VECTOR BUNDLES,
from classical techniques to new perspectives

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LECTURE 3

Tilting Sheaves on Toric Varieties

KING's CONJECTURE

CONJECTURE: Any smooth complete toric variety has a tilting bundle whose summands are line bundles.

We prove the conjecture for the following types of smooth complete toric varieties:

- Any d-dimensional smooth complete toric variety with splitting fan.
- Any d-dimensional smooth complete toric variety with Picard number ≤ 2.
- Any d-dimensional smooth Fano toric variety with (quasi)-maximum Picard number.
- The blow up of any smooth complete minimal toric surface at T-invariants points.

DEFINITION A coherent sheaf $T \in \mathcal{O}_X$-mod on a smooth projective variety X is called a tilting sheaf (or, when it is a locally free sheaf, a tilting bundle) if

(1) It has no higher self-extensions, i.e., $Ext^i(T,T) = 0$ for all $i > 0$,

(2) The endomorphism algebra of T, $A = Hom_X(T,T)$, has finite global homological dimension, and

(3) The direct summands of T generate the derived category $\mathcal{D}^b(\mathcal{O}_X - mod)$ of bounded complexes of coherent sheaves of $\mathcal{O}_X$-mod.

The importance of tilting sheaves realize on the facts:

- Tilting sheaves can be characterized as those sheaves T such that the functors
$$\mathbf{R}Hom_X(T,-) : D^b(\mathcal{O}_X - mod) \longrightarrow D^b(A) \text{ and}$$
$$- \otimes^{\mathbf{L}}_A T : D^b(A) \longrightarrow D^b(\mathcal{O}_X - mod)$$
define mutually inverse equivalences between the derived categories of bounded complexes of coherent sheaves on X and of bounded complexes of finitely generated right A-modules, respectively.

- They play an important role in the problem of characterizing the smooth projective varieties X determined by its derived category of bounded complexes of coherent sheaves $D^b(\mathcal{O}_X - mod)$.

The importance of tilting sheaves realize on the facts:

- Since the fundamental paper of Beilinson where he proves that $T_1 = \oplus_{i=0}^{n} \mathcal{O}_{\mathbb{P}^n}(i)$ and $T_2 = \oplus_{i=0}^{n} \Omega^{i}_{\mathbb{P}^n}(i)$ are tilting bundles on $\mathbb{P}^n$, tilting bundles have become a major tool in classifying vector bundles over smooth projective varieties

PROBLEM

PROBLEM: To characterize smooth projective varieties which have a tilting bundle.

REMARK: The existence of tilting sheaves $T = \oplus_{i=0}^{m} T_i$ imposes rather a strong restriction on X, namely that the Grothendieck group $K_0(X) = K_0(\mathcal{O}_X - mod)$ is isomorphic to $\mathbb{Z}^{m+1}$.

EXAMPLE: Since, the Grothendieck group $K_0(S)$ of a smooth cubic 3-fold $S \subset \mathbb{P}^4$ has torsion, there are no tilting bundles on S.

KING's CONJECTURE

CONJECTURE: Any smooth complete toric variety has a tilting bundle whose summands are line bundles.

REMARK: There are examples of smooth projective varieties X such that any tilting bundle T on X has a summand of higher rank. Example: $Gr(k,n)$, $k < n$.

CONTENTS:

S 1. Basic facts on toric varieties

Let X be a smooth complete toric variety of dimension n over the complex numbers, i.e., X is a smooth variety with an action by the algebraic torus $(\mathbb{C}^*)^n$ and a dense equivariant embedding $(\mathbb{C}^*)^n \longrightarrow X$. X is characterized by a fan $\Sigma := \Sigma(X)$ of strongly convex polyhedral cones in $N \otimes_{\mathbb{Z}} \mathbb{R}$ where N is the lattice $\mathbb{Z}^n$, i.e., N is a free abelian group of rank n.

The cones σ of Σ are rational, i.e. generated by lattice points. For any $0 \leq i \leq n$, we put $\Sigma(i) := \{\sigma \in \Sigma \mid \text{ dim}\sigma = i\}$. Note that to any 1-dimensional cone $\sigma \in \Sigma(1)$ there is a unique generator $v \in N$ such that $\sigma \cap N = \mathbb{Z}_{\geq 0} \cdot v$, we call to v a ray generator.

