Vector bundles, from classical techniques to new perspectives

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LECTURE 2

Cohomological characterization of vector bundles

PROBLEM:

To give a cohomological characterization of vector bundles.

- Horrocks (1964): Cohomological characterization of line bundles $\mathcal{O}_{\mathbb{P}^n}(a)$.
- Horrocks (1980): Cohomological characterization of the p-differential bundle $\Omega^p_{\mathbb{P}^n}$.
- Ottaviani (1987): Cohomological characterization of line bundles $\mathcal{O}_{Q_n}(a)$ and Spinor bundles on $Q_n \subset \mathbb{P}^{n+1}$.
- Miró-Roig (1994): Cohomological characterization of syzygy bundles $syz(x_0^{i_0} \cdots x_n^{i_n}; \quad i_0 + \cdots + i_n = d)$ on $\mathbb{P}^n$.

CONTENTS

1. HORROCKS' THEOREM

Beilinson's Theorem

- $(\mathcal{O}_{\mathbb{P}^n}, \mathcal{O}_{\mathbb{P}^n}(1), \cdots, \mathcal{O}_{\mathbb{P}^n}(n))$ is a full strongly exceptional collection or, equivalently, $(\mathcal{O}_{\mathbb{P}^n}, \mathcal{O}_{\mathbb{P}^n}(1), \cdots, \mathcal{O}_{\mathbb{P}^n}(n))$ is an orthogonal basis of $D^b(\mathcal{O}_{\mathbb{P}^n} - \text{mod})$ and its left dual is $(\mathcal{O}_{\mathbb{P}^n}(n), T_{\mathbb{P}^n}(n-1), \wedge^2 T_{\mathbb{P}^n}(n-2), \cdots, \wedge^n T_{\mathbb{P}^n})$

- Let E be a coherent sheaf on $\mathbb{P}^n$. $\exists$ two spectral sequences situated in $-n \leq p \leq 0$, $0 \leq q \leq n$ and with E_1-term

$${}_IE_1^{pq} = H^q(\mathbb{P}^n, E(p)) \otimes \Omega^{-p}(-p)$$

$${}_{II}E_1^{pq} = H^q(\mathbb{P}^n, E \otimes \Omega^{-p}(-p)) \otimes \mathcal{O}_{\mathbb{P}^n}(p)$$

which converges to

$${}_{II}E_\infty^i = {}_I E_\infty^i = \begin{cases} E \text{ for } i = 0 \\ 0 \text{ for } i \neq 0. \end{cases}$$

Beilinson's Theorem

Let E be a coherent sheaf on $\mathbb{P}^n$. $\exists$ two minimal complexes

$$L^\bullet : \quad 0 \to L^{-n} \xrightarrow{d_n} L^{-n+1} \xrightarrow{d_{-n+1}} \cdots \xrightarrow{d_{-1}} L^0 \xrightarrow{d_0} \to \cdots \xrightarrow{d_{n-1}} L^n \to 0$$

$$F^\bullet : \quad 0 \to F^{-n} \xrightarrow{\delta_n} F^{-n+1} \xrightarrow{\delta_{-n+1}} \cdots \xrightarrow{\delta_{-1}} F^0 \xrightarrow{\delta_0} \to \cdots \xrightarrow{\delta_{n-1}} F^n \to 0$$

with

$$L^k = \oplus_{j+k=i} \Omega^j(j)^{h^i(E(-j))},$$

$$F^k = \oplus_{j+k=i} \mathcal{O}_{\mathbb{P}^n}(-j)^{h^i(E \otimes \Omega^j(j))}$$

and such that

$$\frac{ker(\delta_k)}{Im(\delta_{k-1})} \cong \frac{ker(d_k)}{Im(d_{k-1})} \cong \begin{cases} E & \text{if} \quad k = 0 \\ 0 & \text{if} \quad k \neq 0; \end{cases}$$

Horrocks' Theorem

Let E be a vector bundle on $\mathbb{P}^n$. The following conditions are equivalent:

(i) E splits into a sum of line bundles.

(ii) E has no intermediate cohomology; i.e. $H^i(\mathbb{P}^n, E(t)) = 0$ for $1 \leq i \leq n-1$ and for all $t \in \mathbb{Z}$.

