Vector bundles, from classical techniques to new perspectives

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LECTURE 4

Moduli Spaces of Vector Bundles on Surfaces

and on higher dimensional varieties

- Moduli spaces are one of the fundamental constructions of Algebraic Geometry and they arise in connection to classification problems.
- A moduli space for a collection of objects A and an equivalence relation $\sim$ is a classification space, i.e. a space (in some sense of the word) such that each point corresponds to one, and only one, equivalence class of objects.
- A moduli space of vector bundles on a smooth, algebraic variety X is a scheme whose points are in "natural bijection" to equivalence classes of vector bundles on X.

Let $\mathcal{C}$ be a category (e.g., $\mathcal{C} = (Sch/k)$) and let $\mathcal{M} : \mathcal{C} \longrightarrow (Sets)$ be a contravariant moduli functor.

DEFINITION A moduli functor $\mathcal{M} : \mathcal{C} \longrightarrow (Sets)$ is represented by $M \in Ob(\mathcal{C})$ if it is isomorphic to the functor of points of M, h_M, defined by $h_M(S) = Hom_{\mathcal{C}}(S, M)$. The object M is called a fine moduli space for the moduli functor $\mathcal{M}$.

- If a fine moduli space exists, it is unique up to isomorphism.
- There are very few contravariant moduli functors for which a fine moduli space exits.

DEFINITION

A moduli functor $\mathcal{M} : \mathcal{C} \longrightarrow (Sets)$ is corepresented by $M \in Ob(\mathcal{C})$ if $\exists$ natural transformation $\alpha : \mathcal{M} \longrightarrow h_M$ such that $\alpha(\{pt\})$ is bijective and $\forall\ N \in Ob(\mathcal{C})$ and $\forall$ natural transformation $\beta : \mathcal{M} \longrightarrow h_N$ $\exists!\ \varphi : M \longrightarrow N$ such that $\beta = h_\varphi \alpha$, i.e., the following diagram commutes:

$$\begin{tikzcd} \mathcal{M} \arrow[rr, "\alpha"] \arrow[dr, "\beta"'] & & h_M \arrow[dl, "h_\varphi"] \\ & h_N & \end{tikzcd}$$

The object M is called a coarse moduli space for the contravariant moduli functor $\mathcal{M}$.

- If a coarse moduli space exists, it is unique up to isomorphism.
- A fine moduli space for a given moduli functor $\mathcal{M}$ is always a coarse moduli space but not vice versa.

PROBLEM

- To classify stable vector bundles on smooth, irreducible, projective varieties

- MODULI SPACES OF VECTOR BUNDLES

Let X be a smooth, irreducible, projective variety of dimension n and H an ample divisor on X. Fix $P \in \mathbb{Q}[z]$ and consider the moduli functor

$$\mathcal{M}^P_{X,H}(-) : (Sch/k) \to (Sets); \quad S \longmapsto \mathcal{M}^P_{X,H}(S),$$

$\mathcal{M}^P_{X,H}(S) = \{S-$ flat families $\mathcal{F} \longrightarrow X \times S$ of μ-stable vector bundles on X with Hilbert polynomial $P\}/\sim$, with $\mathcal{F} \sim \mathcal{F}'$ if, and only if, $\mathcal{F} \cong \mathcal{F}' \otimes p^*L$ for some $L \in Pic(S)$ being $p : S \times X \to S$ the natural projection. And if $f : S' \to S$ is a morphism in (Sch/k), let $\mathcal{M}^P_{X,H}(f)(-)$ be the map obtained by pulling-back sheaves via $f_X = f \times id_X$:

$$\mathcal{M}^P_{X,H}(f)(-) : \mathcal{M}^P_{X,H}(S) \longrightarrow \mathcal{M}^P_{X,H}(S'); \quad [\mathcal{F}] \longmapsto [f^*_X \mathcal{F}].$$

MARUYAMA's THEOREM

$\mathcal{M}^P_{X,H}(-)$ has a coarse moduli scheme $M^P_{X,H}$ which is a separated scheme and locally of finite type over k. This means

- $\exists \Psi : \mathcal{M}^P_{X,H}(-) \longrightarrow Hom(-, M^P_{X,H})$, which is bijective for any reduced point x_0.

- $\forall N$ and $\forall \Phi : \mathcal{M}^P_{X,H}(-) \longrightarrow Hom(-, N)$, $\exists ! \varphi : M^P_{X,H} \longrightarrow N$ for which the following diagram commutes

$$\begin{tikzcd}
\mathcal{M}^P_{X,H}(-) \arrow[rr, "\Psi"] \arrow[dr, "\Phi"'] & & Hom(-, M^P_{X,H}) \arrow[dl, "\varphi_*"] \\
& Hom(-, N) &
\end{tikzcd}$$

REMARKS:

(1) $M^P_{X,H}$ is unique (up to isomorphism).

