Chern classes
Splitting of vector bundles on $\mathbb{P}^n$

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PRAGMATIC 09

CHERN CLASSES

For a nonsingular projective variety X consider the Chow ring $A(X)$ of X:

- A cycle Y of codimension r on X is an element of the free abelian group generated by the closed irreducible subvarieties of X of codimension r.
- For each r, let
  $A^r :=$ Group of cycles of codimension r on X modulo rational equivalence.
- Consider an intersection product

$$A^r \times A^s \to A^{r+s}$$

- Define the CHOW RING of X by

$$A(X) := \bigoplus_{r=0}^{n} A^r(X), \quad n = \dim(X).$$

Let $E \to X$ be a rank r vector bundle on X and

$$\pi : \mathbb{P}(E) \to X$$

the associated projective bundle.

Let $\xi \in A^1(\mathbb{P}(E))$ be the class of the divisor corresponding to $\mathcal{O}_{\mathbb{P}(E)}(1)$.

Fact:
$A(\mathbb{P}(E))$ is, via π^*, a free $A(X)$-module generated by $1, \xi, \xi^2, \cdots, \xi^{r-1}$. Thus, ξ^r can be express as a unique linear combination of $1, \xi, \xi^2, \cdots, \xi^{r-1}$.

Definition:

Let E be a rank r vector bundle on X. For each i, $0 \leq i \leq r$, define the *i-th Chern Class* $c_i(E) \in A^i(X)$ by the requirement:

$$c_0(E) = 1, \quad \sum_{i=0}^{r} (-1)^i \pi^* c_i(E) \xi^{r-i} = 0 \quad \text{in} \quad A(\mathbb{P}(E)).$$

The total Chern class of E is defined by

$$c(E) = c_0(E) + c_1(E) + \cdots + c_r(E)$$

and the Chern polynomial by

$$c_t(E) = c_0(E) + c_1(E)t + \cdots + c_r(E)t^r$$

Fact: For any rank r vector bundle E on X,

$$c_i(E) = 0 \quad \text{for} \quad i > \text{mín}\{\dim X, r\}$$

Properties:

(1) If $E = \mathcal{O}_X(D)$, then $c_t(E) = 1 + Dt$.

(2) If $f : X' \to X$ is a morphism, then $c_i(f^*E) = f^*c_i(E)$.

(3) If $0 \to E' \to E \to E'' \to 0$ is an exact sequence of vector bundles, then

$$c_t(E) = c_t(E')c_t(E'').$$

(4) $c_t(E^*) = c_{-t}(E)$.

(5) Let E be a rank r vector bundle and L be any line bundle. Then

$$c_k(E \otimes L) = \sum_{i=0}^{k} \binom{r-i}{k-i} c_i(E)c_1(L)^{k-i}.$$

In particular, $c_1(E \otimes L) = c_1(E) + rc_1(L)$.

Definition:

Let F be a coherent sheaf and let

$$0 \to E^n \to E^{n-1} \to \cdots \to E^0 \to F \to 0$$

be a free resolution by locally free sheaves. Then we define

$$c_t(F) = \prod_{i=0}^{n} c_t(E^i)^{(-1)^i}.$$

Remark: For a coherent sheaf F on X:

$$c_i(F) = 0 \quad i > \dim X$$

Examples:

- Let E be a reflexive ($E \cong E^{**}$) sheaf of rank 2 on $\mathbb{P}^3$. Then

$$c_3(E) = \text{number of points where is not locally free.}$$

- Let $Z \subset X$ a non-empty 0-dimensional subscheme of an algebraic surface X. Then

$$c_2(I_Z) = |Z|.$$

- Let E be a rank 2 vector bundle on a surface X given by the extension

$$0 \to \mathcal{O}_X(D_1) \to E \to \mathcal{O}_X(D_2) \otimes I_Z \to 0$$

being $Z \subset X$ a non-empty 0-dimensional subscheme of length $|Z|$. Then

$$c_1(E) = D_1 + D_2 \quad \text{and} \quad c_2(E) = D_1 D_2 + |Z|.$$

- Since $A(\mathbb{P}^n) = \mathbb{Z}[h]/h^{n+1}$, for any vector bundle E on $\mathbb{P}^n$ we identify $c_i(E) = c_i h^i \in A^i(\mathbb{P}^n)$ with the corresponding integer $c_i \in \mathbb{Z}$.
- Let $E = \bigoplus_{i=1}^r \mathcal{O}_X(D_i)$. Then

$$c_t(E) = \prod_{i=1}^r (1 + D_i t).$$

  In particular, $c_1(E) = D_1 + \cdots + D_r$.
- $c_1(\Omega_{\mathbb{P}^2}) = -3$ and $c_2(\Omega_{\mathbb{P}^2}) = 3$.

Exercise: Compute $c_i(\Omega^j_{\mathbb{P}^n})$.

Splitting of vector bundles on $\mathbb{P}^n$

Theorem (Serre-Grothendieck Theorem):

Let E be a rank r vector bundle on $\mathbb{P}^1$. Then

$$E = \bigoplus_{i=1}^{r} \mathcal{O}_{\mathbb{P}^1}(a_i)$$

with uniquely determined integers $a_1, \cdots, a_r$ with $a_1 \geq a_2 \geq \cdots \geq a_r$.

Remark:

(1) $c_1(E) = a_1 + a_2 + \cdots + a_r$

(2) This theorem is a very important tool for study vector bundles on projective varieties containing lines.

Theorem (Horrocks):

Let E be a vector bundle on $\mathbb{P}^n$. The following conditions are equivalent:

(i) E splits into a direct sum of line bundles.

(ii) E has no intermediate cohomology; i.e. $H^i(\mathbb{P}^n, E(t)) = 0$ for $1 \leq i \leq n-1$ and for all $t \in \mathbb{Z}$.

Corollary:

Let E be a vector bundle on $\mathbb{P}^n$. Then E splits into a direct sum of line bundles if and only if its restriction to some plane $\mathbb{P}^2 \subset \mathbb{P}^n$ splits.

Horrocks theorem is the starting point of two interesting problems:

- Characterize vector bundles on algebraic varieties that split as a direct sum of line bundles.
- Characterize and classify vector bundles on algebraic varieties without intermediate cohomology.