- Cohomological characterization of $\Omega^p_{\mathbb{P}^n}$.
- Cohomological characterization of syzygy bundles $syz(x_0^{i_0} \cdots x_n^{i_n}; \quad i_0 + \cdots + i_n = d)$ on $\mathbb{P}^n$.

2. BEILINSON'S TYPE SPECTRAL SEQUENCE

Let X be a smooth projective variety of dimension n.

- A coherent sheaf F on X is exceptional if $Hom(F, F) = \mathbb{C}$ and $Ext^i_X(F, F) = 0$ for $i > 0$.

- An ordered collection $(F_0, F_1, \ldots, F_m)$ of coherent sheaves on X is an exceptional collection if each sheaf F_i is exceptional and $Ext^i_X(F_k, F_j) = 0$ for $j < k$, $i \geq 0$.

- An exceptional collection $(F_0, F_1, \ldots, F_m)$ is a strongly exceptional collection if in addition $Ext^i_X(F_j, F_k) = 0$ for $i \geq 1$, $j \leq k$.

- An ordered collection $(F_0, \ldots, F_m)$ is a full (strongly) exceptional collection if it is a (strongly) exceptional collection and F_0, $\ldots$, F_m generate $D^b(\mathcal{O}_X - mod)$.

Examples

- $(\mathcal{O}_{\mathbb{P}^n}, \mathcal{O}_{\mathbb{P}^n}(1), \ldots, \mathcal{O}_{\mathbb{P}^n}(n))$ is a full strongly exceptional collection on a projective space $\mathbb{P}^n$.
- $(\mathcal{O}_{\mathbb{P}^n}, \Omega^1_{\mathbb{P}^n}(1), \ldots, \Omega^n_{\mathbb{P}^n}(n))$ is a full strongly exceptional collection on a projective space $\mathbb{P}^n$.
- Let $\pi : \widetilde{\mathbb{P}^2}(1) \to \mathbb{P}^2$ be the blow up of $\mathbb{P}^2$ at one point $p \in \mathbb{P}^2$. Let H be the pullback of the hyperplane divisor in $\mathbb{P}^2$ and let $E = \pi^{-1}(p)$ be the exceptional divisor. Then the collection of divisors $(0, E, H, 2H)$ is a full strongly exceptional collection on $\widetilde{\mathbb{P}^2}(1)$.

Examples

- (Kapranov) Let $Q_n \subset \mathbb{P}^{n+1}$, $n > 2$, be a hyperquadric. If n is even and Σ_1, Σ_2 are the Spinor bundles on Q_n, then

$$(\Sigma_1(-n), \Sigma_2(-n), \mathcal{O}_{Q_n}(-n+1), \cdots, \mathcal{O}_{Q_n}(-1), \mathcal{O}_{Q_n})$$

is a full strongly exceptional collection on Q_n; and if n is odd and Σ is the Spinor bundle on Q_n, then

$$(\Sigma(-n), \mathcal{O}_{Q_n}(-n+1), \cdots, \mathcal{O}_{Q_n}(-1), \mathcal{O}_{Q_n})$$

is a full strongly exceptional collection of sheaves on Q_n

Examples

- (Kapranov) Take $X = Gr(k,n)$. Denotes by $\mathcal{S}$ the tautological k-dimensional bundle and $\Sigma^{\alpha}\mathcal{S}$ the space of the irreducible representations of $GL(\mathcal{S})$ with highest weight $\alpha = (\alpha_1, \ldots, \alpha_s)$. Let $A(k,n)$ be the set of locally free sheaves $\Sigma^{\alpha}\mathcal{S}$ on $Gr(k,n)$ where α runs over Young diagrams fitting inside a $k \times (n-k)$ rectangle. $A(k,n)$ can be totally ordered in such a way that we obtain a full strongly exceptional collection $(E_1, \ldots, E_{\rho(k,n)})$ of sheaves on X.

Remarks

- In all these collections the order is very important.
- The length of any full strongly exceptional collection is $\geq$ dim(X)+1.
- All full strongly exceptional collections on X have the same length and it coincides with the rank of the Grothendieck group $K_0(X)$ as $\mathbb{Z}$-module.
- Not all full strongly exceptional collections are made up of line bundles.

Definition. A GEOMETRIC COLLECTION of coherent sheaves $(E_0, \cdots, E_n)$ on a smooth algebraic variety X is a full exceptional collection of minimal length, dim (X)+1.

Problems

Problem 1. To characterize smooth projective varieties which have a geometric collection.

