An admissibility and asymptotic-preserving scheme on 2D unstructured meshes

UNIVERSITÉ DE NANTES

Laboratoire de Mathématiques Jean Leray

UMR 6629 - Nantes

F. Blachère[1], R. Turpault[2]

[1] Laboratoire de Mathématiques Jean Leray (LMJL),
Université de Nantes,
[2] Institut de Mathématiques de Bordeaux (IMB),
Bordeaux-INP

NumHyp,
19/06/2015, Cortona (Italy)

Outline

Outline

Problematic

Hyperbolic systems of conservation laws with source terms:

$$\partial_t \mathbf{W} + \operatorname{div}(\mathbf{F}(\mathbf{W})) = \gamma(\mathbf{W})(\mathbf{R}(\mathbf{W}) - \mathbf{W}) \quad (1)$$

- $\mathcal{A}$: set of admissible states,
- $\mathbf{W} \in \mathcal{A} \subset \mathbb{R}^N$,
- $\mathbf{F}$: physical flux,
- $\gamma > 0$: controls the stiffness,
- $\mathbf{R}$: $\mathcal{A} \to \mathcal{A}$; smooth function with some compatibility conditions (cf. [Berthon, LeFloch, and Turpault, 2013]).

Problematic

Hyperbolic systems of conservation laws with source terms:

$$\partial_t \mathbf{W} + \mathrm{div}(\mathbf{F}(\mathbf{W})) = \gamma(\mathbf{W})(\mathbf{R}(\mathbf{W}) - \mathbf{W}) \tag{1}$$

Under compatibility conditions on $\mathbf{R}$, when $\gamma t \to \infty$, (1) degenerates into a diffusion equation:

$$\partial_t w - \mathrm{div}\big(\mathbf{D}(w)\nabla w\big) = 0 \tag{2}$$

- $w \in \mathbb{R}$, linked to $\mathbf{W}$,
- $\mathbf{D}$: positive and definite matrix, or positive function.

Example #1: isentropic Euler with friction

$$\begin{cases} \partial_t\rho + \partial_x\rho u + \partial_y\rho v & = & 0 \\ \partial_t\rho u + \partial_x(\rho u^2 + p(\rho)) + \partial_y\rho uv & = & -\kappa\rho u \\ \partial_t\rho v + \partial_x\rho uv + \partial_y(\rho v^2 + p(\rho)) & = & -\kappa\rho v \end{cases}, \text{ with: } p'(\rho) > 0,\ \kappa > 0$$

$$\mathcal{A} = \{(\rho, \rho u, \rho v)^T \in \mathbb{R}^3 / \rho > 0\}$$

Formalism of (1)

- $\mathbf{W} = (\rho, \rho u, \rho v)^T$
- $\mathbf{R}(\mathbf{W}) = (\rho, 0, 0)^T$
- $\mathbf{F}(\mathbf{W}) = \begin{pmatrix} \rho u, & \rho u^2 + p & , \rho uv \\ \rho v, & \rho uv & , \rho v^2 + p \end{pmatrix}^T$
- $\gamma(\mathbf{W}) = \kappa$

Limit diffusion equation

$$\partial_t\rho - \text{div}\left(\frac{1}{\kappa}\nabla p(\rho)\right) = 0$$

Example #2: M_1 model for radiative transfer

$$\left\{\begin{array}{rcl} \partial_t E + \partial_x F_{R,x} + \partial_y F_{R,y} & = & c\sigma^e a T^4 - c\sigma^a E \\ \partial_t F_{R,x} + c^2 \partial_x P_{xx}(E, \mathbf{F}_R) + c^2 \partial_y P_{xy}(E, \mathbf{F}_R) & = & -c\sigma^f F_{R,x} \\ \partial_t F_{R,y} + c^2 \partial_x P_{yx}(E, \mathbf{F}_R) + c^2 \partial_y P_{yy}(E, \mathbf{F}_R) & = & -c\sigma^f F_{R,y} \\ \rho C_v \partial_t T & = & c\sigma^a E - c\sigma^e a T^4 \end{array}\right.$$

$$\sigma = \sigma(E, F_{R,x}, F_{R,y}, T)$$

$$\mathcal{A} = \{(E, F_{R,x}, F_{R,y}, T) \in \mathbb{R}^4 / E > 0,\, T > 0,\, \|\mathbf{F}_R\| < cE\}$$

Formalism of (1):