(2) In general, $M^P_{X,H}$ is not a fine moduli space. In fact, there is no a priori reason why the map

$$\Psi(S) : \mathcal{M}^P_{X,H}(S) \to Hom(S, M^P_{X,H})$$

should be bijective for varieties S other than $\{pt\}$.

(3) $M^P_{X,H}$ decomposes into a disjoint union of schemes $M^s_{X,H}(r; c_1, \cdots, c_{min(r,n)})$ where $M^s_{X,H}(r; c_1, \cdots, c_{min(r,n)})$ is the moduli space of rank r, μ-stable vector bundles on X with Chern classes $(c_1, \cdots, c_{min(r,n)})$ up to numerical equivalence.

PROBLEM:

When $M^s_{X,H}(r;c_1,\cdots,c_{min(r,n)})$ is non-empty?

- If X is a smooth curve of genus $g \geq 2$, then the moduli space of μ-stable vector bundles of rank r and fixed determinant is smooth of dimension $(r^2-1)(g-1)$.

- If $dim(X) \geq 3$, then there are no general results which guarantee the non-emptiness of the moduli space of μ-stable vector bundles on X.

- If $dim(X) = 2$, then the existence conditions are well known whenever X is $\mathbb{P}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1$ and, in general, it is known that the moduli space $M^s_{X,H}(r;c_1,c_2) = \emptyset$ if $\Delta(r;c_1,c_2) < 0$ (Bogomolov's inequality) and non-empty provided $\Delta(r;c_1,c_2) \gg 0$.

THEOREM

Let X be a smooth, irreducible, projective variety of dimension n and let E be a μ-stable vector bundle on X with Chern classes $c_i(E) = c_i \in H^{2i}(X, \mathbb{Z})$, represented by a point $[E] \in M^s_{X,H}(r; c_1, \cdots, c_{min(r,n)})$. Then,

$$T_{[E]} M^s_{X,H}(r; c_1, \cdots, c_{min(r,n)}) \cong Ext^1(E, E).$$

If $Ext^2(E, E) = 0$ then $M^s_{X,H}(r; c_1, \cdots, c_{min(r,n)})$ is smooth at $[E]$. In general, we have the following bounds:

$$dim_k Ext^1(E, E) \geq dim_{[E]} M^s_{X,H}(r; c_1, \cdots, c_{min(r,n)})$$

$$\geq dim_k Ext^1(E, E) - dim_k Ext^2(E, E).$$

If X is a smooth projective surface and $E \in M^s_{X,H}(r; c_1, c_2)$, then (Hirzebruch-Riemann-Roch's Theorem)

$$ext^1(E,E) - ext^2(E,E) = 2rc_2(E) - (r-1)c_1^2 - r^2\chi(\mathcal{O}_X) + 1 + p_g(X).$$

The number $2rc_2(E) - (r-1)c_1^2 - r^2\chi(\mathcal{O}_X) + 1 + p_g(X)$ is called the expected dimension of $M^s_{X,H}(r; c_1, c_2)$.

PROBLEMS

- When $M^s_{X,H}(r; c_1, \cdots, c_{min(r,n)})$ is non-empty?
- To study the local and global structure of the moduli space $M^s_{X,H}(r; c_1, \cdots, c_{min(r,n)})$.
- What does the moduli space look like, as an algebraic variety? Is it, for example, connected, irreducible, rational or smooth? What does it look as a topological space? What is its geometry?

PROPOSITION

Let $X = \mathbb{P}^1_k \times \mathbb{P}^1_k$ be a smooth quadric surface and denote by ℓ and m the standard basis of $Pic(X) \cong \mathbb{Z}^2$. $\forall 0 < c_2 \in \mathbb{Z}$, we fix the ample divisor $L = \ell + (2c_2 - 1)m$. We have

(1) The moduli space $M_{X,L}(2; \ell, c_2)$ is a smooth, irreducible, rational projective variety of dimension $4c_2 - 3$. Even more, $M_{X,L}(2; \ell, c_2) \cong \mathbb{P}^{4c_2-3}$.

(2) For any two ample divisors L_1 and L_2 on $X = \mathbb{P}^1_k \times \mathbb{P}^1_k$, the moduli spaces $M_{X,L_1}(2; \ell, c_2)$ and $M_{X,L_2}(2; \ell, c_2)$ are birational whenever non-empty.