Problem 2.To characterize smooth projective varieties which have a full strongly exceptional collection.

Remark: The existence of full strongly exceptional collection imposes rather a strong restriction on X, namely that the Grothendieck group $K_0(X)$ is isomorphic to $\mathbb{Z}^{m+1}$.

Example: Since, the Grothendieck group $K_0(S)$ of a smooth cubic 3-fold $S \subset \mathbb{P}^4$ has torsion, there are no full strongly exceptional collection on S.

Theorem

Let X be a smooth projective variety of dim n with a geometric collection $(E_0, \cdots, E_n)$ and let F be a coherent sheaf on X. $\exists$ two spectral sequences with E_1-term

$$ {}_I E_1^{pq} = Ext^q(R^{(-p)} E_{n+p}, F) \otimes E_{p+n} $$

$$ {}_{II} E_1^{pq} = Ext^q((E_{n+p})^*, F) \otimes (R^{(-p)} E_{n+p})^* $$

situated in the square $0 \leq q \leq n$, $-n \leq p \leq 0$ which converge to

$$ {}_I E_\infty^i = {}_{II} E_\infty^i = \begin{cases} F \text{ for } i = 0 \\ 0 \text{ for } i \neq 0. \end{cases} $$

Let X be a smooth projective variety of dimension n.

- An exceptional collection $(F_0, F_1, \cdots, F_m)$ of coherent sheaves on X is a block if $Ext^i(F_j, F_k) = 0$ for any i and $j \neq k$.

- An m-block collection of type $(\alpha_0, \alpha_1, \cdots, \alpha_m)$ is an exceptional collection $(\mathcal{E}_0, \mathcal{E}_1, \cdots, \mathcal{E}_m) = (E_1^0, \cdots, E_{\alpha_0}^0, \cdots, E_1^m, \cdots, E_{\alpha_m}^m)$ such that all the subcollections $\mathcal{E}_i = (E_1^i, E_2^i, \cdots, E_{\alpha_i}^i)$ are blocks.

REMARK: Any exceptional collection $(E_0, E_1, \cdots, E_m)$ of length $m+1$ is an m-block collection of type $(1, \cdots, 1)$ where each block has one object.

EXAMPLE

Let $X = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_s}$, $d = n_1 + \cdots + n_s$ and denote

$$\mathcal{O}_X(a_1, a_2, \cdots, a_s) := p_1^* \mathcal{O}_{\mathbb{P}^{n_1}}(a_1) \otimes p_2^* \mathcal{O}_{\mathbb{P}^{n_2}}(a_2) \otimes \cdots \otimes p_s^* \mathcal{O}_{\mathbb{P}^{n_s}}(a_s).$$

For any $0 \leq j \leq d$, denote by $\mathcal{E}_j$ the collection of all line bundles

$$\mathcal{O}_X(a_1^j, a_2^j, \cdots, a_s^j)$$

with $-n_i \leq a_i^j \leq 0$ and $\sum_{i=1}^s a_i^j = j - d$. Each $\mathcal{E}_j$ is a block and

$$(\mathcal{E}_0, \mathcal{E}_1, \cdots, \mathcal{E}_d)$$

is a d-block collection on X.

EXAMPLE

Consider $X = \mathbb{P}^2 \times \mathbb{P}^3$. The collection of line bundles

$$(\mathcal{O}(-2,-3), \mathcal{O}(-2,-2), \mathcal{O}(-1,-3), \mathcal{O}(-2,-1),$$

$$\mathcal{O}(-1,-2), \mathcal{O}(0,-2)\mathcal{O}(-2,0), \mathcal{O}(-1,-1),$$

$$\mathcal{O}(0,-2), \mathcal{O}(-1,0), \mathcal{O}(0,-1), \mathcal{O}(0,0))$$

is a full strongly exceptional collection of length $12 > dim(X) + 1 = 6$ and we pack in 6 blocks:

$$\mathcal{E}_0 = \{\mathcal{O}(-2,-3)\}$$
$$\mathcal{E}_1 = \{\mathcal{O}(-2,-2), \mathcal{O}(-1,-3)\}$$
$$\mathcal{E}_2 = \{\mathcal{O}(-2,-1), \mathcal{O}(-1,-2), \mathcal{O}(0,-3)\},$$
$$\mathcal{E}_3 = \{\mathcal{O}(-2,0), \mathcal{O}(-1,-1), \mathcal{O}(0,-2)\}$$
$$\mathcal{E}_4 = \{\mathcal{O}(-1,0), \mathcal{O}(0,-1)\},$$
$$\mathcal{E}_5 = \{\mathcal{O}(0,0)\}$$