There is a one-to-one correspondence between ray generators v and toric divisors D on X. It holds:

- Given toric divisors D_1, $\ldots$, D_k on X with corresponding ray generators v_1, $\ldots$, v_k we have $D_1 \cap \cdots \cap D_k \neq \emptyset$ iff v_1, $\ldots$, v_k span a cone in Σ.
- If X is a smooth toric variety of dimension n (hence n is also the dimension of the lattice N) and m is the number of toric divisors of X (and hence the number of 1-dimensional rays in Σ) then we have an exact sequence of $\mathbb{Z}$-modules:

$$0 \longrightarrow Hom_{\mathbb{Z}}(N, \mathbb{Z}) \longrightarrow \mathbb{Z}^m \longrightarrow Pic(X) \longrightarrow 0.$$

In particular, the Picard number of X is $b_2(X) = m - n$.

- A set of toric divisors $\{D_1, ..., D_k\}$ on X is called a primitive set if $D_1 \cap \cdots \cap D_k = \emptyset$ but $D_1 \cap \cdots \cap \widehat{D_j} \cap \cdots \cap D_k \neq \emptyset$ for all j. Equivalently, $< v_1, ..., v_k > \notin \Sigma$ but $< v_1, ..., \widehat{v_j}, ..., v_k > \in \Sigma$ for all j and we call to $P = \{v_1, ..., v_k\}$ a primitive collection.

- Let X be a d-dimensional smooth complete toric variety and let Σ be the corresponding fan. We say that Σ is a splitting fan if any two primitive collections have no common elements.

EXAMPLE

Consider a Hirzebruch surface $X = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n))$, $n \geq 0$. X is a toric variety. Its fan Σ in $N = \mathbb{Z}^2$ with basis e_1 and e_2 has the following set of one dimensional cones (ray generators):

$$v_1 = e_1, \quad v_2 = -e_1 + ne_2, \quad v_3 = e_2, \quad v_4 = -e_2$$

and the corresponding toric divisors Z_1, Z_2, Z_3 and Z_4. The set of primitive collections of Σ is given by

$$\mathcal{P} = \{\{v_1, v_2\}, \{v_3, v_4\}\}.$$

Since there is no common elements between the two primitive collections, the fan Σ associated to the Hirzebruch surface X is a splitting fan.

- Kleinschmidt: Let X be a d-dimensional smooth complete toric variety and let Σ be the corresponding fan. If the Picard number of X is two then Σ is a splitting fan.
- Batyrev: Let X be a d-dimensional smooth complete toric variety and let Σ be the corresponding fan. Then Σ is a splitting fan if and only if there exists a sequence of toric varieties $X = X_r$, $\ldots$, X_0 such that $X_0 = \mathbb{P}^n$ for a certain n and for $1 \leq i \leq r$, X_i is a projectivization of a decomposable vector bundle over X_{i-1}.
- Any d-dimensional smooth complete toric variety X with Picard number 2 is a projectivization of a decomposable vector bundle over a projective space.

The search for tilting bundles

Let X be a smooth projective variety of dimension n.

- A coherent sheaf F on X is exceptional if $Hom(F,F) = \mathbb{C}$ and $Ext^i_X(F,F) = 0$ for $i > 0$.

- An ordered collection $(F_0, F_1, \ldots, F_m)$ of coherent sheaves on X is an exceptional collection if each sheaf F_i is exceptional and $Ext^i_X(F_k, F_j) = 0$ for $j < k$, $i \geq 0$.

- An exceptional collection $(F_0, F_1, \ldots, F_m)$ is a strongly exceptional collection if in addition $Ext^i_X(F_j, F_k) = 0$ for $i \geq 1$, $j \leq k$.

- An ordered collection $(F_0, \ldots, F_m)$ is a full (strongly) exceptional collection if it is a (strongly) exceptional collection and F_0, ..., F_m generate $D^b(\mathcal{O}_X - mod)$.

REMARK

Any full strongly exceptional collection $(F_0, F_1, \ldots, F_m)$ of coherent sheaves on X defines a tilting sheaf $T = \oplus_{i=0}^{m} F_i$ because the endomorphism algebra of $T = \oplus_{i=0}^{m} F_i$ is a "triangular" algebra and it has global dimension at most m. And vice versa, each tilting bundle whose direct summands are line bundles gives rise to a full strongly exceptional collection.

The search for tilting sheaves on a smooth projective variety X naturally splits into two parts:

- First, we have to find a strongly exceptional collection of coherent sheaves on X, $(F_0, F_1, \ldots, F_m)$; and
- Second, we have to show that F_0, F_1, ..., F_m generate the derived category $D^b(X)$ of bounded complexes.

Example: ($\mathcal{O}_{\mathbb{P}^n}$, $\mathcal{O}_{\mathbb{P}^n}(1)$, $\ldots$, $\mathcal{O}_{\mathbb{P}^n}(n)$) is a full strongly exceptional collection on a projective space $\mathbb{P}^n$. So, $T = \oplus_{i=0}^{n}\mathcal{O}(i)$ is a tilting bundle on $\mathbb{P}^n$.

