Vector bundles, from classical techniques to new perspectives

Catania, September - October, 2009

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LECTURE 1

Vector Bundles and their Classification problem

- Moduli spaces are one of the fundamental constructions of Algebraic Geometry and they arise in connection to classification problems.
- A moduli space for a collection of objects A and an equivalence relation $\sim$ is a classification space, i.e. a space (in some sense of the word) such that each point corresponds to one, and only one, equivalence class of objects.
- A moduli space of vector bundles on a smooth, algebraic variety X is a scheme whose points are in "natural bijection" to equivalence classes of vector bundles on X.

Any moduli problem or classification problem naturally splits in two parts:

1st STEP. Existence and construction of the moduli space ("parameter space").

2nd STEP. What does the moduli space look like, as an algebraic variety? Is it, for example, connected, irreducible, rational or smooth? What does it look as a topological space? What is its geometry?

GENERAL PHILOSOPHY:

These days we will review some of the known results on moduli spaces of stable vector bundles on a smooth projective variety which nicely reflect the general philosophy that moduli spaces inherit a lot of properties of the underlying variety; essentially when the underlying variety is a surface and we will suggest you some precise problems whose solution will give more support to this idea

PROGRAM OF THIS WEEK

MONDAY. Vector Bundles and their Classification Problem

TUESDAY. Cohomological Characterization of Vector Bundles.

WEDNESDAY. Orthogonal Basis of the Derived Category of Toric Varieties.

THURSDAY. Moduli Spaces of Vector Bundles.

FRIDAY. Students' Problem I.

LECTURE 1

Vector Bundles and their Classification problem

- Vector bundles and locally free sheaves.
- First examples.
- Existence and construction of vector bundles with "nice" properties.
- Stability. Criteria of stability.
- Classification of vector bundles: Moduli problems.

- $Char(k) = 0$, $k = \bar{k}$, X smooth irreducible algebraic variety.
- A linear fibration of rank r on X is an algebraic variety E and a surjective map $p : E \longrightarrow X$ such that for each $x \in X$, $E_x := p^{-1}(x)$ is a k-vector space of rank r.
- Given two fibrations $p : E \longrightarrow X$ and $p' : E' \longrightarrow X$, a morphism of varieties $f : E \longrightarrow E'$ is a map of linear fibrations if it is compatible with the projections p and p', i.e. $p'f = p$, and, for each $x \in X$, the induced map $f_x : E_x \longrightarrow E'_x$ is linear.

- The fibration $X \times k^r \longrightarrow X$ given by projection to the first factor is called the trivial fibration of rank r.
- For each open set $U \subset X$ and fibration $E \longrightarrow X$, we write $E_{|U}$ for the fibration $p^{-1}(U) \longrightarrow U$ given by restriction to U.
- An algebraic vector bundle of rank r on X is a linear fibration $E \longrightarrow X$ which is locally trivial, that is, for any $x \in X$ there exists an open neighborhood U of x and an isomorphism of fibrations $\varphi : E_{|U} \longrightarrow U \times k^r$.

DEFINITION. Let $p : E \longrightarrow X$ be a vector bundle of rank r. A **regular section** of E over an open subset $U \subset X$ is a morphism $s : U \longrightarrow E$ such that $p(s(x)) = x$ for all $x \in U$.

- The set $\Gamma(U, E)$ of regular sections of E over U has a structure of module over the algebra $\mathcal{O}_X(U)$. So, we obtain a sheaf of $\mathcal{O}_X$-modules $\mathcal{E} = \mathcal{O}_X(E)$ over X, locally isomorphic to $\mathcal{O}_X^r$; i.e. a locally free sheaf of rank r.

PROPOSITION. The functor $E \longrightarrow \mathcal{E} = \mathcal{O}_X(E)$ is an equivalence of categories between the category of vector bundles of rank r over X and the category of locally free sheaves of rank r on X.

EXAMPLES

- Line bundles $\mathcal{O}_{\mathbb{P}^N}(t)$.
- Tangent bundle $T_{\mathbb{P}^N}$, $\Omega^1_{\mathbb{P}^N}$ and $\Omega^p_{\mathbb{P}^N}$.
- Let X be a non singular variety of dimension n and let $Y \subset X$ be a non singular subvariety of codimension r.
  Conormal bundle: $\mathcal{I}_{Y,X}/\mathcal{I}^2_{Y,X}$; and
  Normal bundle: $\mathcal{N}_{Y,X} := \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{I}_{Y,X}/\mathcal{I}^2_{Y,X}, \mathcal{O}_Y)$.

$$0 \to T_Y \to T_X \otimes \mathcal{O}_Y \to \mathcal{N}_{Y,X} \to 0$$

- Syzygy bundles $E_{d,n} := \ker(\mathcal{O}_{\mathbb{P}^N}(-d)^n \longrightarrow \mathcal{O}_{\mathbb{P}^N})$.

