A unified approach for hyperbolic systems with multi scale relaxation

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A prototypical multi-scale relaxation model

We consider the usual prototype hyperbolic system

$$\partial_\tau u + \partial_\xi V = 0,$$
$$\partial_\tau V + \partial_\xi p(u) = -\frac{1}{\varepsilon}(V - F(u)),$$

Under the recalling

$$t = \varepsilon^{1+\alpha}\tau, \quad x = \varepsilon\xi, \quad v(x,t) = V(\xi,\tau)/\varepsilon^\alpha, \quad f(u) = F(u)/\varepsilon^\alpha,$$

with $\alpha \in [0,1]$, we get a multi-scale relaxation system:

$$\partial_t u + \partial_x v = 0,$$
$$\varepsilon^\alpha \partial_t v + \frac{1}{\varepsilon^\alpha}\partial_x p(u) = -\frac{1}{\varepsilon}(v - f(u)),$$

Particular cases for multi-scale relaxation system

- $\alpha = 0$,

$$\partial_t u + \partial_x v = 0,$$
$$\partial_t v + \partial_x p(u) = -\frac{1}{\varepsilon}(v - f(u)),$$

this system is hyperbolic with two distinct real characteristics speeds $\pm\sqrt{p'(u)}$ and the limit system ($\varepsilon \to 0$) becomes

$$u_t + f(u)_x = 0, \qquad v = f(u),$$

The stability condition of the system reduces to $|f'(u)|^2 < p'(u)$, i.e., sub-characteristic condition by Liu for hyperbolic system.

- $\alpha = 1$,

$$\partial_t u + \partial_x v = 0,$$
$$\partial_t v + \frac{1}{\varepsilon^2}\partial_x p(u) = -\frac{1}{\varepsilon^2}(v - f(u)),$$

two distinct real characteristics speed $\pm\sqrt{p'(u)}/\varepsilon$, and the limit system ($\varepsilon \to 0$) becomes

$$u_t + f(u)_x = p(u)_{xx}, \quad \text{and} \quad v = f(u) - p(u)_x$$

The stability condition: $|f'(u)|^2 < \frac{p'(u)}{\varepsilon^2}$, and it is satisfied in the limit $\varepsilon \to 0$.

Issues related to the relaxation model $\alpha = 1$

- Stiffness both in the convection and in the relaxation terms. Characteristic speed of the hyperbolic part is of order $1/\varepsilon$ leading to a stability condition like $\Delta t \approx \varepsilon \Delta x$. This condition is too restrictive in the hyperbolic relaxation.
- Special care must be taken to assure that the numerical schemes possess the correct zero-relaxation limit.
- Naive application of most of the popular methods fail to capture the correct behavior of the solution in the relaxation limit $(\varepsilon \to 0)$

Some References:

- G. Naldi, L. Pareschi: *Numerical Schemes for hyperbolic systems of conservation laws with stiff Diffusive relaxation*. SIAM. J. Num. Anal. Vol. 37, No. 4 (2000), pp. 1246-1270.
- S. Jin, L. Pareschi and G. Toscani: *Diffusive relaxation for multiscale Discrete-Velocity Kinetic Equations*. SIAM. J. Num. Anal. Vol. 35, No. 6 (1998), pp. 2405-2439.

Separation of the hyperbolic part into a non stiff and a stiff part. Bring the stiff part to the r.h.s., treat it "implicitly".
Other related works

- Klar *An asymptotic-induced scheme for nonstationary transport equations in the diffusive limit*, SIAM J Numer. Anal. Vol. 35, No. 3, pp. 1073-1094 (1998).
- Mieussens, Lemou [SISC, 2008] *A new asymptotic preserving scheme based on micro-macro dormulation for linear kinetic equations in the diffusion limit*, SISC, 2008.

