P.R.A.G.MAT.I.C. 2008
FREE RESOLUTIONS AND HILBERT SERIES: ALGEBRAIC, COMBINATORIAL AND GEOMETRIC ASPECTS

JÜRGEN HERZOG AND VOLKMAR WELKER

Introduction: For a standard graded k-algebra $A = \bigoplus_{n\geq 0} A_n$ let $H(A,t) = \sum_{n\geq 0} \dim_k A_n t^n$ be its Hilbert-series. By classical results $H(A,t) = \frac{h_A(t)}{(1-t)^d}$ is a rational function, where d is the Krull dimension of A and $h_A(t) = h_0 + \cdots + h_r t^r$ a polynomial with integer coefficients with $h_0 = 1$. Since the work of Stanley (see e.g. [41], [37]) in the 70's enumerative properties of the coefficient series $(h_0, \ldots, h_r)$ have been studied intensively. This includes the classification the coefficient series of specific classes of combinatorially interesting algebras (see e.g. [7], [6]) and the explicit control over the coefficient series for classical algebras A (see e.g. [14], [19]). Despite many groundbreaking results some of the most fundamental questions remain open. Among them the g-theorem for Stanley-Reisner rings $k[\Delta]$ of Gorenstein* simplicial complexes [46], the Charney-Davis conjecture for $k[\Delta]$ of flag Gorenstein* simplicial complexes [13] and the unimodality property of the coefficient series for classical rings, such a quotients by determinantal ideals.

Using Gröbner bases techniques (see [42] for aspects relevant to the school topic) it has been shown [35], [4], [12], [30] or conjectured that questions about the unimodality property for some classes of algebras can been resolved with the help of g-theorems for suitable classes of simplicial complexes. On the other hand new g-theorems continue to emerge. In addition flag-ness has been linked to the Koszul property of A and the question of real rootedness of the polynomial $h_A(t)$ [35], [34].

Alexander duality has become an important tool in the study of squarefree monomial ideals, due to the remarkable result of Eagon-Reiner [15] and extensions by Terai [44] , relating data of the resolution of the Stanley-Reisner ring of a simplicial complex to that of its Alexander dual. This fundamental result and further extensions to squarefree modules [36], [45], and even more generally to multigraded modules by Miller [33] has many applications. For example, by using this duality, a classification of all Cohen-Macaulay bipartite graphs has been given [20] and a relationship between the Hilbert-Burch theorem and Dirac's theorem on chordal graphs has been exhibited [21].

In graph theory vertex covers are classical topic of research. In recent papers [22], [23], [24] vertex covers of higher order, not only for graphs but for hypergraphs as well, have been introduced, and attached to these vertex covers so-called vertex cover algebras have been considered which may as well be interpreted as symbolic power algebras of suitable monomial ideals. The research in this direction so far focussed mostly on the question when these algebras are standard graded. It turns out that the property of being standard graded reflects fundamental properties of the associated hypergraphs, such as being Mengerian or unimodular. It is an interesting open problem to study the Hilbert function and the relation type of these algebras. Vertex cover algebras form a different point of view have also been considered in [16] and [18].

A challenging open problem is a conjecture of Stanley [38] concerning so-called Stanley decompositions of multigraded modules and a related conjecture of Stanley [40], [41] concerning partionable simplicial simplicial complexes. The conjecture asserts that there exist Stanley decompositions whose 'size' can be bounded below by the depth of the module. Only very partial results are known [2], [3], [25], [28], [29], [1], as well as an algorithm which allows to compute the so-called Stanley depth [26]. The concept of Stanley decompositions in the special case of the residue class ring of a monomial ideal was first considered by Janet (1936) in his study of linear partial differential equations. Janet divisions also have applications to control theory.

School Topics. During the course we will introduce the basic notions and techniques from commutative algebra and combinatorics for studying the polynomials $h_A(t)$, for the use of Alexander duality and the analysis of vertex covers and Stanley decompositions. We will also exhibit the basic techniques on important examples and example classes and relate the subject to neighboring fields.

(i) Generalities on Hilbert-series, rational generating functions and real rooted polynomials [11], [39].

(ii) Generalities on simplicial complexes and their Stanley-Reisner rings [41].

(iii) Alexander duality, multigraded modules and finite free resolutions [15],[44], [36], [45],[33],[20],[21].

(iv) Koszul-algebras and infinite free resolutions [34], [5].