Definition. Let $\sigma = (\mathcal{E}_0, \cdots, \mathcal{E}_m)$ be an m-block collection of coherent sheaves which generates $D^b(X)$. The m-block $\mathcal{H} = (\mathcal{H}_0, \cdots, \mathcal{H}_m)$ is called the left dual m-block collection of σ if

$$Ext^t(H^i_j, E^k_l) = 0$$

except for $Ext^k(H^k_i, E^{m-k}_i) = \mathbb{C}$.
The m-block $\mathcal{G} = (\mathcal{G}_0, \cdots, \mathcal{G}_m)$ is called the right dual m-block collection of σ if

$$Ext^t(E^k_l, G^i_j) = 0$$

except for $Ext^{m-k}(E^{m-k}_i, G^k_i) = \mathbb{C}$.
Remark. Left and right dual block collections can be constructed by means of left and right mutations.

EXAMPLE

Let V be a $\mathbb{C}$-vector space of dim $n+1$ and $\mathbb{P}^n = \mathbb{P}(V)$. We consider the n-block collection

$$\mathcal{B} = (\mathcal{O}_{\mathbb{P}^n}, \mathcal{O}_{\mathbb{P}^n}(1), \cdots, \mathcal{O}_{\mathbb{P}^n}(n)).$$

Using the exterior powers

$$0 \longrightarrow \wedge^{k-1} T_{\mathbb{P}^n} \longrightarrow \wedge^k V \otimes \mathcal{O}_{\mathbb{P}^n}(k) \longrightarrow \wedge^k T_{\mathbb{P}^n} \longrightarrow 0$$

of the Euler sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n} \longrightarrow V \otimes \mathcal{O}_{\mathbb{P}^n}(1) \longrightarrow T_{\mathbb{P}^n} \longrightarrow 0$$

we compute the left dual n-block collection of $\mathcal{B}$ and we get

$$\begin{aligned} \mathcal{H} &= (\mathcal{H}_0, \mathcal{H}_1, \cdots, \mathcal{H}_j, \cdots, \mathcal{H}_n) \\ &= (\mathcal{O}_{\mathbb{P}^n}(n), T_{\mathbb{P}^n}(n-1), \cdots, \wedge^j T_{\mathbb{P}^n}(n-j), \cdots, \wedge^n T_{\mathbb{P}^n}). \end{aligned}$$

Proposition

Let $X = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_s}$ be a multiprojective space of dimension $d = n_1 + \cdots + n_s$. Consider $\mathcal{B} = (\mathcal{E}_0, \cdots, \mathcal{E}_d)$ the d-block collection where for any $0 \leq j \leq d$, $\mathcal{E}_j$ is the set of line bundles

$$\mathcal{O}_X(a_1^j, a_2^j, \cdots, a_s^j)$$

with $-n_i \leq a_i^j \leq 0$ and $\sum_{i=1}^s a_i^j = j - d$. Then, for any $E_i^{d-k} = \mathcal{O}_X(t_1, \cdots, t_s) \in \mathcal{E}_{d-k}$ and any $0 \leq k \leq d$, $(\mathcal{H}_0, \mathcal{H}_1, \cdots, \mathcal{H}_j, \cdots, \mathcal{H}_d)$, with

$$H_i^k = \bigwedge^{-t_1} T_{\mathbb{P}^{n_1}}(t_1) \boxtimes \cdots \boxtimes \bigwedge^{-t_s} T_{\mathbb{P}^{n_s}}(t_s) \in \mathcal{H}_k$$

is the left dual d-block collection of $\mathcal{B} = (\mathcal{E}_0, \cdots, \mathcal{E}_d)$.

