Toric varieties

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- Toric varieties provide a quite different yet elementary way to see many examples and phenomena in algebraic geometry.
- In the classification scheme are special but they provide a remarkably fertile testing ground for general theories.
- Toric varieties correspond to combinatorial objects and this makes everything much more computable and concrete than usual.

We will work over $\mathbb{C}$.

BASIC REFERENCES

- W. Fulton, *Introduction to toric varieties*, Ann. of Math. Studies, Princeton, **131** (1993).
- T. Oda, *Convex Bodies and Algebraic Geometry*, Springer-Verlag (1988).
- V. I. Danilov, *The Geometry of toric varieties*, Russian Math. Surveys 33:2 (1978), 97-154.

Definition:

A complete Toric Variety of dimension n over $\mathbb{C}$ is a smooth variety X that contains a torus $T = (\mathbb{C}^*)^n$ as a dense open subset, together with an action of T on X:

$$T \times X \longrightarrow X$$

that extends the natural action of T on itself.

Example: $\mathbb{P}^n$ is a toric variety:

$$(\mathbb{C}^*)^n \subset \mathbb{C}^n \subset \mathbb{P}^n.$$

Fact:
Any toric variety will be constructed from a lattice $N \cong \mathbb{Z}^n$ for some $n \in \mathbb{Z}$ and a fan Δ in N.

Definition:

Let $N = \mathbb{Z}^n$ be a lattice and $V = N \otimes_{\mathbb{Z}} \mathbb{R}$ the real vector space with dual $V^* = M \otimes_{\mathbb{Z}} \mathbb{R}$ being $M = \mathrm{Hom}(N, \mathbb{Z})$ the dual lattice.
A convex polyhedral cone is a set

$$\sigma = \{r_1 v_1 + \cdots + r_s v_s \in V | r_i \geq 0\}$$

generated by any finite set of vectors $v_1, \cdots, v_s$ in V. Such vectors are called generators of the cone σ.
The dimension $\dim(\sigma)$ of σ is the dimension of the linear space spanned by σ.
The dual σ^* of σ is

$$\sigma^* := \{u \in V^* | \langle u, v \rangle \geq 0 \quad \text{for all } v \in \sigma\}.$$

Definition:

A face τ of σ is the intersection of σ with any supporting hyperplane

$$\tau = \sigma \cap u^{\perp} = \{v \in \sigma | \langle u, v \rangle = 0\}$$

for some $u \in \sigma^*$.
A cone is said to be rational if its generators can be taken from N.

A face $\tau = \sigma \cap u^{\perp}$ is generated by those v_i in a generating set of σ such that $\langle u, v_i \rangle = 0$. So,

a cone has only finitely many faces.

Main Basic Facts:

- $(\sigma^*)^* = \sigma$.
- Any face is also a convex polyhedral cone.
- Any intersection of faces is a face.
- Any proper face is contained in some face of codimension one.
- The dual of a convex polyhedral cone is a convex polyhedral cone.

Proposition: (Gordon's Lemma)

If σ is a rational cenvex polihedral cone, then

$$S_\sigma := \sigma^* \cap M$$

is a finitely generated semigroup.

Definition:

A cone is said to be strongly convex if it satisfies one of the following equivalent conditions:

- $\sigma \cap (-\sigma) = \{0\}$.
- σ contains no nonzero linear subspace.
- There is $u \in \sigma^*$ with $\sigma \cap u^\perp = \{0\}$.
- σ^* spans V^*.

Notation:

From now on, a cone in N will be a rational strongly convex polyhedral cone in N.

Definition:

A fan Δ in N is a set of rational strongly convex polyhedral cones in $N_{\mathbb{R}}$ such that:

- Each face of a cone in Δ is also a cone in Δ.
- The intersection of two cones in Δ is a face of each.

We will assume that a fan is finite: that consists of a finite number of cones.

By Gordon's lemma, for any $\sigma \in \Delta$

$$S_\sigma := \sigma^* \cap M, \qquad \sigma^* := \{u \in V^* | \langle u, v \rangle \geq 0 \quad \text{for all } v \in \sigma\}$$

is a finitely generated semigroup thus the group algebra

$$\mathbb{C}[S_\sigma]$$

is a finitely generated commutative $\mathbb{C}$-algebra. This algebra corresponds to an affine variety

$$U_\sigma := Spec(\mathbb{C}[S_\sigma]).$$

If τ is a face of σ, $S_\sigma \hookrightarrow S_\tau$ and this gives a map

$$U_\tau \to U_\sigma.$$

From a fan Δ, the toric variety $X(\Delta)$ is constructed by taking the disjoint union of the affine toric varieties U_σ, one for each $\sigma \in \Delta$ and gluing.

Conversely,

any toric variety X can be realized as $X(\Delta)$ for a unique fan Δ in N.