Related Issues and Goals

- In all above approaches the resulting scheme, when $\varepsilon \to 0$ is an explicit scheme with the usual CFL parabolic condition $\Delta t \approx \Delta x^2$.
- **Goals**:
    - Overcome the CFL parabolic restriction and require $\Delta t \propto \Delta x$, with corse grid $\Delta t \ \Delta x \gg \varepsilon$.
    - construct numerical schemes with the correct asymptotic limit. (AP property)

Removing parabolic stiffness restriction

- (Penalization technique)[1] [2] based on adding two opposite terms to the first equation and treating one explicitly and one implicitly., i.e.

$$
\begin{aligned}
u_t &= \underbrace{-(v_x + \mu(\varepsilon)p(u)_{xx})}_{explicit} + \underbrace{\mu(\varepsilon)p(u)_{xx}}_{implicit} \\
\varepsilon^2 v_t &= \underbrace{-p(u)_x}_{explicit} - \underbrace{(v - f(u))}_{implicit}
\end{aligned}
$$

Here $\mu(\varepsilon) \in [0, 1]$ is a free parameter such that $\mu(0) = 1$, $\mu(1) = 0$[3].

- IMEX (implicit-explicit) RK scheme are used for time integration.
- In the diffusive limit ($\varepsilon \to 0$), the system relaxes to the convection–diffusion equation

$$
u_t + \underbrace{f(u)_x}_{explicit} = \underbrace{p(u)_{xx}}_{implicit}
$$

classical CFL hyperbolic condition for the time step is required $\Delta t \propto \Delta x$.

[1] S.B. L. Pareschi, G. Russo, SIAM JSC 2013
[2] S.B. and G. Russo SINUM 2013
[3] S.B. P.G. LeFloch, G. Russo SIAM SISC, 2014

Different approach

Now a new idea here is to treat in the first equation the component v implicit. We consider $\alpha = 1$ and $p(u) = u$ and $f(u) = 0$ with limit system $u_t = u_{xx}$ and $v = -u_x$.

$$\frac{u^{n+1} - u^n}{\Delta t} = -v_x^{n+1}$$
$$\varepsilon^2 \frac{v^{n+1} - v^n}{\Delta t} = -(u_x^n + v^{n+1}).$$

Solving the second equation for v^{n+1} and making use in the first equation, one obtains the following system

$$\frac{u^{n+1} - u^n}{\Delta t} + \frac{\varepsilon^2}{\varepsilon^2 + \Delta t} v_x^n = \frac{\Delta t}{\varepsilon^2 + \Delta t} u_{xx}$$
$$\frac{v^{n+1} - v^n}{\Delta t} + \frac{1}{\varepsilon^2} u_x^n = -\frac{1}{\varepsilon^2} v^{n+1}.$$

Remarks

- The left part is hyperbolic with characteristic speeds $\lambda = \pm\frac{1}{\sqrt{\varepsilon^2+\Delta t}}$, if $\varepsilon \to 0$, characteristic speeds of the hyperbolic part do not diverge, (with Δt fixed), and the discrete system relaxes to

$$\frac{u^{n+1}-u^n}{\Delta t} = u^n_{xx}, \quad \text{with} \quad v_{n+1} \quad = -u^n_x.$$

  i.e. the limit scheme is a consistent discretization of the limit equation when $\varepsilon \to 0$, i.e. the heat equation $u_t = u_{xx}$, (Asymptotic preserving property, AP)
- If $\Delta t \to 0$ (with ε fixed) the characteristics speeds converge to the usual ones, i.e. $\lambda = \pm\frac{1}{\varepsilon}$, and the system converges to the original one:

$$u_t + v_x = 0$$
$$v_t + \frac{1}{\varepsilon^2}u_x = -\frac{1}{\varepsilon^2}v.$$

We can generalize this result by using IMEX R-K schemes....

