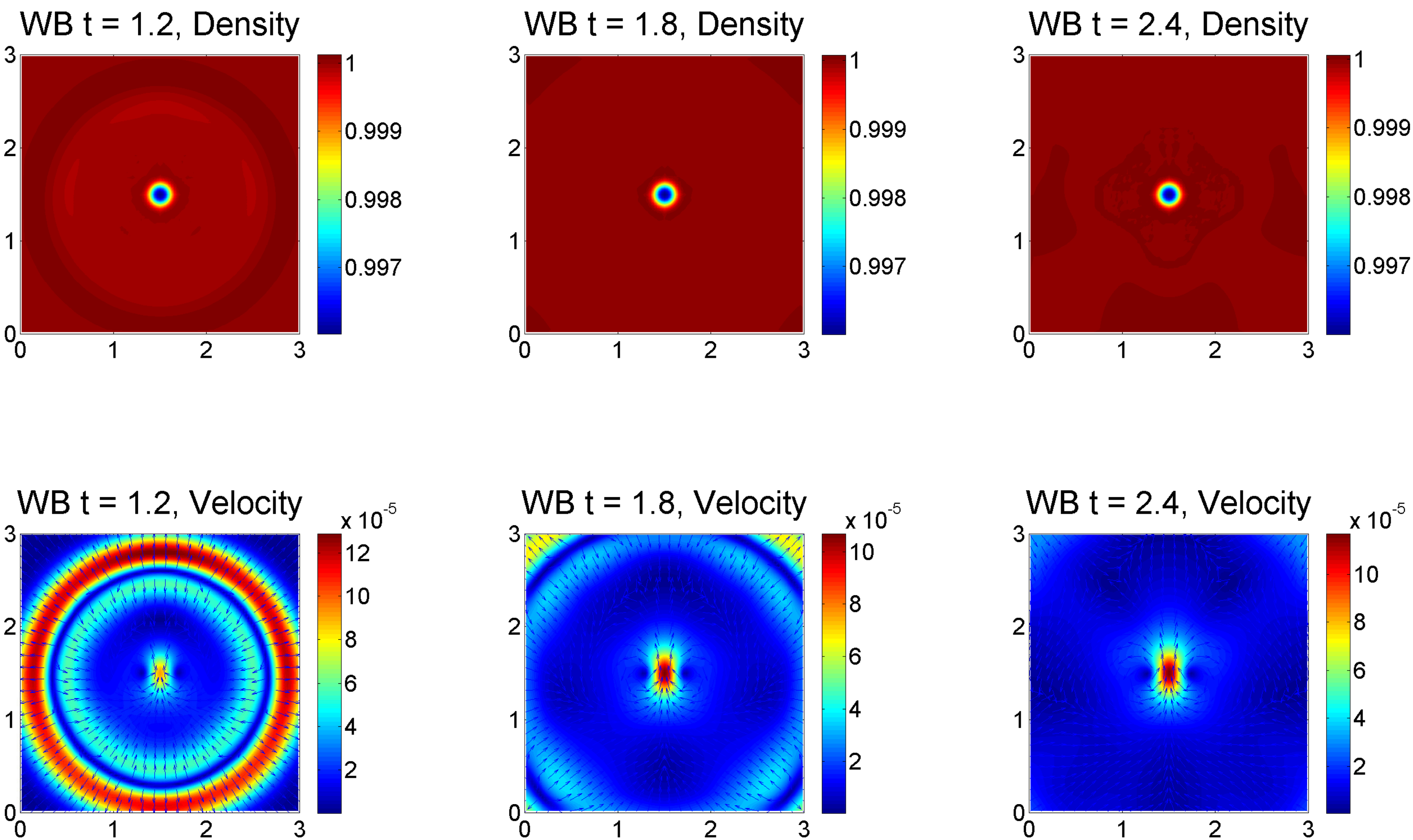
- The ideal gas with $\gamma = 1.4$; domain domain $[0,3] \times [0,3]$
- The gravitational force is $\phi_y = \mathrm{g} = 1$
- The steady-state initial conditions are

$$\rho(x,y,0) \equiv 1, \quad u(x,y,0) = v(x,y,0) \equiv 0$$

$$p(x,y,0) = 1 - gy + \begin{cases} 0.005, & (x-1.5)^2 + (y-1.5)^2 < 0.01, \\ 0, & \text{otherwise.} \end{cases}$$

- A zero-order extrapolation at the boundaries is used


NWB t = 1.2, Density
NWB t = 1.8, Density
NWB t = 2.4, Density
NWB t = 1.2, Velocity
x 10^-4
NWB t = 1.8, Velocity
x 10^-5
NWB t = 2.4, Velocity
x 10^-5


WB t = 1.2, Density
WB t = 1.8, Density
WB t = 2.4, Density
1
0.999
0.998
0.997
WB t = 1.2, Velocity
WB t = 1.8, Velocity
WB t = 2.4, Velocity
x 10^-5

THANK YOU!