Let X be a smooth irreducible projective variety of dimension d, H an ample line bundle on X and F a torsion free sheaf on X. Set

$$\mu_H(F) := \frac{c_1(F)H^{d-1}}{rk(F)}, \qquad P_F(m) := \frac{\chi(F \otimes \mathcal{O}_X(mH))}{rk(F)}.$$

DEFINITION. F is μ-semistable (resp. GM-semistable) with respect to H if and only if

$$\mu_H(E) \leq \mu_H(F) \qquad \text{(resp. } P_E(m) \leq P_F(m) \quad for \quad m \gg 0)$$

for all non-zero subsheaves $E \subset F$ with $rk(E) < rk(F)$. If strict inequality holds then F is μ-stable (resp. GM-stable) with respect to H.

Remarks

- $\mu - stable \Rightarrow GM - stable \Rightarrow GM - semistable \Rightarrow \mu - semistable$
- The notion of stability depends on the choice of H
- Harder-Narasihan filtration
- Cohomological criteria
- Which Chern classes are realized by μ-(semi-)stable vector bundles?

Criterium of Stability.

Let E be a rank r reflexive sheaf on $\mathbb{P}^n$. Set $E_{norm} := E(k_E)$ where k_E is the unique integer such that $c_1(E(k_E)) \in \{-r+1, \cdots, 0\}$.

LEMMA Let E be a rank 2 reflexive sheaf on $\mathbb{P}^n$. Then, E is μ-stable if and only if $H^0(\mathbb{P}^n, E_{norm}) = 0$. If $c_1(E)$ is even, then E is μ-semistable if and only if $H^0(\mathbb{P}^n, E_{norm}(-1)) = 0$.

PROPOSITION

Let E be a rank r locally-free sheaf on a smooth projective variety X with $Pic(X) = \mathbb{Z}$.

(a) If $H^0((\Lambda^q E)_{\mathrm{norm}}) = 0$ for $1 \leq q \leq r-1$, then E is μ-stable.

(b) If $H^0((\Lambda^q E)_{\mathrm{norm}}(-1)) = 0$ for $1 \leq q \leq r-1$, then E is μ-semistable.

QUESTION:
Which Chern classes are realized by μ-(semi-)stable vector bundles?

THEOREM.
Let X be a smooth projective variety of dimension $n \geq 2$. Then, for any rank r torsion-free sheaf E, μ-semistable with respect to H, we have (Bogomolov's inequality)

$$\Delta(E) := (2rc_2(E) - (r-1)c_1^2(E))H^{n-2} \geq 0.$$

The class $\Delta(E) := (2rc_2(E) - (r-1)c_1^2(E))H^{n-2}$ is called the discriminant of E.

- Line bundles are always μ-stable.
- The sum $E_1 \oplus E_2$ of two μ-semistable sheaves is μ-semistable if and only if $\mu(E_1) = \mu(E_2)$.
- E is μ-(semi)stable if and only if E^* is.
- If E_1 and E_2 are μ-semistable with respect to H then $E_1 \otimes E_2$ is also μ-semistable with respect to H.
- E is μ-(semi)stable if and only if $E \otimes \mathcal{O}_X(k)$ is.
- For rank r vector bundles with $(c_1(E)H^{d-1}, r) = 1$ the concepts of μ-stability and μ-semistability with respect to H coincide.

MODULI SPACES OF VECTOR BUNDLES

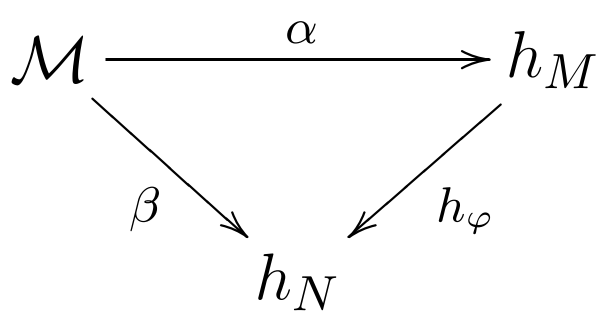
Let $\mathcal{C}$ be a category (e.g., $\mathcal{C} = (Sch/k)$) and let $\mathcal{M} : \mathcal{C} \longrightarrow (Sets)$ be a contravariant moduli functor.