MODULI SPACES

- Moduli space of stable vector bundles on surfaces
- Moduli space of vector bundles on higher dimensional varieties

Surfaces

- X smooth, irreducible, algebraic surface / k and H an ample line bundle on X.
- $M_{X,H}(r; L, n) =$ moduli space of rank r vector bundles E on X, μ-stable with respect to H with $det(E) = L \in Pic(X)$ and $c_2(E) = n \in \mathbb{Z}$.
- $\overline{M}_{X,H}(r; L, n) =$ moduli space of rank r torsion free sheaves E on X, GM-semistable with respect to H with $det(E) = L \in Pic(X)$ and $c_2(E) = n \in \mathbb{Z}$.

REMARK

From now on, we will assume that the discriminant

$$\Delta(r; L, n) := 2rn - (r-1)L^2 \gg 0.$$

- If $\Delta(r; L, n) < 0$ then $M_{X,H}(r; L, n) = \emptyset$ (Bogomolov's inequality)
- If $\Delta(r; L, n) \gg 0$ then $M_{X,H}(r; L, n) \neq \emptyset$

THEOREM

Let X be a smooth, irreducible, projective surface and let H be an ample line bundle on X.

If $\Delta(r; L, n) \gg 0$, then the moduli space $M_{X,H}(r; L, n)$ is a normal, generically smooth, irreducible, quasi-projective variety of dimension $2rn - (r-1)L^2 - (r^2-1)\chi(\mathcal{O}_X)$.

QUESTION

Let X be a smooth, irreducible, projective variety and let H and H' be two different ample line bundles on X.

- What is the difference between the moduli spaces $M_{X,H}(r; L, n)$ and $M_{X,H'}(r; L, n)$?
- Are $M_{X,H}(r; L, n)$ and $M_{X,H'}(r; L, n)$ isomorphic or, at least, birational?

- The ample cone C_X of X has a chamber structure such that the moduli space $M_{X,H}(r;L,n)$ only depends on the chamber of H.
- In general, the moduli space $M_{X,H}(r;L,n)$ changes when H crosses a wall between two chambers.

DEFINITION

Let C_X be the ample cone in $\mathbb{R} \otimes Num(X)$. $\forall \xi \in Num(X)$, we define

$$W^\xi := C_X \cap \{x \in Num(X) \otimes \mathbb{R} \ \ s.t. \ \ x \cdot \xi = 0\}.$$

W^ξ is called the wall of type (c_1, c_2) determined by ξ if and only if $\exists G \in Pic(X)$ with $G \equiv \xi$ such that $G + c_1$ is divisible by 2 in $Pic(X)$ and $c_1^2 - 4c_2 \leq G^2 < 0$.

W^ξ is a non-empty wall of type (c_1, c_2) if $\exists$ ample line bundle L with $L\xi = 0$.

Let $W(c_1, c_2)$ be the union of the walls of type (c_1, c_2). A chamber of type (c_1, c_2) is a connected component of $C_X \setminus W(c_1, c_2)$.

- The ample cone C_X of X has a chamber structure such that the moduli space $M_{X,H}(r; L, n)$ only depends on the chamber of H.
- In general, the moduli space $M_{X,H}(r; L, n)$ changes when H crosses a wall between two chambers.

THEOREM Let X be a smooth, irreducible, projective surface and let H and H' be ample line bundles on X. If $\Delta(r; L, n) \gg 0$, then the moduli spaces $M_{X,H}(r; L, n)$ and $M_{X,H'}(r; L, n)$ are birational.

- When X is a smooth Fano surface and $r = 2$, the assumption $\Delta(r; L, n) \gg 0$ can be avoided.

REMARK The birational map between $M_{X,H}(2; L, c_2)$ and $M_{X,H'}(2; L, c_2)$ is not, in general, an isomorphism

- We can reduce the study of the rationality of the moduli space $M_{X,H}(r; L, n)$ for any ample line bundle H to the study of the rationality of $M_{X,H}(r; L, n)$ for a suitable ample divisor H.

EXAMPLE $X = \mathbb{P}^1_k \times \mathbb{P}^1_k$, $Pic(X) =< \ell, m >$, $L = \ell + 5m$. The moduli space $M_{X,L}(2; \ell, 3) \cong \mathbb{P}^9$ and hence it is rational. Applying the last results, we conclude that for any other ample line bundle H on X the moduli space $M_{X,H}(2; \ell, 3)$ is rational whenever non-empty.

RATIOINALITY OF $M_{X,H}(r; L, k(n))$

- There is at present no counterexample known to the question whether the moduli spaces are always rational provided the underlying surface is rational.