Sketch of the proof

For any $0 \le i \le d$, we take $\mathcal{O}_X(a_1^i, \cdots, a_s^i) \in \mathcal{E}_i$ and we apply the Künneth formula,

$$H^\alpha(X, \bigwedge^{-t_1} \Omega_{\mathbb{P}^{n_1}}(-t_1) \boxtimes \cdots \boxtimes \bigwedge^{-t_s} \Omega_{\mathbb{P}^{n_s}}(-t_s) \otimes \mathcal{O}_X(a_1^i, \cdots, a_s^i))$$

$$= \bigoplus_{\alpha_1+\cdots+\alpha_s=\alpha} H^{\alpha_1}(\mathbb{P}^{n_1}, \bigwedge^{-t_1} \Omega(a_1^i - t_1)) \otimes \cdots \otimes H^{\alpha_s}(\mathbb{P}^{n_s}, \bigwedge^{-t_s} \Omega(a_s^i - t_s)).$$

Using Bott's formula, it is zero unless $\alpha = k$, $i = d - k$ and $\mathcal{O}_X(a_1^i, \cdots, a_s^i) = \mathcal{O}_X(t_1, \cdots, t_s)$, which proves what we want.

Beilinson type spectral sequence

Let X be a smooth projective variety of dim n with an n-block collection $(\mathcal{E}_0, \mathcal{E}_1, \cdots, \mathcal{E}_n)$, $\mathcal{E}_i = (E^i_1, \dots, E^i_{\alpha_i})$ of coherent sheaves which generates $D^b(X)$. Denote by $(\mathcal{H}_0, \mathcal{H}_1, \cdots, \mathcal{H}_n)$, $\mathcal{H}_i = (H^i_1, \dots, H^i_{\alpha_i})$ the left dual n-block collection. $\forall F$ coherent sheaf , $\exists$ spectral sequences

$$ {}_IE_1^{pq} = \begin{cases} \bigoplus_{i=1}^{\alpha_{p+n}} Ext^q(H^p_i, F) \otimes E^{p+n}_i & \text{if} \quad -n \leq p \leq -1 \\ \bigoplus_{i=1}^{\alpha_n} Ext^q(E^n_i, F) \otimes E^n_i & \text{if} \quad p = 0 \end{cases} $$

$$ {}_{II}E_1^{pq} = \begin{cases} \bigoplus_{i=1}^{\alpha_{p+n}} Ext^q((E^{p+n}_i)^*, F) \otimes (H^p_i)^* & \text{if} \quad -n \leq p \leq -1 \\ \bigoplus_{i=1}^{\alpha_n} Ext^q((E^n_i)^*, F) \otimes (E^n_i)^* & \text{if} \quad p = 0 \end{cases} $$

which converge to ${}_IE^i_\infty = {}_{II}E^i_\infty = \begin{cases} F \text{ for } i = 0 \\ 0 \text{ for } i \neq 0. \end{cases}$.

3. COHOMOLOGICAL CHARACTERIZATION OF VECTOR BUNDLES

- Vector bundles on multiprojective spaces
- Steiner vector bundles on algebraic varieties

THEOREM

Let $X=\mathbb{P}^{n_1}\times\cdots\times\mathbb{P}^{n_s}$, $d=n_1+\cdots+n_s$ and $(\mathcal{E}_0,\cdots,\mathcal{E}_d)$ the d-block collection described before. Assume $\exists F$ a rank $\binom{d}{j}$, $0<j<d$, vector bundle on X s.t.
$H^{-p-1}(X,F\otimes E_i^{p+d})=0$ for $-d\le p\le -j-1$ and $1\le i\le\alpha_p$,

$H^{-p+1}(X,F\otimes E_i^{p+d})=0$ for $-j+1\le p\le 0$ and $1\le i\le\alpha_p$,

$H^j(F\otimes E_i^{d-j})=\mathbb{C}$ for $1\le i\le\alpha_{d-j}$. Then

$$F\cong\bigoplus_{t_1+\cdots+t_s=j-d}\bigwedge^{-t_1}\Omega_{\mathbb{P}^{n_1}}(-t_1)\boxtimes\cdots\boxtimes\bigwedge^{-t_s}\Omega_{\mathbb{P}^{n_s}}(-t_s)$$

$\cong\bigwedge^{d-j}(\Omega_{\mathbb{P}^{n_1}\times\cdots\times\mathbb{P}^{n_s}}(1,\cdots,1))$ with $-n_i\le t_i\le 0$

.