- $\mathbf{W} = (E, F_{R,x}, F_{R,y}, T)^T$
- $\mathbf{R}(\mathbf{W})$
- $\mathbf{F}(\mathbf{W}) = \begin{pmatrix} F_{R,x}, & c^2 P_{xx} & , c^2 P_{yx} & , 0 \\ F_{R,y}, & c^2 P_{xy} & , c^2 P_{yy} & , 0 \end{pmatrix}^T$
- $\gamma(\mathbf{W}) = c\sigma^m(\mathbf{W})$

Limit diffusion equation: *equilibrium diffusion equation*

$$\partial_t(\rho C_v T + aT^4) - \operatorname{div}\left(\frac{c}{3\sigma^r}\nabla(aT^4)\right) = 0$$

Conservation laws:

$$\partial_t \mathbf{W} + \mathrm{div}(\mathbf{F}(\mathbf{W})) = \gamma(\mathbf{W})(\mathbf{R}(\mathbf{W}) - \mathbf{W})$$

$\gamma t \to \infty$

Diffusion equation:

$$\partial_t w - \mathrm{div}\big(\mathbf{D}(w)\nabla w\big) = 0$$

consistent:

$\Delta t, \Delta x \to 0$

consistent?

Numerical scheme

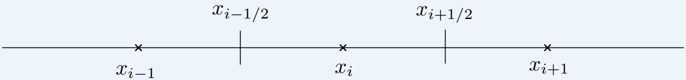
$\gamma t \to \infty$

Limit scheme

Example of a non AP scheme in 1D

$$\partial_t \mathbf{W} + \partial_x(\mathbf{F}(\mathbf{W})) = \gamma(\mathbf{W})(\mathbf{R}(\mathbf{W}) - \mathbf{W}) \tag{1}$$

$$\begin{array}{rclrcl} \mathbf{W} & = & (\rho, \rho u)^T & \mathbf{F}(\mathbf{W}) & = & (\rho u, \rho u^2 + p)^T \\ \gamma(\mathbf{W}) & = & \kappa & \mathbf{R}(\mathbf{W}) & = & (\rho, 0)^T \end{array}$$



$$\frac{\mathbf{W}_i^{n+1} - \mathbf{W}_i^n}{\Delta t} = -\frac{1}{\Delta x}\left(\mathcal{F}_{i+1/2} - \mathcal{F}_{i-1/2}\right) + \gamma(\mathbf{W}_i^n)(\mathbf{R}(\mathbf{W}_i^n) - \mathbf{W}_i^n)$$

Limit

$$\frac{\rho_i^{n+1} - \rho_i^n}{\Delta t} = \frac{b_{i+1/2}\Delta x(\rho_{i+1}^n - \rho_i^n) - b_{i-1/2}\Delta x(\rho_i^n - \rho_{i-1}^n)}{2\Delta x^2}$$

State-of-the-art for AP schemes in 1D

1. control of numerical diffusion:
   - telegraph equations: [Gosse and Toscani, 2002],
   - M1 model: [Buet and Després, 2006], [Buet and Cordier, 2007], [Berthon, Charrier, and Dubroca, 2007], . . .
   - Euler with gravity and friction: [Chalons, Coquel, Godlewski, Raviart, and Seguin, 2010],
2. ideas of hydrostatic reconstruction used in 'well-balanced' scheme used to have AP properties:
   - Euler with friction: [Bouchut, Ounaissa, and Perthame, 2007],
3. using convergence speed and finite differences:
   - [Aregba-Driollet, Briani, and Natalini, 2012],
4. generalization of Gosse and Toscani:
   - [Berthon and Turpault, 2011],
   - [Berthon, LeFloch, and Turpault, 2013].

State-of-the-art for AP schemes in 2D

- Cartesian and admissible meshes $\Longrightarrow$ 1D
- unstructured meshes:
  1. MPFA based scheme:
     - [Buet, Després, and Franck, 2012],
  2. using the diamond scheme (Coudière, Vila, and Villedieu) for the limit scheme:
     - [Berthon, Moebs, and Turpault, 2014],
  3. SW with Manning-type friction:
     - [Duran, Marche, Turpault, and Berthon, 2015].