DEFINITION A moduli functor $\mathcal{M} : \mathcal{C} \longrightarrow (Sets)$ is represented by $M \in Ob(\mathcal{C})$ if it is isomorphic to the functor of points of M, h_M, defined by $h_M(S) = Hom_{\mathcal{C}}(S, M)$. The object M is called a fine moduli space for the moduli functor $\mathcal{M}$.

- If a fine moduli space exists, it is unique up to isomorphism.
- There are very few contravariant moduli functors for which a fine moduli space exits.

DEFINITION

A moduli functor $\mathcal{M} : \mathcal{C} \longrightarrow (Sets)$ is corepresented by $M \in Ob(\mathcal{C})$ if $\exists$ natural transformation $\alpha : \mathcal{M} \longrightarrow h_M$ such that $\forall\ N \in Ob(\mathcal{C})$ and $\forall$ natural transformation $\beta : \mathcal{M} \longrightarrow h_N$ $\exists!\ \varphi : M \longrightarrow N$ such that $\beta = h_\varphi \alpha$, i.e., the following diagram commutes:



The object M is called a coarse moduli space for the contravariant moduli functor $\mathcal{M}$.

- If a coarse moduli space exists, it is unique up to isomorphism.
- A fine moduli space for a given moduli functor $\mathcal{M}$ is always a coarse moduli space but not vice versa.

PROBLEM

To classify stable vector bundles on smooth, irreducible, projective varieties

Let X be a smooth, irreducible, projective variety of dimension n and H an ample divisor on X. Fix $P \in \mathbb{Q}[z]$ and consider the moduli functor

$$\mathcal{M}^P_{X,H}(-) : (Sch/k) \to (Sets); \quad S \longmapsto \mathcal{M}^P_{X,H}(S),$$

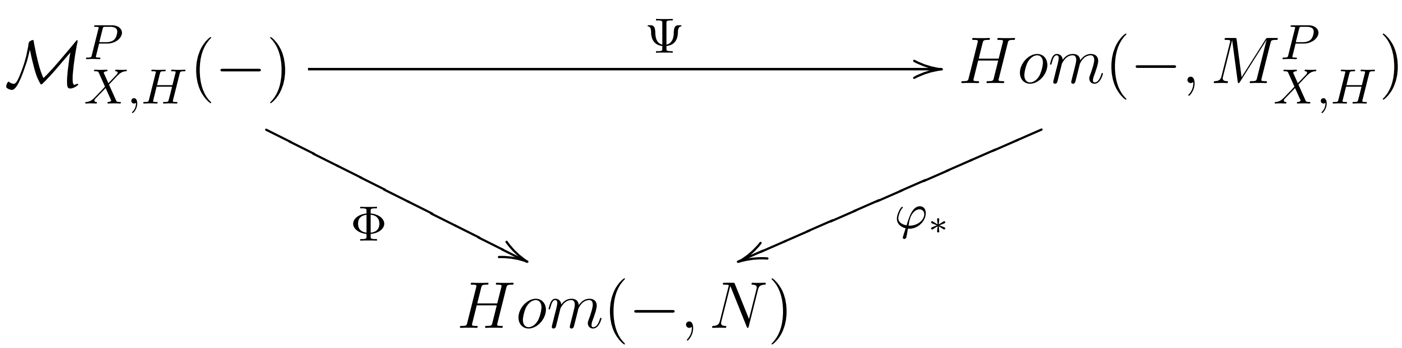
$\mathcal{M}^P_{X,H}(S) = \{S-$ flat families $\mathcal{F} \longrightarrow X \times S$ of μ-stable vector bundles on X with Hilbert polynomial $P\}/\sim$, with $\mathcal{F} \sim \mathcal{F}'$ if, and only if, $\mathcal{F} \cong \mathcal{F}' \otimes p^*L$ for some $L \in Pic(S)$ being $p : S \times X \to S$ the natural projection. And if $f : S' \to S$ is a morphism in (Sch/k), let $\mathcal{M}^P_{X,H}(f)(-)$ be the map obtained by pulling-back sheaves via $f_X = f \times id_X$:

$$\mathcal{M}^P_{X,H}(f)(-) : \mathcal{M}^P_{X,H}(S) \longrightarrow \mathcal{M}^P_{X,H}(S'); \quad [\mathcal{F}] \longmapsto [f^*_X \mathcal{F}].$$

MARUYAMA's THEOREM

$\mathcal{M}^P_{X,H}(-)$ has a coarse moduli scheme $M^P_{X,H}$ which is a separated scheme and locally of finite type over k. This means

- $\exists \Psi : \mathcal{M}^P_{X,H}(-) \longrightarrow Hom(-, M^P_{X,H})$, which is bijective for any reduced point x_0.
- $\forall N$ and $\forall \Phi : \mathcal{M}^P_{X,H}(-) \longrightarrow Hom(-, N)$, $\exists ! \varphi : M^P_{X,H} \longrightarrow N$ for which the following diagram commutes



REMARKS:

(1) $M_{X,H}^{P}$ is unique (up to isomorphism).