Sketch of the proof

We apply to F the spectral sequence with E_1-term

$$ {}_{II}E_1^{pq} = \begin{cases} \bigoplus_{i=1}^{\alpha_{p+d}} Ext^q((E_i^{p+d})^*, F) \otimes (H_i^p)^* & \text{if} \quad -d \leq p \leq -1 \\ \bigoplus_{i=1}^{\alpha_d} Ext^q(E_i^{d^*}, F) \otimes E_i^{d^*} & \text{if} \quad p = 0 \end{cases} $$

By assumption, there is an integer j, $0 < j < d$, such that ${}_{II}E_1^{p,-p-1} = 0$ for any $-n \leq p \leq -j-1$ and ${}_{II}E_1^{p,-p+1} = 0$ for any $-j+1 \leq p \leq 0$.
So, F contains ${}_{II}E_1^{jj}$, i.e. F contains

$$ \Big(\Big(\bigoplus_{t_1+\cdots+t_s=j-d} \bigwedge^{-t_1} T_{\mathbb{P}^{n_1}}(-t_1) \boxtimes \cdots \boxtimes \bigwedge^{-t_s} T_{\mathbb{P}^{n_s}}(-t_s)\Big)^*\Big)^{\alpha_i} $$

with $\alpha_i = h^j(F \otimes E_i^{n-j})$ as a direct summand.

Since $rank F = \binom{d}{j}$, we get

$$
\begin{aligned}
F &\cong \left(\bigoplus_{t_1+\cdots+t_s=j-d} \bigwedge^{-t_1} T_{\mathbb{P}^{n_1}}(-t_1) \boxtimes \cdots \boxtimes \bigwedge^{-t_s} T_{\mathbb{P}^{n_s}}(-t_s)\right)^* \\
&\cong \bigoplus_{t_1+\cdots+t_s=j-d} \bigwedge^{-t_1} \Omega_{\mathbb{P}^{n_1}}(-t_1) \boxtimes \cdots \boxtimes \bigwedge^{-t_s} \Omega_{\mathbb{P}^{n_s}}(-t_s) \\
&\cong \bigwedge^{d-j} (\Omega_{\mathbb{P}^{n_1}\times\cdots\times\mathbb{P}^{n_s}}(1,\cdots,1)).
\end{aligned}
$$

Steiner bundles

Steiner bundles were first defined by Dolgachev and Kapranov as vector bundles E on $\mathbb{P}^n$ defined by an exact sequence of the form (Schwarzenberger: $t = s + n$)

$$(*) \quad 0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-1)^s \longrightarrow \mathcal{O}_{\mathbb{P}^n}^t \longrightarrow E \longrightarrow 0.$$

- They used Steiner bundles to study logarithmic bundles $\Omega(\log \mathcal{H})$ of meromorphic forms on $\mathbb{P}^n$ having at most logarithmic poles on a finite union $\mathcal{H}$ of hyperplanes with normal crossing.
- Dolgachev - Kapranov: A vector bundle E on $\mathbb{P}^n$ is a Steiner bundle defined by an exact sequence (*) if and only if $H^q(E \otimes \Omega^p_{\mathbb{P}^n}(p)) = 0$ for $q > 0$ and also for $q = 0$, $p > 1$.

Steiner bundles

Definition A vector bundle E on a smooth irreducible algebraic variety X is called a Steiner bundle if it is defined by an exact sequence of the form

$$0 \longrightarrow F_0^s \xrightarrow{\varphi} F_1^t \longrightarrow E \longrightarrow 0,$$

where $s, t \geq 1$ and (F_0, F_1) is an ordered pair of vector bundles on X satisfying the following two conditions:

(i) (F_0, F_1) is strongly exceptional;

(ii) $F_0^{\vee} \otimes F_1$ is generated by global sections.

- When $X = \mathbb{P}^n$, $F_0 = \mathcal{O}_{\mathbb{P}^n}(-1)$ and $F_1 = \mathcal{O}_{\mathbb{P}^n}$ we obtain the classical Steiner bundles as defined by Dolgachev and Kapranov.

Examples of Steiner bundles

- Vector bundles E on $\mathbb{P}^n$ given by

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(a)^s \longrightarrow \mathcal{O}_{\mathbb{P}^n}(b)^t \longrightarrow E \longrightarrow 0,$$

where $1 \leq b - a \leq n$, are Steiner bundles on $\mathbb{P}^n$.