Outline

Aim of the development

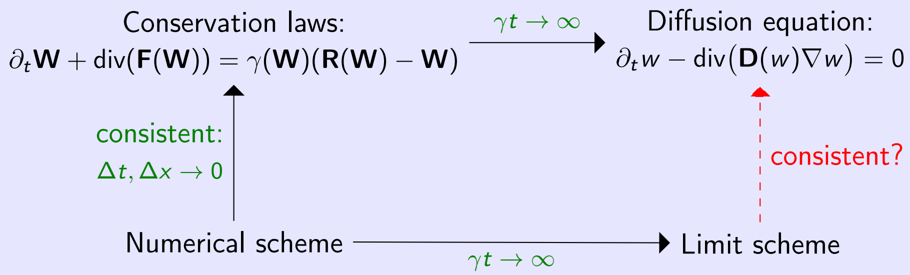
- for any 2D unstructured meshes,
- for any system of conservation laws which could be written as (1),
- under a 'hyperbolic' CFL condition:
  - stability,
  - preservation of $\mathcal{A}$,
  - preservation of the asymptotic behaviour,

$$\max_{\substack{K\in\mathcal{M}\\ i\in\mathcal{E}_K}} \left(b_{K,i}\frac{\Delta t}{\Delta x}\right) \leq \frac{1}{2}.$$

Outline

1. General context and examples
2. Development of an admissibility & asymptotic preserving FV scheme
   - Choice of a limit scheme
   - Hyperbolic part
   - Numerical results for the hyperbolic part
   - Scheme for the complete system
   - Results for the complete system
3. Conclusion and perspectives

Aim of an AP scheme



Choice of the limit scheme

FV scheme to discretize diffusion equations:

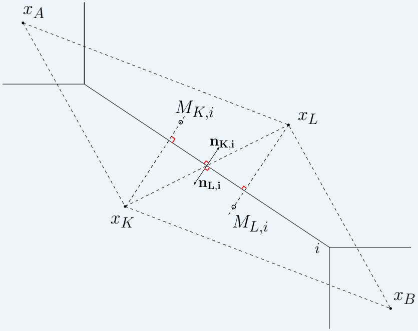
$$\partial_t w - \mathrm{div}(\mathbf{D}(w)\nabla w) = 0. \quad (2)$$

Choice: scheme developed in [Droniou and Le Potier, 2011] (DLP)

- conservative and consistent,
- preserves $\mathcal{A}$,
- nonlinear,

$$(\mathbf{D}(w_K)\nabla_i w_K)\cdot \mathbf{n}_{K,i} = \sum_{J\in S_{K,i}} \overline{\nu}_{K,i}^{J}(w)(w_J - w_K),$$

- $S_{K,i}$ the set of points used for the reconstruction on edges i of cell K,
- $\overline{\nu}_{K,i}^{J}(w) \geq 0$.



$$\begin{aligned} M_{K,i} &= \sum_{J \in S_{K,i}} \omega_{K,i}^J \, \mathbf{x}_J & \qquad w_{M_{K,i}} &= \sum_{J \in S_{K,i}} \omega_{K,i}^J \, w_J \\ M_{L,i} &= \sum_{J \in S_{L,i}} \omega_{L,i}^J \, \mathbf{x}_J & \qquad w_{M_{L,i}} &= \sum_{J \in S_{L,i}} \omega_{L,i}^J \, w_J \end{aligned}$$

Presentation of the DLP scheme

Two approximations

$$\begin{aligned} \nabla_i w_K \cdot \mathbf{n}_{K,i} &= \frac{w_{M_{K,i}} - w_K}{|KM_{K,i}|} \\ \nabla_i w_L \cdot \mathbf{n}_{L,i} &= \frac{w_{M_{L,i}} - w_L}{|LM_{L,i}|} \end{aligned}$$

Convex combination: $\gamma_{K,i} + \gamma_{L,i} = 1$, $\gamma_{K,i} \geq 0$, $\gamma_{L,i} \geq 0$

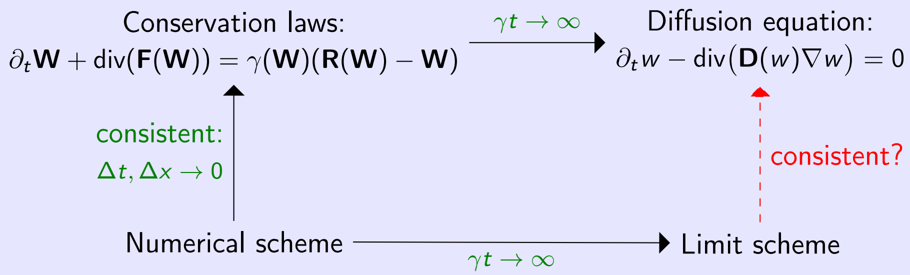
$$\begin{aligned} \nabla_i w_K \cdot \mathbf{n}_{K,i} &= \gamma_{K,i}(w) \nabla_i w_K \cdot \mathbf{n}_{K,i} + \gamma_{L,i}(w) \nabla_i w_L \cdot \mathbf{n}_{L,i} \\ &= \sum_{J \in S_{K,i}} \overline{\nu}_{K,i}^J(w)(w_J - w_K), \text{ with : } \overline{\nu}_{K,i}^J(w) \geq 0 \end{aligned}$$