IMEX R-K schemes for the time discretization

An s-stage IMEX (IMplicit-EXplicit) Runge-Kutta method is characterized by the $s \times s$ matrices $\tilde{A}$, A and vectors $\tilde{c}$; c; $\tilde{b}$; $b \in \mathbb{R}^s$, represented by the double Butcher tableau:

$$\text{Double Butcher } \mathit{Tableau}: \quad \begin{array}{c|c} \tilde{c} & \tilde{A} \\ \hline & \tilde{b}^T \end{array} \qquad \begin{array}{c|c} c & A \\ \hline & b^T \end{array}.$$

Reason: $u' = f(u) + \frac{1}{\varepsilon} g(u)$. In most cases $f(u)$ is non stiff and $\frac{1}{\varepsilon} g(u)$ contains the stiffness.

Classical order conditions + Coupling order conditions

An important ingredient for AP

Definition

We say that an IMEX R-K scheme is globally stiffly accurate, **GSA**, if

1. The implicit R-K scheme is *stiffly accurate*, **SA**, if $e_s^T A = b^T$, with $e_s^T = (0, ..., 0, \underbrace{1}_{sth-comp.})$. This property is important for the *L*-stability of the scheme.
2. The *s*-stage explicit R-K scheme satisfies the condition $e_s^T \tilde{A} = \tilde{b}^T$ (**FSAL**, *First Same As Last*).

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$$\text{Explicit: } \begin{array}{c|c} 0 & \cdots \\ \vdots & \ddots \\ 1 & \tilde{b}_1, \tilde{b}_2, \cdots, 0 \\ \hline & \tilde{b}_1, \tilde{b}_2, \cdots, 0 \end{array} \qquad \text{Implicit: } \begin{array}{c|c} 0 & \cdots \\ \vdots & \ddots \\ 1 & b_1, b_2, \cdots, b_s \\ \hline & b_1, b_2, \cdots, b_s \end{array}.$$

GSA property guarantees that the numerical solution is identical to the last internal stage values of the scheme.

Removing parabolic stiffness restriction

We consider $\alpha = 1$ and $p(u) = \mathrm{u}$.
Applying an GSA IMEX RK scheme we obtain

$$U = u^n e - \Delta t A V_x$$
$$V = v^n e + \frac{\Delta t}{\varepsilon^2}\tilde{A}(f(U) + U_x) - \frac{\Delta t}{\varepsilon^2} A V.$$

$$u^{n+1} = u^n - \Delta t b^T V_x$$
$$v^{n+1} = v^n - \frac{\Delta t}{\varepsilon^2}\tilde{b}^T f(U) + U_x) - \frac{\Delta t}{\varepsilon^2} b^T V.$$

After algebraic manipulations for the variable u in the limit case $\varepsilon = 0$ we obtain,

$$U = u^n + \Delta t \tilde{A}(U_x - f(U))_x,$$
$$u^{n+1} = u^n + \Delta t \tilde{b}^T (U_x - f(U))_x.$$

We note that the scheme relaxes to an explicit RK scheme, i.e. $\Delta t \approx \Delta x^2$. In order to remove the parabolic stability condition we choose an alternative time discretization.

Alternative time discretization

$$U = u^n e - \Delta t A V_x$$
$$V = v^n e + \frac{\Delta t}{\varepsilon^2}\tilde{A} f(U) - \frac{\Delta t}{\varepsilon^2} A(V + U_x).$$

and

$$u^{n+1} = u^n - \Delta t b^T V_x$$
$$v^{n+1} = v^n - \frac{\Delta t}{\varepsilon^2}\tilde{b}^T f(U) - \frac{\Delta t}{\varepsilon^2} b^T (V + U_x).$$

After algebraic manipulations for the variable u in the limit case $\varepsilon = 0$ we obtain,

$$U = u^n - \Delta t \tilde{A} f(U)_x + \Delta t A U_{xx},$$
$$u^{n+1} = u^n - \Delta t b^T A^{-1} \tilde{A} f(U)_x + \Delta t b^T U_{xx}.$$