QUESTION

Whether the moduli spaces $M_{X,H}(r; c_1, c_2)$ are rational provided X is rational?

THEOREM

Let X be a smooth rational surface, $L \in Pic(X)$ and $n \in \mathbb{Z}$. Assume that $\Delta(2; L, n) \gg 0$. Then, there exists an ample line bundle H on X such that the moduli space $M_{X,H}(2; L, n)$ is rational.

Higher dimensional varieties

- X is a smooth, projective, n-dimensional variety over an algebraically closed field of characteristic 0
- We denote by $M_{X,L}(r; c_1, \cdots, c_{min\{r,n\}})$ the moduli space of rank r, vector bundles E on X, μ-stable with respect to an ample line bundle L with fixed Chern classes $c_i(E) = c_i \in H^{2i}(X, \mathbb{Z})$.
- It was a major result in the theory of vector bundles on an algebraic surface S the proof that the moduli space $M_{S,L}(r; c_1, c_2)$ of rank r vector bundles E on S, μ-stable with respect to a fixed ample line bundle L and with given Chern classes $c_i \in H^{2i}(S, \mathbb{Z})$ is irreducible and generically smooth provided $c_2 \gg 0$.

Examples:

- Ein proved that the minimal number of irreducible components of the moduli space of rank 2 stable vector bundles on $\mathbb{P}^3$ with fixed c_1 and c_2 going to infinity grows to ∞.

- For all integers $k \geq 3$ and $n \geq 2$, the moduli spaces, $MI_{2n+1}(k)$, of mathematical instanton bundles over $\mathbb{P}^{2n+1}$ with second Chern class $c_2 = k$ are singular.

Example

Take $\mathcal{E} := \bigoplus_{i=0}^{d} \mathcal{O}_{\mathbb{P}^1}(a_i)$ with $0 = a_0 \leq a_1 \leq ... \leq a_d$ and $a_d > 0$. Let

$$\pi : X = \mathbb{P}(\mathcal{E}) = Proj(Sym(\mathcal{E})) \longrightarrow \mathbb{P}^1$$

be the projectived vector bundle and let $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ be the tautological line bundle. $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ defines a birational map

$$f : X = \mathbb{P}(\mathcal{E}) \longrightarrow \mathbb{P}^N,$$

where $N = d + \sum_{i=0}^{d} a_i$. The image of f is a variety Y of dimension $d+1$ and minimal degree ($deg(Y) = \sum_{i=0}^{d} a_i$) called rational normal scroll. By abusing, we shall also call to X rational normal scroll.

Let H be the class in $Pic(X)$ associated to the tautological line bundle $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ on X and let F be the fiber of π. We have

$$Pic(X) \cong \mathbb{Z}^2 \cong < H, F > \text{ with } H^{d+1} = \sum_{i=0}^{d} a_i; H^d F = 1; F^2 = 0.$$

- A divisor $L = aH + bF$ on X is ample if and only if $a > 0$ and $b > 0$.

Our approach will be to write μ-stable with respect to L, rank 2 vector bundles E on X, as an extension of two line bundles.
It is well known is that any vector bundle of rank $r \geq 2$ on a curve can be written as an extension of lower rank vector bundles.
For higher dimensional varieties we may not be able to get such a nice result. (For instance, it is not true for vector bundles on $X = \mathbb{P}^n$).
However it turns to be true for certain μ-stable with respect to L, rank 2 vector bundles E on rational normal scrolls. In fact, using this idea we construct big enough families of μ-stable with respect to L, rank 2 vector bundles E on rational normal scrolls.

Construction Let X be a $(d+1)$-dimensional, rational, normal scroll, $0 \ll c_2 \in \mathbb{Z}$. We construct a rank 2 vector bundle E on X as a non-trivial extension

$$e: \qquad 0 \longrightarrow \mathcal{O}_X(H - c_2 F) \longrightarrow E \longrightarrow \mathcal{O}_X(c_2 F) \longrightarrow 0.$$

Theorem: Let X be a $(d+1)$-dimensional, rational, normal scroll and $0 \ll c_2 \in \mathbb{Z}$. We fix the ample divisor $L = dH + bF$ on X with $b = 2c_2 - H^{d+1} - 1$. Then $M_L(2; H, c_2 HF)$ is a smooth, irreducible, rational, projective variety of dimension

$$2(d+1)c_2 - H^{d+1} - (d+2).$$

- The last Theorem reflects nicely the general philosophy that (at least for suitable choice of the Chern classes and the ample line bundle) the geometry of the underlying variety and of the moduli spaces are intimately related.
- We hope that phenomena of this sort will be true for other high dimensional varieties.