Properties of the DLP scheme

- consistent with the diffusion equation on any mesh,
- satisfies the maximum principle.

Outline

Aim of an AP scheme



Scheme for the hyperbolic part

$$\mathbf{W}_K^{n+1} = \mathbf{W}_K^n - \frac{\Delta t}{|K|} \sum_{i \in \mathcal{E}_K} \boldsymbol{\mathcal{F}}_i(\mathbf{W}_K, \mathbf{W}_L, \dots) \cdot \mathbf{n}_{K,i} \tag{3}$$

Theorem

We assume that the conservative flux $\boldsymbol{\mathcal{F}}_i$ has the following properties:

1. *Consistency: if $\mathbf{W}_K^n \equiv \mathbf{W}$ then $\boldsymbol{\mathcal{F}}_i \cdot \mathbf{n}_{K,i} = \mathbf{F}(\mathbf{W}) \cdot \mathbf{n}_{K,i}$,*
2. *Admissibility:*
   - a $\exists\ \nu_{K,i}^J \geq 0,\ \boldsymbol{\mathcal{F}}_i \cdot \mathbf{n}_{K,i} = \sum_{J \in \mathcal{S}_{\boldsymbol{K},\boldsymbol{i}}} \nu_{K,i}^J \boldsymbol{\mathcal{F}}_{KJ} \cdot \boldsymbol{\eta}_{KJ}$
   - b $\sum_{i \in \mathcal{E}_{\boldsymbol{K}}} |e_i| \sum_{J \in \mathcal{S}_{\boldsymbol{K},\boldsymbol{i}}} \nu_{K,i}^J \cdot \boldsymbol{\eta}_{KJ} = 0.$

Then the scheme (3) is stable, and preserves $\mathcal{A}$ under the classical following CFL condition:

$$\max_{\substack{\boldsymbol{K} \in \mathcal{M} \\ J \in \mathcal{E}_{\boldsymbol{K}}}} \left(b_{KJ} \frac{\Delta t}{\delta_{KJ}} \right) \leq 1. \tag{4}$$

Example of fluxes (with Rusanov)

1. TP flux:

$$\mathcal{F}_i(\mathbf{W}_K, \mathbf{W}_L) \cdot \mathbf{n}_{K,i} = \frac{\mathbf{F}(\mathbf{W}_K) + \mathbf{F}(\mathbf{W}_L)}{2} \cdot \mathbf{n}_{K,i} - b_{KL}(\mathbf{W}_L - \mathbf{W}_K)$$

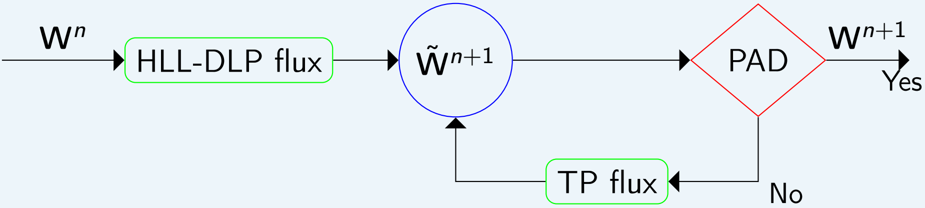
2. HLL-DLP flux:

$$\mathcal{F}_i(\mathbf{W}_K, \mathbf{W}_L, \mathbf{W}_J) \cdot \mathbf{n}_{K,i} = \sum_{J \in \mathcal{S}_{\boldsymbol{K},\boldsymbol{i}}} \nu_{K,i}^J \left(\frac{\mathbf{F}(\mathbf{W}_K) + \mathbf{F}(\mathbf{W}_J)}{2} \cdot \boldsymbol{\eta}_{KJ} - b_{KJ}(\mathbf{W}_J - \mathbf{W}_K) \right)$$

But. . .