AP IMEX R-K scheme

Then if the IMEX Runge-Kutta scheme is **GSA**, i.e. $b^T A^{-1}\tilde{A} = e_s^T \tilde{A} = \tilde{b}^T$ such scheme relaxes to the same IMEX R-K scheme, i.e.,

$$U = u^n - \Delta t \tilde{A} f(U)_x + \Delta t A U_{xx},$$
$$u^{n+1} = u^n - \Delta t \tilde{b}^T f(U)_x + \Delta t b^T U_{xx}.$$

Theorem

AP property: If the IMEX R-K scheme applied to the diffusive relaxation system ($\alpha = 1$) is GSA then in the limit case, i.e. $\varepsilon \to 0$, the IMEX RK scheme relaxes to the same scheme characterized by the pair $(\tilde{A}, \tilde{b})$, (A, b) applied to the limit equation $u_t + f(u)_x = p(u)_{xx}$ and any CFL parabolic restriction is required.

Remarks

- Any penalized technique we require, i.e. no adding and subtracting any quantity in the first equation, in order to overcome the classical CFL parabolic condition $\Delta t \approx \Delta x^2$.
- The property of **GSA** for the IMEX RK scheme guarantees that the scheme is Asymptotic Preserving (AP), i.e. the limit scheme is a consistent discretization of the limit equation when $\varepsilon \to 0$.
- Note that the property of AP does not imply that the scheme preserves the order of accuracy in time in the stiff limit $\varepsilon \to 0$. In the latter case the scheme is said to Asymptotically Accurate (AA).
- The classical order conditions for IMEX RK schemes guarantee the AA only for the u-component, but not for the v-component.
- In order to have a GSA IMEX RK scheme AA for both variables, we require that the coefficients of the scheme satisfy some additional order conditions together to the classical order ones.

Additional order conditions for the v-component

In order to reduce the number of the additional order conditions we required $\tilde{c} = c$.

- $b^T A^{-2} \tilde{A} e = 1$, consistency,
- $b^T A^{-2} \tilde{A} c = 1$, first order,
- $b^T A^{-2} \tilde{A} c^2 = 1, \quad b^T A^{-2} \tilde{A} A c = 1/2, \quad b^T A^{-2} \tilde{A} \tilde{A} c = 1/2$, second order.

We propose a new second order GSA IMEX RK that satisfies this additional order condition and we compare it with other classical second order GSA IMEX RK scheme presented in the literature.

Numerical results

Accuracy test:

$$\begin{cases} \partial_t u + \partial_x v = 0, \\ \partial_t v + \dfrac{1}{\varepsilon^2}\partial_x u = -\dfrac{1}{\varepsilon^2}(v-u), \end{cases}$$

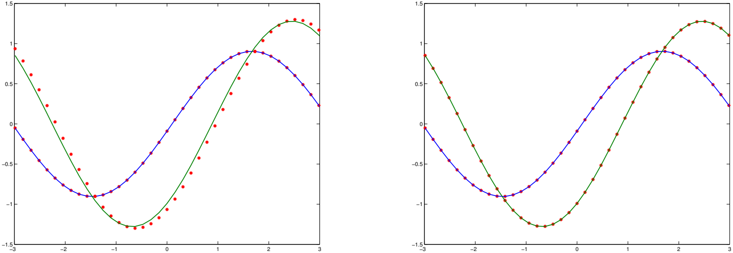
In the limit $\varepsilon \to 0$

$$u_t + u_x = u_{xx}, \quad v = u - \frac{\partial u}{\partial x}$$

Note that the limiting advection-diffusion equation admits the following exact solution

$$u(x,t) = e^{-t}\sin(x-t), \quad v(x,t) = e^{-t}(\sin(x-t) - \cos(x-t)),$$

on the domain $[-\pi, \pi]$ with periodic boundary conditions. We choose $\varepsilon = 10^{-6}$ with final time $T = 0.1$.



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Comparison between classical second order ARS(2,2,2)[4] scheme (right) and the New-IMEX2 GSA scheme [5] (left) with $N = 40$ and $\Delta t = \Delta x$. Green line v-component, blue line u-component.

