# Brill-Noether theory for moduli spaces of sheaves on algebraic varieties

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Pragmatic 09

- Let $X$ be an $n$-dimensional smooth projective variety / $K = \overline{K}$, $char(K) = 0$.
- Let

$$M_{X,H}(r; c_1, \cdots, c_s)$$

be the moduli space of rank $r$, $H$-stable vector bundles $E$ on $X$ with fixed Chern classes

$$c_i(E) = c_i \quad \text{for} \quad i = 1, \cdots, s := min\{r, n\}.$$

- Moduli spaces of stable vector bundles were constructed in the 1970's by Maruyama.
- Since then they have been extensively studied from different points of view.
- Unfortunately, except in the classical case of vector bundles on curves, relatively little is known about their geometry in terms of the existence and structure of their subvarieties.

# Classical Brill-Noether Theory

- Deals with the case of line bundles on smooth projective curves $C$ of genus $g$.
- It is concerned with the subvarieties $W^k$ of $Pic^d(C)$ whose line bundles have at least $k+1$ independent global sections.
- Basic questions concerning non-emptiness, connectedness, irreducibility, dimension, singularities....have been answered when the curve $C$ is generic on the moduli space of curves of genus $g$.

# Ways to generalize the classical Brill-Noether theory

Consider vector bundles of any rank on curves

- Giving rise to the Brill-Noether loci

$$W^k(r,d) = \{E \in M(r,d) | h^0(E) \geq k\}$$

  in the moduli space of stable rank $r$ and degree $d$ vector bundles on curves.
- During the last two decades, a great amount of job has been made around this Brill-Noether stratification giving rise to nice and interesting descriptions of these subvarieties.
- Many questions concerning their geometry still remain open.

Consider line bundles on varieties of arbitrary dimension

# Ways to generalize the classical theory of Brill-Noether

Go in both directions simultaneously

- Consider $X$ a smooth projective variety of arbitrary dimension.
- Consider the moduli space

$$M_{X,H}(r; c_1, \cdots, c_s)$$

of rank $r$, $H$-stable vector bundles $E$ on $X$ with fixed Chern classes $c_i$.

- Try to study the subschemes in $M_{X,H}(r; c_1, \cdots, c_s)$ defined by conditions $\{\dim H^j(X, E) \geq n_j\}$.

Remark:

If we are dealing with vector bundles $E$ on curves, we only have at most two non-vanishing cohomology groups $H^0(E)$ and $H^1(E)$ which are related by Riemman-Roch Theorem. Thus, the condition $h^0(E) \geq k$ defines a filtration of the moduli space $M(r, d)$.

If we are dealing with vector bundles $E$ on $n$-dimensional projective varieties, a priory we have $n + 1$ non-vanishing cohomology groups $H^i(E)$ and with the conditions $\{\dim H^j(X, E) \geq n_j\}$ we get a multigraded filtration.

# MAIN GOAL

**GOAL:**

The main GOAL is to introduce a Brill-Noether theory for moduli spaces of rank $r$, $H$-stable vector bundles on algebraic varieties of arbitrary dimension, extending, in particular, the Brill-Noether theory on curves to higher dimensional varieties.

# WHAT WE WILL DO:

- We will define the Brill-Noether locus

$$W_H^k(r; c_1, \cdots, c_s)$$

  in $M_{X,H}(r; c_1, \cdots, c_s)$ as the set of vector bundles in $M_{X,H}(r; c_1, \cdots, c_s)$ having at least $k$ independent sections.
- Associated to this locus we will consider the generalized Brill-Noether number $\rho_H^k(r; c_1, \cdots, c_s)$.
- We will prove that $W_H^k(r; c_1, \cdots, c_s)$ has a natural structure of algebraic variety and that any of its non-empty components has dimension $\geq \rho_H^k(r; c_1, \cdots, c_s)$
- We will address the main problems and we will analyze them for several concrete moduli problems.

## DEFINITION:

Let $H$ be an ample line bundle on $X$. For a torsion free sheaf $F$ on $X$ we set

$$\mu(F) = \mu_H(F) := \frac{c_1(F)H^{n-1}}{rk(F)}.$$

The sheaf $F$ is said to be *$H$-semistable* if

$$\mu_H(E) \leq \mu_H(F)$$

for all non-zero subsheaves $E \subset F$ with $rk(E) < rk(F)$; if strict inequality holds then $F$ is *$H$-stable*.