1. Fully respect the theorem, but not consistent in the diffusion limit
2. Does not respect the second property of admissibility, but consistent in the limit

Procedure to preserve $\mathcal{A}$



1. $\tilde{\mathbf{W}}^{n+1}$ is computed with the HLL-DLP flux and with the CFL condition (4),
2. *Physical Admissiblility Detection* (PAD): if $\tilde{\mathbf{W}}^{n+1} \in \mathcal{A}$ then the time iterations can continue, else:
    - property 2b is enforced by using the TP flux on all not-admissible cells,
    - Δt and $\tilde{\mathbf{W}}^{n+1}$ are re-computed with the TP flux.

Outline

Convergence

Advection equation

$$\partial_t W + \text{div}(\mathbf{a} W) = 0, \text{ with: } \mathbf{a} = (1,1)^T.$$

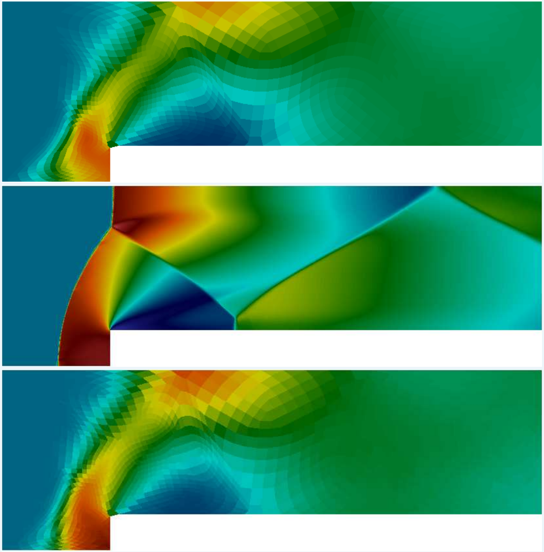
- $W_0(x,y) = \sin(2\pi x)\sin(2\pi y)$
- $W(x,y,t) = \sin(2\pi(x - a_x t))\sin(2\pi(y - a_y t))$

Convergence results

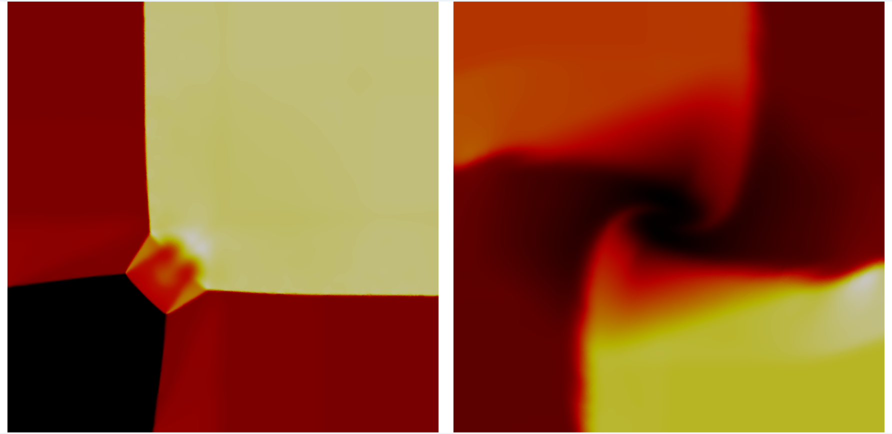
Mesh		HLL-DLP flux		TP flux	
Nb. cells	Size	Error	Slope	Error	Slope
24 705	6.68×10^{-4}	6.68×10^{-2}	—	6.83×10^{-2}	—
98 561	3.34×10^{-4}	3.58×10^{-2}	**0.90**	3.64×10^{-2}	**0.91**
393 729	1.67×10^{-4}	1.86×10^{-2}	**0.95**	1.89×10^{-2}	**0.95**
1 573 889	8.35×10^{-5}	9.49×10^{-3}	**0.97**	9.62×10^{-3}	**0.97**
6 293 505	4.17×10^{-5}	4.81×10^{-3}	**0.98**	4.97×10^{-3}	**0.98**

Wind tunnel with step [Woodward and Colella, 1984]

- fluid at Mach 3
- TP flux correction $< 1\%$



2D Riemann problems [Kurganov and Tadmor, 2002]