(2) In general, $M_{X,H}^{P}$ is not a fine moduli space. In fact, there is no a priori reason why the map

$$\Psi(S) : \mathcal{M}_{X,H}^{P}(S) \rightarrow Hom(S, M_{X,H}^{P})$$

should be bijective for varieties S other than $\{pt\}$.

(3) $M_{X,H}^{P}$ decomposes into a disjoint union of schemes $M_{X,H}^{s}(r; c_1, \cdots, c_{min(r,n)})$ where $M_{X,H}^{s}(r; c_1, \cdots, c_{min(r,n)})$ is the moduli space of rank r, μ-stable vector bundles on X with Chern classes $(c_1, \cdots, c_{min(r,n)})$ up to numerical equivalence.

PROBLEM:

When $M^s_{X,H}(r; c_1, \cdots, c_{min(r,n)})$ is non-empty?

- If X is a smooth curve of genus $g \geq 2$, then the moduli space of μ-stable vector bundles of rank r and fixed determinant is smooth of dimension $(r^2-1)(g-1)$.
- If $dim(X) \geq 3$, then there are no general results which guarantee the non-emptiness of the moduli space of μ-stable vector bundles on X.
- If $dim(X) = 2$, then the existence conditions are well known whenever X is $\mathbb{P}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1$ and, in general, it is known that the moduli space $M^s_{X,H}(r; c_1, c_2) = \emptyset$ if $\Delta(r; c_1, c_2) < 0$ (Bogomolov's inequality) and non-empty provided $\Delta(r; c_1, c_2) \gg 0$.

THEOREM

Let X be a smooth, irreducible, projective variety of dimension n and let E be a μ-stable vector bundle on X with Chern classes $c_i(E) = c_i \in H^{2i}(X, \mathbb{Z})$, represented by a point $[E] \in M^s_{X,H}(r; c_1, \cdots, c_{min(r,n)})$. Then,

$$T_{[E]} M^s_{X,H}(r; c_1, \cdots, c_{min(r,n)}) \cong Ext^1(E, E).$$

If $Ext^2(E, E) = 0$ then $M^s_{X,H}(r; c_1, \cdots, c_{min(r,n)})$ is smooth at $[E]$. In general, we have the following bounds:

$$dim_k Ext^1(E, E) \geq dim_{[E]} M^s_{X,H}(r; c_1, \cdots, c_{min(r,n)})$$

$$\geq dim_k Ext^1(E, E) - dim_k Ext^2(E, E).$$

If X is a smooth projective surface and $E \in M^s_{X,H}(r; c_1, c_2)$, then (Hirzebruch-Riemann-Roch's Theorem)

$$ext^1(E,E) - ext^2(E,E) = 2rc_2(E) - (r-1)c_1^2 - r^2\chi(\mathcal{O}_X) + 1 + p_g(X).$$

The number $2rc_2(E) - (r-1)c_1^2 - r^2\chi(\mathcal{O}_X) + 1 + p_g(X)$ is called the expected dimension of $M^s_{X,H}(r; c_1, c_2)$.

PROBLEMS

- When $M^s_{X,H}(r; c_1, \cdots, c_{min(r,n)})$ is non-empty?
- To study the local and global structure of the moduli space $M^s_{X,H}(r; c_1, \cdots, c_{min(r,n)})$.
- What does the moduli space look like, as an algebraic variety? Is it, for example, connected, irreducible, rational or smooth? What does it look as a topological space? What is its geometry?

PROPOSITION

Let $X = \mathbb{P}^1_k \times \mathbb{P}^1_k$ be a smooth quadric surface and denote by ℓ and m the standard basis of $Pic(X) \cong \mathbb{Z}^2$. $\forall 0 < c_2 \in \mathbb{Z}$, we fix the ample divisor $L = \ell + (2c_2 - 1)m$. We have

(1) The moduli space $M_{X,L}(2; \ell, c_2)$ is a smooth, irreducible, rational projective variety of dimension $4c_2 - 3$. Even more, $M_{X,L}(2; \ell, c_2) \cong \mathbb{P}^{4c_2-3}$.

(2) For any two ample divisors L_1 and L_2 on $X = \mathbb{P}^1_k \times \mathbb{P}^1_k$, the moduli spaces $M_{X,L_1}(2; \ell, c_2)$ and $M_{X,L_2}(2; \ell, c_2)$ are birational whenever non-empty.