(v) Lefschetz-algebras and g-theorems [43], [31], [32], [17].

(vi) Vertex cover algebras [22], [23], [24].

(vii) Gröbner bases with squarefree initial ideals [42], [35], [4], [12], [30].

(viii) Transformations preserving real rootedness and applications to algebra [10],[9], [8].

(ix) Stanley decompositions, partionable simpicial complexes and prime filtrations [38], [40], [27], [28], [29], [25], [26].

Background for research problems. As general prerequisite we expect familiarity with basic commutative and homological algebra. Also the knowledge of the fundamental facts of Gröbner basis theory is desirable. We intend to propose several research problems which naturally arise in the discussion of the above mentioned themes and which in principle should be accessible with the methods introduced.

References

[1] I. Anwar, D. Popescu. Stanley Conjecture in small embedding dimension. J. of Algebra **318** 1027-1031 (2007).

[2] J. Apel. On a conjecture of R. P. Stanley. Part I-Monomial Ideals. J. of Algebraic Combin. **17** 3659 (2003).

[3] J. Apel. On a conjecture of R. P. Stanley. Part II-Quotients Modulo Monomial Ideals J. of Algebraic Combin. **17** 57-74 (2003).

[4] C.A. Athanasiadis. Ehrhart polynomials, simplicial polytopes, magic squares and a conjecture of Stanley. J. Reine Angew. Math. **583** 163-174 (2005).

[5] L. Avramov. Infinite free resolutions. In: Elias, J. (ed.) et al., Six lectures on commutative algebra. Basel: Birkhäuser. Prog. Math. **166**, 1-118 (1998).

[6] L.J. Billera, C.W. Lee. Sufficiently of McMullen's conditions for f-vectors of simplicial polytopes. Bull. Am. Math. Soc., New Ser. **2** 181-185 (1980).

[7] A. Björner, G. Kalai. An extended Euler-Poincaré theorem. Acta Math. **161** 279-303 (1988).

[8] J. Borcea, P. Branden, B. Shapiro. Polya-Schur master theorems for circular domains and their boundaries. arXiv:math/0607416, Ann. Math. to appear. preprint 2006.

[9] F. Brenti. Unimodal, log-concave and Polya frequency sequences in combinatorics. Mem. Am. Math. Soc. **413** (1989).

[10] F. Brenti, V. Welker. f-vectors of barycentric subdivisions. Math. Z. to appear. DOI 10.1007/s00209-007-0251-z. preprint 2006.

[11] W. Bruns, J. Herzog. Cohen-Macaulay rings. Cambridge Studies in Advanced Mathematics. **39**. Cambridge: Cambridge University Press. (1993).

[12] W. Bruns, T. Römer. h-vectors of Gorenstein polytopes. J. Comb. Theory, Ser. A **114**, 65-76 (2007).

[13] R. Charney, M. Davis. The Euler characteristic of a nonpositively curved, piecewise Euclidean manifold. Pac. J. Math. **171** 117-137 (1995).

[14] A. Conca, J. Herzog. On the Hilbert function of determinantal rings and their canonical module. Proc. Am. Math. Soc. **122**, 677-681 (1994).

[15] J. Eagon, V. Reiner. Resolutions of StanleyReisner rings and Alexander duality. J. Pure Appl Algebra **130** 265275 (1998).

[16] C.A. Escobar, R. Villarreal and Y. Yoshino. Torsion freeness and normality of blowup rings of monomial ideals. Commutative algebra, 6984, Lect. Notes Pure Appl. Math. 244, Chapman & Hall, 2006.

[17] A. V. Geramita, T. Harima, J. C. Migliore, Yong Su Shin. The Hilbert Function of a Level Algebra. Mem. Am. Math. Soc. **186** (2007).

[18] I. Gitler, C. E. Valencia and R. Villarreal. A note on Rees algebras and the MFMC property. Beiträge Algebra Geom. **48** 141-150 (2007).

[19] S.R.Ghorpade, C. Krattenthaler. The Hilbert series of Pfaffian rings. – Appendix: Geometry of degeneracy loci and a plethora of multiplicity formula. In: Christensen, Chris (ed.) et al., Algebra, arithmetic and geometry with applications. Berlin: Springer. 337-350; appendix 350–356 (2003).

[20] J. Herzog, T. Hibi. Distributive Lattices, Bipartite Graphs and Alexander Duality. J. Algebraic Combin. **22** 289–302 (2005).