- TP flux correction $< 1\%$
- No TP flux correction

Outline

Scheme for the complete model

Complete system

$$\partial_t \mathbf{W} + \mathrm{div}(\mathbf{F}(\mathbf{W})) = \gamma(\mathbf{W})(\mathbf{R}(\mathbf{W}) - \mathbf{W}) \tag{1}$$

$$\mathbf{W}_K^{n+1} = \mathbf{W}_K^n - \frac{\Delta t}{|K|} \sum_{i \in \mathcal{E}_K} |e_i| \overline{\boldsymbol{\mathcal{F}}}_{K,i} \cdot \mathbf{n}_{K,i}, \tag{5}$$

Construction of $\overline{\boldsymbol{\mathcal{F}}}_{K,i}$

$$\overline{\boldsymbol{\mathcal{F}}}_{K,i} \cdot \mathbf{n}_{K,i} = \sum_{J \in \mathcal{S}_{K,i}} \nu_{K,i}^J \overline{\boldsymbol{\mathcal{F}}}_{KJ} \cdot \boldsymbol{\eta}_{KJ}, \ \alpha_{KJ} = \frac{b_{KJ}}{b_{KJ} + \gamma_K \delta_{KJ}},$$

$$\begin{aligned} \overline{\boldsymbol{\mathcal{F}}}_{KJ} \cdot \boldsymbol{\eta}_{KJ} = & \alpha_{KJ} \boldsymbol{\mathcal{F}}_{KJ} \cdot \boldsymbol{\eta}_{KJ} - (\alpha_{KJ} - \alpha_{KK}) \mathbf{F}(\mathbf{W}_K^n) \cdot \boldsymbol{\eta}_{KJ} \\ & - (1 - \alpha_{KJ}) b_{KJ} (\mathbf{R}(\mathbf{W}_K^n) - \mathbf{W}_K^n). \end{aligned}$$

Scheme for the complete model

- Is the scheme with the source term AP?

  $\Rightarrow$ generally not ...

Equivalent formulation

Rewrite (1) into:

$$
\begin{aligned}
\partial_t \mathbf{W} + \mathrm{div}(\mathbf{F}(\mathbf{W})) &= \gamma(\mathbf{W})(\mathbf{R}(\mathbf{W}) - \mathbf{W}), \qquad (1)\\
&= \gamma(\mathbf{W})(\mathbf{R}(\mathbf{W}) - \mathbf{W}) + (\overline{\gamma} - \overline{\gamma})\mathbf{W},\\
\partial_t \mathbf{W} + \mathrm{div}(\mathbf{F}(\mathbf{W})) &= (\gamma(\mathbf{W}) + \overline{\gamma})(\bar{\mathbf{R}}(\mathbf{W}) - \mathbf{W}). \qquad (6)
\end{aligned}
$$

with:

- $\gamma(\mathbf{W}) + \overline{\gamma} > 0$
- $\bar{\mathbf{R}}(\mathbf{W}) = \frac{\gamma \mathbf{R}(\mathbf{W}) + \overline{\gamma}\mathbf{W}}{\gamma + \overline{\gamma}}$

Limit scheme

Introduction of $\overline{\gamma}$

$$\partial_t \mathbf{W} + \mathrm{div}(\mathbf{F}(\mathbf{W})) = (\gamma(\mathbf{W}) + \overline{\gamma}(\mathbf{W}))(\bar{\mathbf{R}}(\mathbf{W}) - \mathbf{W})$$

Rescaling

$$\begin{cases} \gamma & \leftarrow \dfrac{\gamma}{\varepsilon} \\ \Delta t & \leftarrow \dfrac{\Delta t}{\varepsilon} \end{cases}$$

- ε^{-1}: $\mathbf{R}(\mathbf{W}) = \mathbf{W}$
- ε^{0}:

$$\mathbf{W}_K^{n+1} = \mathbf{W}_K^n - \sum_{i \in \mathcal{E}_K} \frac{\Delta t}{|K|} |e_i| \sum_{J \in \mathcal{S}_{K,i}} \frac{\nu_{K,i}^J}{\gamma_K + \overline{\gamma}_{K,i}^J} \left[\frac{b_{KJ}}{\delta_{KJ}} \mathcal{F}_{KJ} - \left(\frac{b_{KJ}}{\delta_{KJ}} - \frac{b_{KK}}{\delta_{KK}} \right) \mathbf{F}(\mathbf{W}_K^n) \right]_{|\mathbf{R}(\mathbf{W}) = \mathbf{W}} \cdot \boldsymbol{\eta}_{KJ}$$

Limit scheme for the Euler equations

- $\mathbf{R}(\mathbf{U}) = \mathbf{U} \Rightarrow \rho u = \rho v = 0$
- $\gamma + \overline{\gamma} := \kappa + \bar{\kappa}$