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[4]U. Asher, S. Ruuth, and R. J. Spiteri, ANM 1997

[5]S. Boscarino, L. Pareschi, G. Russo, paper in preparation

Method	N	L_∞ u-error	Order of u	v-error	Order v
IMEX2-GSA	40	$1.4911e-04$	$--$	$1.3156e-02$	$--$
IMEX2-GSA	80	$3.9405e-05$	1.9199	$1.4897e-02$	-0.1793
IMEX2-GSA	160	$1.1356e-05$	1.7949	$1.1520e-03$	3.6928
IMEX2-GSA	320	$2.8331e-06$	2.0030	$1.2584e-03$	-0.1274
IMEX2-GSA	640	$7.0874e-07$	1.9991	$1.2877e-03$	-0.0332

IMEX3 GSA	40	$5.8318e-06$	$--$
IMEX3 GSA	80	$7.8658e-07$	2.8903
IMEX3 GSA	160	$1.2095e-07$	2.7012
IMEX3 GSA	320	$1.5297e-08$	2.9831
IMEX3 GSA	640	$1.9253e-09$	2.9901

Converge rates for u component for ARS(2,2,2) with $\varepsilon = 10^{-6}$.

N	L_∞ ρ-error	Order u	L_∞ v-error	Order v
40	$1.9129e-04$	$--$	$2.8704e-04$	$--$
80	$4.9963e-05$	1.9368	$8.0261e-05$	1.8385
160	$1.4374e-05$	1.7974	$2.0603e-05$	1.9618
320	$3.5895e-06$	2.0016	$5.2702e-06$	1.9669
640	$9.0120e-07$	1.9939	$1.4011e-06$	1.9113

Converge rates for u and v components for the New IMEX-GSA RK scheme with $\varepsilon = 10^{-6}$.

Viscous Burgers equation

$$\begin{cases} \partial_t \rho + \partial_x j = 0\,, \\ \partial_t j + \dfrac{1}{\varepsilon^2}\partial_x \rho = \dfrac{1}{\varepsilon^{2+\alpha}}\left\{-j + \dfrac{1}{2}\left(\rho^2 - \varepsilon^2 j^2\right)\right\}. \end{cases}$$

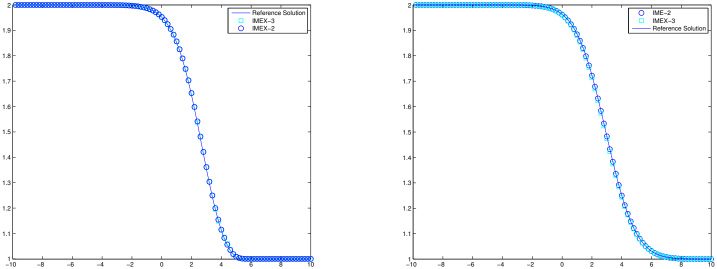
For small values of ε we have the viscous Burgers equation

$$\partial_t \rho + \partial_x \left(\frac{\rho^2}{2}\right) = \varepsilon^\alpha \partial_{xx}\rho \quad \text{and} \quad j = \frac{1}{2}\rho^2 - \varepsilon^\alpha \partial_x \rho.$$

We consider two different initial conditions. The first initial condition is given by two local Maxwelliam characterized by

$$\begin{cases} \rho_L = 1.0, & j_L = 0, & -10 < x < 0, \\ \rho_R = 2.0, & j_R = 0, & 0 < x < 10, \end{cases}$$

with $j = [(1 + \rho^2\varepsilon^2/2)^{1/2} - 1]/2\varepsilon^2$ and $\alpha = 1$.