[21] J. Herzog, T. Hibi, X. Zheng. Dirac's theorem on chordal graphs and Alexander duality. European J. Comb. **25** 949–960 (2004).

[22] J. Herzog, T. Hibi, N.V. Trung. Symbolic powers of monomial ideals and vertex cover algebras. Adv. Math. **210** 304–322 (2007).

[23] J. Herzog, T. Hibi, N.V. Trung, X. Zheng. Standard graded vertex cover algebras, cycles and leaves. arXiv:math.AC/0606357. Trans. Amer. Math. Soc. to appear, preprint 2006.

[24] J. Herzog, T. Hibi, N.V. Trung. Vertex cover algebras of unimodular hypergraphs. math.AC/0703577. Proc. AMS. to appear, preprint 2007.

[25] J. Herzog, A. Soleyman Jahan, S. Yassemi. Stanley decompositions and partitionable simplicial complexes. arXiv:math.AC/0612848. J.Algebraic. Combin. to appear, preprint 2006.

[26] J. Herzog, M. Vladoiu, X. Zheng. How to compute the Stanley depth of a monomial ideal. arXiv: arXiv:0712.2308. J. of Algebra to appear, preprint 2007.

[27] J. Herzog, D. Popescu. Finite filtrations of modules and shellable multicomplexes. Manuscripta Math. **121** 385-410 (2006).

[28] A. Soleyman Jahan. Prime filtrations of monomial ideals and polarizations. J. Algebra, **312** 1011-1032 (2007).

[29] A. Soleyman Jahan. Stanley decompositions of squarefree modules and Alexander duality. arXiv: arXiv:0709.4145, preprint 2007.

[30] J. Jonsson, V. Welker. A spherical initial ideal for Pfaffians. arXiv:math/0601335, Illinois J. Math. to appear, preprint 2006.

[31] M. Kubitzke, E. Nevo. The Lefschetz property for barycentric subdivisions of shellable complexes. arXiv:0712.1560, preprint 2007.

[32] D.M. Meyer, L. Smith. Poincaré duality algebras, Macaulay's dual systems, and Steenrod operators. Cambridge Tracts in Mathematics **167**. Cambridge: Cambridge University Press. (2005).

[33] E. Miller. The Alexander Duality Functors and Local Duality with Monomial Support. J. of Algebra **231** 180-234 (2000).

[34] A. Polishchuk, L. Posicelskii. Quadratic algebras. University Lecture Series. **37**. Providence, RI: American Mathematical Society (AMS). (2005).

[35] V. Reiner, V. Welker. On the Charney-Davis and Neggers-Stanley conjectures. J. Comb. Theory, Ser. A **109** 247-280 (2005).

[36] T. Römer. Generalized Alexander duality and applications. Osaka J. Math. **38** 469485 (2001).

[37] R.P. Stanley. Hilbert functions of graded algebras. Adv. Math. **28** 57-83 (1978).

[38] R. P. Stanley. Linear Diophantine equations and local cohomology. Invent.Math. **68** 175-193 (1982).

[39] R.P. Stanley. Enumerative combinatorics. Vol. 1. Cambridge Studies in Advanced Mathematics. **49**. Cambridge: Cambridge University Press. (1999).

[40] R. P. Stanley. Positivity Problems and Conjectures in Algebraic Combinatorics. In Mathematics: Frontiers and Perspectives (V. Arnold, M. Atiyah, P. Lax, and B.Mazur, eds.), American Mathematical Society, Providence, RI, 2000, pp. 295-319.

[41] R.P Stanley. Combinatorics and commutative algebra. 2nd ed. Prog. Math. **41**. Basel: Birkhäuser (2005).

[42] B. Sturmfels. Gröbner bases and convex polytopes. University Lecture Series. **8**. Providence, RI: American Mathematical Society (AMS). (1996).

[43] E. Swartz. g-elements, finite buildings and higher Cohen-Macaulay connectivity. J. Comb. Theory, Ser. A **113** 1305-1320 (2006).

[44] N. Terai. Generalization of EagonReiner theorem and h-vectors of graded rings. Preprint 2000.

[45] K. Yanagawa. Alexander duality for Stanley-Reisner rings and squarefree N^n-graded modules. J. Algebra, **225**, 630-645 (2000).

[46] G.M. Ziegler. Lectures on polytopes. Graduate Texts in Mathematics. **152**. Berlin: Springer-Verlag. (1995).