Actual limit scheme

$$\rho_K^{n+1} = \rho_K^n + \sum_{i \in \mathcal{E}_K} \frac{\Delta t}{|K|} |e_i| \sum_{J \in \mathcal{S}_{K,i}} \nu_{K,i}^J \frac{b_{KJ}^2}{2(\kappa_K + \overline{\kappa}_{K,i}^J)\delta_{KJ}} (\rho_J - \rho_K).$$

AP correction $\overline{\kappa}$

$$\frac{\nu_{K,i}^J b_{KJ}^2}{2(\kappa_K + \overline{\kappa}_{K,i}^J)\delta_{KJ}} (\rho_J - \rho_K) = \frac{\overline{\nu}_{K,i}^J}{\kappa} (p_J - p_K) \Rightarrow \kappa_K + \overline{\kappa}_{K,i}^J = \kappa_K \frac{\nu_{K,i}^J b_{KJ}^2}{2\overline{\nu}_{K,i}^J \delta_{KJ}} \frac{\rho_J - \rho_K}{p_J - p_K} \quad (7)$$

$$\rho_K^{n+1} = \rho_K^n + \sum_{i \in \mathcal{E}_K} \frac{\Delta t}{|K|} |e_i| \sum_{J \in \mathcal{S}_{K,i}} \frac{\overline{\nu}_{K,i}^J}{\kappa} (p_J - p_K) \longrightarrow \partial_t \rho - \mathrm{div} \left(\frac{1}{\kappa} \nabla p(\rho) \right) = 0$$

Theorem

$$\mathbf{W}_K^{n+1} = \mathbf{W}_K^n - \frac{\Delta t}{|K|} \sum_{i \in \mathcal{E}_K} |e_i| \overline{\mathcal{F}}_{K,i} \cdot \mathbf{n}_{K,i}$$

$$= \sum_{J \in \overline{\mathcal{E}_K}} \omega_{KJ} \left(\mathbf{W}_K^n - \frac{\Delta t}{\delta_{KJ}} [\tilde{\mathcal{F}}_{KJ} - \tilde{\mathcal{F}}_{KK}] \cdot \boldsymbol{\eta}_{KJ} \right) \tag{5}$$

Theorem

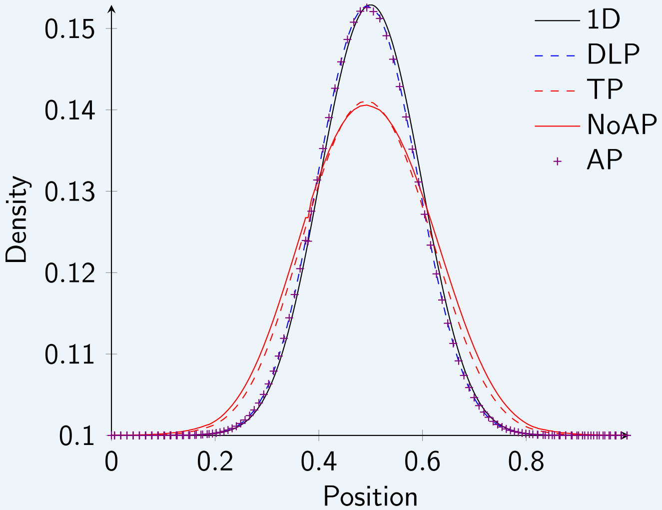
The scheme (5) is consistent with the system of conservation laws (1), under the same assumptions of the previous theorem. Moreover, it preserves the set of admissible states $\mathcal{A}$ under the CFL condition:

$$\max_{\substack{K \in \mathcal{M} \\ J \in \overline{\mathcal{E}_K}}} \left(b_{KJ} \frac{\Delta t}{\delta_{KJ}} \right) \leq 1. \tag{4}$$

Outline

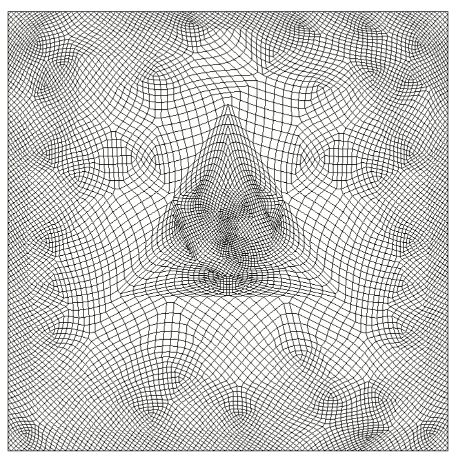
Comparison in the diffusion limit (pseudo 1D)

$$\rho_0(x,y) = 0.1\exp\left(\left(\frac{x-0.5}{0.01}\right)^2\right) + 0.1,\ \mathbf{u} = 0,\ \kappa = 2000,\ t_f = 10$$