Remarks:

- To bigger $n-r$ is the more difficult it becomes to find rank r vector bundles on $\mathbb{P}^n$.

PROBLEM: For which n and r are there indecomposable rank r vector bundles on $\mathbb{P}^n$?

HARTSHORNE's CONJECTURE There are no rank 2 vector bundles on $\mathbb{P}^n$, $n \geq 6$.

Some of the techniques we use to construct/study vector bundles on smooth projective varieties and to address their classification problem are

- Serre's construction,
- Beilinson's type spectral sequences,
- Elementary transformations, and
- Monads.

DEFINITION

Let X be a smooth projective variety. A monad on X is a complex of vector bundles:

$$M_\bullet : \quad 0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$$

which is exact at A and at C. The sheaf $E := Ker(\beta)/Im(\alpha)$ is called the cohomology sheaf of the monad $M_\bullet$.

- Monads were first introduced by Horrocks who showed that all vector bundles E on $\mathbb{P}^3$ can be obtained as the cohomology bundle of a monad of the following kind:

$$0 \longrightarrow \oplus_i \mathcal{O}_{\mathbb{P}^3}(a_i) \longrightarrow \oplus_j \mathcal{O}_{\mathbb{P}^3}(b_j) \longrightarrow \oplus_n \mathcal{O}_{\mathbb{P}^3}(c_n) \longrightarrow 0.$$

LINEAR MONADS

DEFINITION Let X be a nonsingular projective variety A linear monad on X is the short complex of sheaves

$$M_\bullet \;:\; 0 \to \mathcal{O}_X(-1)^a \xrightarrow{\alpha} \mathcal{O}_X^b \xrightarrow{\beta} \mathcal{O}_X(1)^c \to 0 \tag{1}$$

which is exact on the first and last terms. EXAMPLE Manin and Drinfield proved that mathematical instanton bundles E on $\mathbb{P}^3$ with quantum number k correspond to linear monads

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^3}(-1)^k \longrightarrow \mathcal{O}_{\mathbb{P}^3}^{2k+2} \longrightarrow \mathcal{O}_{\mathbb{P}^3}(1)^k \longrightarrow 0.$$

PROPOSITION (Fløystead)

Let $n \geq 1$. There exist monads on $\mathbb{P}^n$ whose entries are linear maps, i.e. linear monads

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-1)^a \xrightarrow{\alpha} \mathcal{O}_{\mathbb{P}^n}^b \xrightarrow{\beta} \mathcal{O}_{\mathbb{P}^n}(1)^c \longrightarrow 0$$

if and only if at least one of the following conditions holds:

(1) $b \geq 2c + n - 1$ and $b \geq a + c$.

(2) $b \geq a + c + n$.

If so, there actually exists a linear monad with the map α degenerating in expected codimension $b - a - c + 1$.

- If $n = 3$, $a = c = k$ and $b = 2k + 2$ we have an instanton bundle.

Characterization of linear sheaves

- Cohomological characterization of linear sheaves on $\mathbb{P}^n$, $n \geq 1$ (i.e. cohomology sheaves of linear monads).

Fix integers a, b and c such that

(1) $b \geq 2c + n - 1$ and $b \geq a + c$, or

(2) $b \geq a + c + n$.

We have:

PROPOSITION

Let E be a rank $b-a-c$ torsion free sheaf on $\mathbb{P}^n$ with Chern polynomial $c_t(E) = \frac{1}{(1-t)^a(1+t)^c}$. It holds:

(1) If $b < c(n+1)$ and E has natural cohomology in the range $-n \leq j \leq 0$, then E is the cohomology sheaf of a linear monad

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-1)^a \longrightarrow \mathcal{O}_{\mathbb{P}^n}^b \longrightarrow \mathcal{O}_{\mathbb{P}^n}(1)^c \longrightarrow 0.$$

(2) If E is the cohomology sheaf of a linear monad

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-1)^a \longrightarrow \mathcal{O}_{\mathbb{P}^n}^b \longrightarrow \mathcal{O}_{\mathbb{P}^n}(1)^c \longrightarrow 0$$

and $H^0(\mathbb{P}^n, E) = 0$, then E has natural cohomology in the range $-n \leq j \leq 0$.

QUESTION:

QUESTION Is any linear sheaf on a cyclic variety μ-stable or at least μ-semistable?

QUESTION: Is any instanton bundle on $\mathbb{P}^{2n+1}$ μ-stable?

PROBLEM: Cohomological characterization of linear shaves on projective varieties