Proposition: Let X be a smooth projective variety. Assume that $(E_1, \cdots, E_r)$ is an exceptional collection in $D^b(X)$. Then, the following are equivalent

(a) $(E_1, \cdots, E_r)$ is full, i.e. $\langle E_1, \cdots, E_r\rangle = D^b(X)$;

(b) $0 = {}^{\perp}\langle E_1, \cdots, E_r\rangle := \{F \in D^b(X) | Ext^{\bullet}(F, E_i) = 0 \quad \forall i\}$;

(c) $0 = \langle E_1, \cdots, E_r\rangle^{\perp} := \{F \in D^b(X) | Ext^{\bullet}(E_i, F) = 0 \quad \forall i\}$;

(d) $0 = {}^{\perp}\langle E_1, \cdots, E_k\rangle \cap \langle E_{k+1}, \cdots, E_r\rangle^{\perp}$ for all k.

Examples

- $(\mathcal{O}_{\mathbb{P}^n}, \mathcal{O}_{\mathbb{P}^n}(1), \ldots, \mathcal{O}_{\mathbb{P}^n}(n))$ and $(\mathcal{O}_{\mathbb{P}^n}, \Omega^1(1), \ldots, \Omega^n(n))$ are full strongly exceptional collections on $\mathbb{P}^n$. So, $T_1 = \oplus_{i=0}^n \mathcal{O}(i)$ and $T_2 = \oplus_{i=0}^n \Omega^i(i)$ are tilting bundles on $\mathbb{P}^n$.

- Consider a Hirzebruch surface $X = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n))$, $n \geq 0$ and all its toric divisors Z_1, Z_2 Z_3 and Z_4. It is not difficult to see that $(\mathcal{O}, \mathcal{O}(Z_1), \mathcal{O}(Z_4), \mathcal{O}(Z_1 + Z_4))$ is a full strongly exceptional collection on X.

- Let $\pi : \widetilde{\mathbb{P}^2}(1) \to \mathbb{P}^2$ be the blow up of $\mathbb{P}^2$ at one point $p \in \mathbb{P}^2$. Let H be the pullback of the hyperplane divisor in $\mathbb{P}^2$ and let $E = \pi^{-1}(p)$ be the exceptional divisor. Then the collection of divisors $(0, E, H, 2H)$ is a full strongly exceptional collection on $\widetilde{\mathbb{P}^2}(1)$.

Examples

- (Kapranov) Let $Q_n \subset \mathbb{P}^{n+1}$, $n > 2$, be a hyperquadric. If n is even and Σ_1, Σ_2 are the Spinor bundles on Q_n, then

$$(\Sigma_1(-n), \Sigma_2(-n), \mathcal{O}_{Q_n}(-n+1), \cdots, \mathcal{O}_{Q_n}(-1), \mathcal{O}_{Q_n})$$

is a full strongly exceptional collection on Q_n; and if n is odd and Σ is the Spinor bundle on Q_n, then

$$(\Sigma(-n), \mathcal{O}_{Q_n}(-n+1), \cdots, \mathcal{O}_{Q_n}(-1), \mathcal{O}_{Q_n})$$

is a full strongly exceptional collection of sheaves on Q_n

Examples

- (Kapranov) Take $X = Gr(k,n)$. Denotes by $\mathcal{S}$ the tautological k-dimensional bundle and $\Sigma^\alpha \mathcal{S}$ the space of the irreducible representations of $GL(\mathcal{S})$ with highest weight $\alpha = (\alpha_1, \ldots, \alpha_s)$. Let $A(k,n)$ be the set of locally free sheaves $\Sigma^\alpha \mathcal{S}$ on $Gr(k,n)$ where α runs over Young diagrams fitting inside a $k \times (n-k)$ rectangle. $A(k,n)$ can be totally ordered in such a way that we obtain a full strongly exceptional collection ($E_1, \ldots, E_{\rho(k,n)}$) of sheaves on X.

Remarks

- In all these collections the order is very important.
- The length of any full strongly exceptional collection is $\geq$ dim(X)+1.
- All full strongly exceptional collections on X have the same length and it coincides with the rank of the Grothendieck group $K_0(X)$ as $\mathbb{Z}$-module.
- Not all full strongly exceptional collections are made up of line bundles.