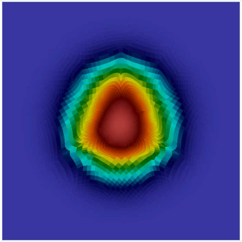


Comparison in the diffusion limit (2D)

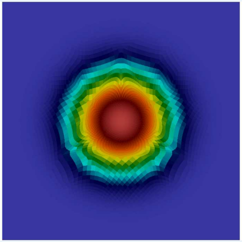
$$\rho_0(x,y) = \begin{cases} 1 & \text{if } (x-\frac{1}{2})^2 + (y-\frac{1}{2})^2 < 0.1^2 \\ 0.1 & \text{otherwise} \end{cases}, \ \mathbf{u} = 0,\ \kappa = 2000,\ t_f = 10$$



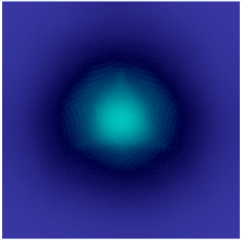
Comparison in the diffusion limit (2D)



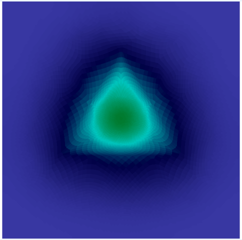
(a) HLL-DLP-AP



(b) DLP



(c) HLL-DLP-NoAP



(d) HLL-TP

Space probe test case (**Euler**-M_1)

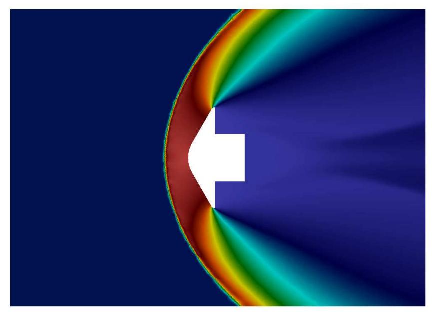


Figure: Density: ρ

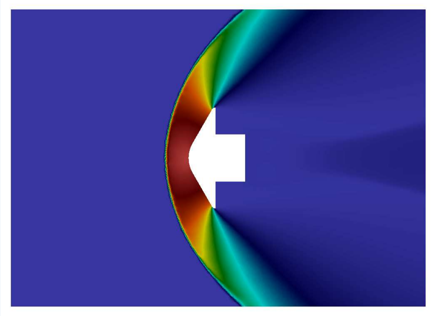


Figure: Pressure: p

Space probe test case (Euler-$\mathbf{M_1}$)

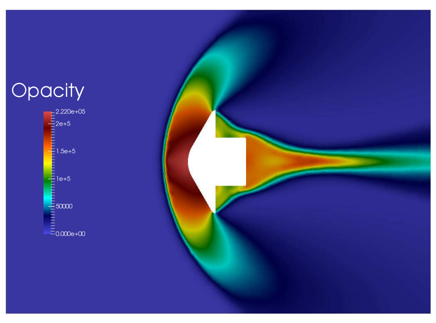


Figure: Opacity: σ^f

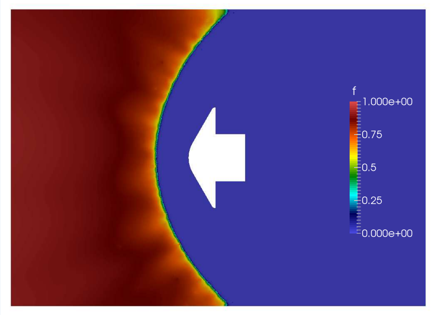


Figure: Anistropy factor: $f = \frac{\|\mathbf{F_R}\|}{cE}$

Outline

Conclusion and perspectives

Conclusion

- generic theory for various hyperbolic problems with asymptotic behaviours,
- first order scheme that preserve $\mathcal{A}$ and the asymptotic limit.

Perspectives

- extend the limit scheme to take care of diffusion systems and nonlinear diffusion equation,
- high-order schemes.

Thanks for your attention.