The toric variety $X(\Delta)$ is smooth iff any cone $\sigma \in \Delta$ is generated by a part of a basis of N.

Notation:

Let N be the lattice $\mathbb{Z}^n$ and denote by $e_1, \ldots, e_n$ a $\mathbb{Z}$-basis of N. For any $0 \leq i \leq n$, we put

$$\Delta(i) := \{\sigma \in \Delta \mid \dim \sigma = i\}.$$

In particular, to any 1-dimensional cone $\sigma \in \Delta(1)$ there is a unique generator $v \in N$ such that $\sigma \cap N = \mathbb{Z}_{\geq 0} \cdot v$ and it is called ray generator.

Examples:

Let us see the fan associated to:

- $\mathbb{P}^2$ and $\mathbb{P}^n$.
- Hirzebruch surfaces.
- $Bl_p(\mathbb{P}^2)$.
- Products of toric varieties.

Proposition:

There is a one-to-one correspondence between the ray generators $v_1, \cdots, v_k$ of Δ and toric divisors $D_1, \cdots, D_k$ on $X(\Delta)$ and any Weil divisor D on $X(\Delta)$ is of the form

$$D = \sum_{i=1}^{k} a_i D_i.$$

Moreover, $D_{i_1} D_{i_2} \cdots D_{i_j} = 0$ if and only if the corresponding vectors $v_{i_1}, v_{i_2}, \cdots, v_{i_j}$ do not span a cone in Δ.

Proposition:

If $X(\Delta)$ is a smooth toric variety of dimension n (hence n is also the dimension of the lattice N) and m is the number of toric divisors (and hence the number of 1-dimensional rays in Δ) then we have

$$0 \to M \to \mathbb{Z}^m \to \mathrm{Pic}(X(\Delta)) \to 0.$$

In particular, the Picard number of $X(\Delta)$ is $\rho(X(\Delta)) = m - n$ and the anticanonical divisor $-K_{X(\Delta)}$ is given by

$$-K_{X(\Delta)} = D_1 + \cdots + D_m.$$

The relations among the toric divisors are given by

$$\sum_{i=1}^{m} \langle u, v_i \rangle D_i = 0$$

for u in a basis of $M = \mathrm{Hom}(N, \mathbb{Z})$.

Definition:

A variety X is called Fano if the anticanonical divisor $-K_X$ is ample.

Definition:

A set of toric divisors $\{D_1, ..., D_k\}$ on $X(\Delta)$ is called a primitive set if $D_1 \cap \cdots \cap D_k = \emptyset$ but $D_1 \cap \cdots \cap \widehat{D_j} \cap \cdots \cap D_k \neq \emptyset$ for all j. Equivalently, this means $< v_1, ..., v_k > \notin \Delta$ but $< v_1, ..., \widehat{v_j}, ..., v_k > \in \Delta$ for all j and we call to $P = \{v_1, ..., v_k\}$ a primitive collection. If $S := \{D_1, ..., D_k\}$ is a primitive set, the element $v := v_1 + ... + v_k$ lies in the relative interior of a unique cone of Δ, say the cone generated by $v'_1, ..., v'_s$ and $v_1 + ... + v_k = a_1 v'_1 + ... + a_s v'_s$ with $a_i > 0$ is the corresponding primitive relation.

In terms of primitive collections and relations we have a nice criterion for checking if a smooth toric variety is Fano or not. In fact, $X(\Delta)$ is Fano if and only if for every primitive relation

$$v_{i_1} + \cdots + v_{i_k} - c_1 v_{j_1} - \cdots - c_r v_{j_r} = 0$$

one has $k - \sum_{i=1}^{r} c_i > 0$.

Classification of toric varieties:

There are different ways to approach the problem of classifying smooth Fano toric varieties:

- By its dimension
- By its Picard number.
- Special properties like pseudosymmetric toric varieties or toric varieties with a splitting fan.
- Etc...

By the dimension are classified Fano toric varieties up to dimension 4.

By its Picard Number:

- If $\rho = 1$, then $X(\Delta) \cong \mathbb{P}^n$.
- Assume $\rho = 2$. In that case $X(\Delta)$ has a splitting fan and is completely classified: there exists a sequence of toric varieties $X = X_r, \ldots, X_0$ such that $X_0 = \mathbb{P}^n$ for a certain n and for $1 \leq i \leq r$, X_i is a projectivization of a decomposable vector bundle over X_{i-1}

- Assume $\rho = 3$. By Batyrev, Δ has three or five primitive collections.

  If Δ has three primitive collections, then $X(\Delta)$ is a projectivization of a decomposable vector bundle over a smooth toric variety Y with Picard number 2.

  If Δ has exactly 5 primitive collections, $X(\Delta)$ is completely classified by Batyrev.
- Fano toric varieties with maximal and almost maximal Picard number are also classified.