- The exact sequences define Steiner bundles on $\mathbb{P}^n$:

$$0 \longrightarrow \Omega^p_{\mathbb{P}^n}(p)^s \longrightarrow \mathcal{O}^t_{\mathbb{P}^n} \longrightarrow E \longrightarrow 0, \quad 1 \leq p \leq n,$$

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-1)^s \longrightarrow \Omega^p_{\mathbb{P}^n}(p)^t \longrightarrow F \longrightarrow 0, \quad 0 \leq p \leq n-1.$$

Examples of Steiner bundles

- Let $Q_n \subset \mathbb{P}^{+1}$, $n \geq 2$, be the smooth hyperquadric. Let Σ_* denote the Spinor bundle Σ on Q_n if n is odd, and one of the Spinor bundles Σ_1 or Σ_2 on Q_n if n is even. The short exact sequences

$$0 \to \mathcal{O}_{Q_n}(a)^s \to \Sigma_*(n-1)^t \to E \to 0,$$

where $0 \leq a \leq n-1$, and

$$0 \to \Sigma_*(-n)^s \to \mathcal{O}_{Q_n}(a)^t \to F \to 0,$$

where $-n+1 \leq a \leq 0$, define Steiner bundles on Q_n.

THEOREM

Let X be a smooth projective variety of dim n with an n-block collection $\mathcal{B} = (\mathcal{E}_0, \mathcal{E}_1, \dots, \mathcal{E}_n)$, $\mathcal{E}_i = (E^i_1, \dots, E^i_{\alpha_i})$, of vector bundles on X which generate $\mathcal{D}^b(X)$. Let $E^a_{i_0} \in \mathcal{E}_a$, $E^b_{j_0} \in \mathcal{E}_b$, where $0 \leq a < b \leq n$ and $1 \leq i_0 \leq \alpha_a$, $1 \leq j_0 \leq \alpha_b$, and let E be a vector bundle on X. Then E is a Steiner bundle of type $(E^a_{i_0}, E^b_{j_0})$ defined by

$$0 \longrightarrow (E^a_{i_0})^s \longrightarrow (E^b_{j_0})^t \longrightarrow E \longrightarrow 0,$$

iff $(E^a_{i_0})^\vee \otimes E^b_{j_0}$ is globally generated and all $h^k(E \otimes (R^{(m)} E^{n-m}_i)^\vee)$ vanish, with the only exceptions of

$$h^{n-a-1}(E \otimes (R^{(n-a)} E^a_{i_0})^\vee) = s \text{ and } h^{n-b}(E \otimes (R^{(n-b)} E^b_{j_0})^\vee) = t.$$

COROLLARY

Let X be a smooth projective variety X with an n-block collection $\mathcal{B} = (\mathcal{E}_0, \mathcal{E}_1, \dots, \mathcal{E}_n)$, $\mathcal{E}_i = (E_1^i, \dots, E_{\alpha_i}^i)$, of locally free sheaves on X which generate $\mathcal{D}$. Let $E_{i_0}^a \in \mathcal{E}_a$, $E_{j_0}^b \in \mathcal{E}_b$, where $0 \leq a < b \leq n$ and $1 \leq i_0 \leq \alpha_a$, $1 \leq j_0 \leq \alpha_b$.

If E and F are Steiner bundles of type $(E_{i_0}^a, E_{j_0}^b)$ on X then any extension G of E by F,

$$0 \longrightarrow F \longrightarrow G \longrightarrow E \longrightarrow 0,$$

is a Steiner bundle of type $(E_{i_0}^a, E_{j_0}^b)$ on X.

OPEN PROBLEMS

- To characterize smooth projective varieties X of dimension n with an n-block collection (of line bundles) which generates the derived category $\mathcal{D}$.
- To characterize smooth projective varieties X which have a full strongly exceptional collection (made up of line bundles).
- To characterize smooth projective varieties X which have a geometric collection (of line bundles).
- To give new cohomological characterization of vector bundles E on projective varieties X.

Steiner bundles

Steiner bundles were first defined by Dolgachev and Kapranov as vector bundles E on $\mathbb{P}^n$ defined by an exact sequence of the form (Schwarzenberger: $t = s + n$)

$$(*) \quad 0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-1)^s \longrightarrow \mathcal{O}_{\mathbb{P}^n}^t \longrightarrow E \longrightarrow 0.$$

- Dolgachev - Kapranov: A vector bundle E on $\mathbb{P}^n$ is a Steiner bundle defined by an exact sequence (*) if and only if $H^q(E \otimes \Omega^p_{\mathbb{P}^n}(p)) = 0$ for $q > 0$ and also for $q = 0$, $p > 1$.