Let $\mathcal{E}$ be a rank r vector bundle on a smooth projective variety X, denote by $p : \mathbb{P}(\mathcal{E}) \longrightarrow X$ the corresponding projective bundle and $\mathcal{O}_{\mathcal{E}}(1)$ the tautological line bundle on $\mathbb{P}(\mathcal{E})$. If $(F_0, F_1, \ldots, F_m)$ is a full exceptional collection of coherent sheaves on X, then

$$(*) \qquad (p^*F_0 \otimes \mathcal{O}_{\mathcal{E}}(-r+1), p^*F_1 \otimes \mathcal{O}_{\mathcal{E}}(-r+1), \ldots,$$

$$p^*F_m \otimes \mathcal{O}_{\mathcal{E}}(-r+1), \ldots, p^*F_0, p^*F_1, \ldots, p^*F_m)$$

is a full exceptional collection of coherent sheaves on $\mathbb{P}(\mathcal{E})$.

Proposition With the above notation, if $H^i(X, S^a\mathcal{E} \otimes F_t \otimes F_l^{\vee}) = 0$ for $i > 0$, $0 \leq a \leq r-1$ and $0 \leq l \leq t \leq m$ then (*) is a full strongly exceptional collection of coherent sheaves on $\mathbb{P}(\mathcal{E})$.

King's Conjecture

CONJECTURE (A. King): Any smooth complete toric variety has a tilting bundle whose summands are line bundles.

CONTRIBUTIONS TO KING'S CONJECTURE:

- Beilinson (1979): $\mathbb{P}^n$.
- King (1997): Hirzebruch surfaces.
- Costa and Miró-Roig (2004):
  - Smooth complete toric varieties with splitting fan,
  - Smooth complete toric varieties with Picard number ≤ 2
  - The blow-up of any smooth complete minimal toric surface at T-invariants points,
  - Products of toric varieties.

Contributions to King's Conjecture

- Kawamata (2005)
- Bondal (In preparation)

Proposition

Let Y be a smooth complete toric variety which is the projectivization of a rank r vector bundle $\mathcal{E}$ over a smooth complete toric variety X. Assume that X has a full strongly exceptional collection of locally free sheaves. Then, Y has a full strongly exceptional collection of locally free sheaves.

Theorems:

1.- Any d-dimensional, smooth, complete toric variety V with a splitting fan $\Sigma(V)$ has a tilting bundle whose summands are line bundles.

2.- Any d-dimensional, smooth, complete toric variety V with Picard number 2 has a tilting bundle whose summands are line bundles.

3.- Any 3-dimensional pseudo-symmetric toric Fano variety has a tilting bundle whose summands are line bundles.

Theorem:

Let X_1 and X_2 be two smooth projective varieties. Assume that X_i has a tilting bundle T_i whose direct summands are line bundles. Then $T_1 \otimes T_2$ is a tilting bundle of $X_1 \times X_2$ whose direct summands are line bundles.

Theorem:

Let X_1 and X_2 be two smooth projective varieties and let $(F_0^i, F_1^i, \dots, F_{n_i}^i)$ be a full strongly exceptional collection of locally free sheaves on X_i, $i = 1, 2$. Then,

$$(F_0^1 \otimes F_0^2, F_1^1 \otimes F_0^2, \dots, F_{n_1}^1 \otimes F_0^2, \dots, F_0^1 \otimes F_{n_2}^2, F_1^1 \otimes F_{n_2}^2, \dots, F_{n_1}^1 \otimes F_{n_2}^2)$$

is a full strongly exceptional collection of locally free sheaves on $X_1 \times X_2$ where

$$F_k^1 \otimes F_l^2 := p_1^* F_k \otimes p_2^* F_l$$

with $p_i : X_1 \times X_2 \to X_i$, $i = 1, 2$ the natural projections.

THEOREM:

Let X_1 and X_2 be two smooth projective varieties. Assume X_i has a tilting bundle T_i whose summands are line bundles. Then $T_1 \otimes T_2$ is a tilting bundle of $X_1 \times X_2$ whose direct summands are line bundles.

In particular,

COROLLARY:

Any multiprojective space $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}$ has a tilting bundle whose summands are line bundles.

King’s Conjecture:

Any smooth complete toric variety has a tilting bundle whose summands are line bundles.

CONJECTURE:

Any smooth complete FANO toric variety has a tilting bundle whose summands are line bundles.

LAST RESULTS:

The Conjecture is true in the following cases:

- X is the blow up of $\mathbb{P}^{n-r} \times \mathbb{P}^r$ along a multilinear subspace $\mathbb{P}^{n-r-1} \times \mathbb{P}^{r-1}$ of codimension 2 of $\mathbb{P}^{n-r} \times \mathbb{P}^r$,
- X is a (Fano) Y-toric fibration over $\mathbb{P}^n$ and Y has a titling bundle whose summands are line bundles.
- X is a d-dimensional smooth Fano toric variety with (almost) maximal Picard number (i.e. $2d - 1 \leq b_2(X) \leq 2d$.)