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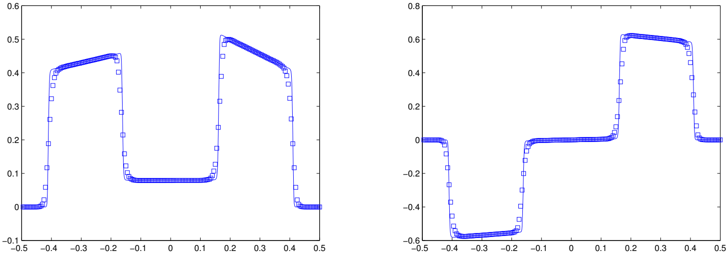
Numerical solutions for the density ρ of Burgers' equation at $T = 2.0$ with $\Delta t = 0.5\Delta x$. Left: the rarefied regime for $\varepsilon = 0.4$. Right: the parabolic regime for $\varepsilon = 10^{-6}$. We use a second order IMEX2-GSA (IMEX-2) and a classical third order IMEX3-GSA ARS(4,4,3)[6]. Numerical solutions are in very good agreement with the reference solutions, solid line.

[6] U. Asher, S. Ruuth, and R. J. Spiteri, ANM 1997

The last test we consider, it is the propagation of an initial square wave. The initial profile is specified as

$$\begin{aligned} \rho &= 1.0, \quad j = 0.0, \quad |x| < 0.125, \\ \rho &= 0.0, \quad j = 0.0, \quad |x| > 0.125 \end{aligned}$$

with reflecting boundary conditions.
We integrate the equations over $[-0.5, 0.5]$ with 200 spatial cells. In Figure we plot the behaviour of the system in the rarefied regime for $\varepsilon = 0.7$ and $\alpha = 0$ at time $T = 0.2$ and in the parabolic regime we take $\varepsilon = 10^{-8}$, $\alpha = 0.5$ and 0.25 and $T = 0.5$. We use third order IMEX3-GSA RK scheme.



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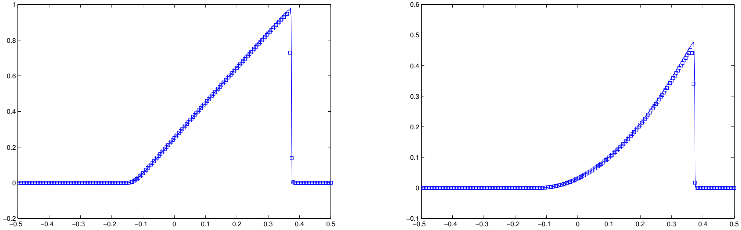
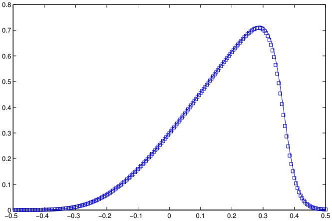
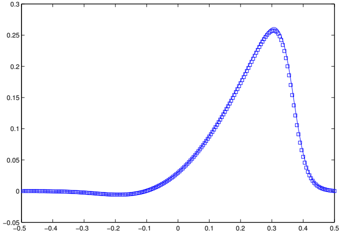
Numerical solutions for the mass density ρ (left) and the momentum (j) (right) in the rarefied regime with $\varepsilon = 0.7$, $\alpha = 0$ and $\Delta t = 0.0025$ and $\Delta x = 0.005$ at $T = 0.2$. *Reference* solution, obtained using fine grids with $\Delta x = 0.001$ (1000 spatial cells)



In the parabolic regime $\varepsilon = 10^{-8}$, with $\alpha = 0.5$ with $\Delta t = 0.004$ (i.e. $\Delta t = 0.8\Delta x$) at time $T = 0.5$.





In the parabolic regime $\varepsilon = 10^{-8}$, with $\alpha = 0.25$ with $\Delta t = 0.004$ (i.e. $\Delta t = 0.8\Delta x$) at time $T = 0.5$.

Conclusions

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- A time step $\Delta t \propto \Delta x$ in the limit case $\varepsilon \to 0$
- GSA property for the IMEX RK scheme guarantees The AP property.
- Additional order conditions to guarantee AA property.

THANK YOU! FOR YOUR ATTENTION