DEFINITION

Let X be a smooth projective variety. A monad on X is a complex of vector bundles:

$$M_\bullet : \quad 0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$$

which is exact at A and at C. The sheaf $E := Ker(\beta)/Im(\alpha)$ is called the cohomology sheaf of the monad $M_\bullet$.

A monad $M_\bullet : \quad 0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$ has a so-called display:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A & \longrightarrow & K & \longrightarrow & E & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A & \xrightarrow{\alpha} & B & \longrightarrow & Q & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & C & = & C & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

where $K := Ker(\beta)$ and $Q := Coker(\alpha)$.

From the display of a monad $M_\bullet$ one easily computes the rank and the Chern character of its cohomology sheaf. We have

(i) $rk(E) = rk(B) - rk(A) - rk(C)$, and

(ii) $c_t(E) = c_t(B)c_t(A)^{-1}c_t(C)^{-1}$.

- Monads were first introduced by Horrocks who showed that all vector bundles E on $\mathbb{P}^3$ can be obtained as the cohomology bundle of a monad of the following kind:

$$0 \longrightarrow \oplus_i \mathcal{O}_{\mathbb{P}^3}(a_i) \longrightarrow \oplus_j \mathcal{O}_{\mathbb{P}^3}(b_j) \longrightarrow \oplus_n \mathcal{O}_{\mathbb{P}^3}(c_n) \longrightarrow 0.$$

LINEAR MONADS

DEFINITION Let X be a nonsingular projective variety A linear monad on X is the short complex of sheaves

$$M_\bullet \ : \ 0 \to \mathcal{O}_X(-1)^a \xrightarrow{\alpha} \mathcal{O}_X^b \xrightarrow{\beta} \mathcal{O}_X(1)^c \to 0 \tag{1}$$

which is exact on the first and last terms. EXAMPLE Manin and Drinfield proved that mathematical instanton bundles E on $\mathbb{P}^3$ with quantum number k correspond to linear monads

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^3}(-1)^k \longrightarrow \mathcal{O}_{\mathbb{P}^3}^{2k+2} \longrightarrow \mathcal{O}_{\mathbb{P}^3}(1)^k \longrightarrow 0.$$

PROPOSITION (Fløystead)

Let $n \geq 1$. There exist monads on $\mathbb{P}^n$ whose entries are linear maps, i.e. linear monads

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-1)^a \xrightarrow{\alpha} \mathcal{O}_{\mathbb{P}^n}^b \xrightarrow{\beta} \mathcal{O}_{\mathbb{P}^n}(1)^c \longrightarrow 0$$

if and only if at least one of the following conditions holds:

(1) $b \geq 2c + n - 1$ and $b \geq a + c$.

(2) $b \geq a + c + n$.

If so, there actually exists a linear monad with the map α degenerating in expected codimension $b - a - c + 1$.

- If n is odd ($n = 2m + 1$), $a = c = k$ and $b = 2k + 2m$ we have an instanton bundle

Characterization of linear sheaves

- Cohomological characterization of linear sheaves on $\mathbb{P}^n$, $n \geq 1$.

Fix integers a, b and c such that

(1) $b \geq 2c + n - 1$ and $b \geq a + c$, or

(2) $b \geq a + c + n$.

We have:

PROPOSITION

Let E be a rank $b-a-c$ torsion free sheaf on $\mathbb{P}^n$ with Chern polynomial $c_t(E) = \frac{1}{(1-t)^a(1+t)^c}$. It holds:

(1) If $b < c(n+1)$ and E has natural cohomology in the range $-n \leq j \leq 0$, then E is the cohomology sheaf of a linear monad

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-1)^a \longrightarrow \mathcal{O}_{\mathbb{P}^n}^b \longrightarrow \mathcal{O}_{\mathbb{P}^n}(1)^c \longrightarrow 0.$$

(2) If E is the cohomology sheaf of a linear monad

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-1)^a \longrightarrow \mathcal{O}_{\mathbb{P}^n}^b \longrightarrow \mathcal{O}_{\mathbb{P}^n}(1)^c \longrightarrow 0$$

and $H^0(\mathbb{P}^n, E) = 0$, then E has natural cohomology in the range $-n \leq j \leq 0$.