### Remark:

The definition of stability depends on the choice of the ample line bundle $H$.

**THEOREM:**

Let $X$ be an $n$-dimensional smooth projective variety and $M_H = M_{X,H}(r; c_1, \cdots, c_s)$. Assume that for any $E \in M_H$, $H^i(E) = 0$ for $i \geq 2$. Then, for any $k \geq 0$, there exists a determinantal variety $W^k_H(r; c_1, \cdots, c_s)$ such that

$$Supp(W^k_H(r; c_1, \cdots, c_s)) = \{E \in M_H | h^0(E) \geq k\}.$$

Moreover, each non-empty irreducible component of $W^k_H(r; c_1, \cdots, c_s)$ has dimension at least $\dim(M_H) - k(k - \chi(r; c_1, \cdots, c_s))$, and $W^{k+1}_H(r; c_1, \cdots, c_s) \subset Sing(W^k_H(r; c_1, \cdots, c_s))$ whenever $W^k_H(r; c_1, \cdots, c_s) \neq M_{X,H}(r; c_1, \cdots, c_s)$.

# Sketch of the Proof.

Assume that $M_H$ has a universal family $\mathcal{U} \to X \times M_H$ such that for any $t \in M_H$,

$$\mathcal{U}|_{X\times\{t\}} = E_t$$

is an $H$-stable rank $r$ vector bundle on $X$ with Chern classes $c_i$. Let $D$ be an effective divisor on $X$ such that for any $t \in M_H$,

$$h^0(E_t(D)) = \chi(E_t(D)), \quad H^i(E_t(D)) = 0, \quad i \geq 1.$$

Consider $\mathcal{D} = D \times M_H$ the corresponding product divisor on $X \times M_H$ and denote by

$$\nu : X \times M_H \to M_H$$

the natural projection.

We get the exact sequence

$$0 \to \nu_* \mathcal{U} \to \nu_* \mathcal{U}(\mathcal{D}) \xrightarrow{\gamma} \nu_* (\mathcal{U}(\mathcal{D})/\mathcal{U}) \to R^1 \nu_* \mathcal{U} \to 0.$$

The map $\gamma$ is a morphism between locally free sheaves on $M_H$ of rank $\chi(E_t(D))$ and $\chi(E_t(D)) - \chi(E)$ respectively.
The $(\chi(E_t(D)) - k)$-th determinantal variety

$$W_H^k(r; c_1, \cdots, c_s) \subset M_H$$

associated to it has support

$$\{E_t \in M_H | \operatorname{rank} \gamma_{E_t} \leq \chi(E_t(D)) - k\}$$

i.e. $W_H^k(r; c_1, \cdots, c_s)$ is the locus where the fiber of $R^1 \nu_* \mathcal{U}$ has dimension at least
$(\chi(E_t(D)) - \chi(E_t)) - (\chi(E_t(D)) - k) = k - \chi(E_t)$.
For any $E_t \in M_H$ the assumption $h^i(E_t) = 0$, $i \geq 2$, implies

$$h^1(E_t) = h^0(E_t) - \chi(E_t).$$

Thus,

$$\begin{aligned} Supp(W_H^k(r;c_1,\cdots,c_s)) &= \{E \in M_H | h^1(E) \geq k - \chi(E)\} \\ &= \{E \in M_H | h^0(E) \geq k\}. \end{aligned}$$

Finally, since $W_H^k(r;c_1,\cdots,c_s)$ is a $(\chi(E_t(D)) - k)$-determinantal variety associated to a morphism between locally free sheaves of rank $\chi(E_t(D))$ and $\chi(E_t(D)) - \chi(E)$ respectively, any of its non-empty irreducible components has dimension greater or equal to $\dim(M_H) - k(k - \chi(E))$ and

$$W_H^{k+1}(r;c_1,\cdots,c_s) \subset Sing(W_H^k(r;c_1,\cdots,c_s))$$

whenever $W_H^k(r;c_1,\cdots,c_s) \neq M_{X,H}(r;c_1,\cdots,c_s)$ $\square$.

Remark: The cohomological assumptions are natural if we want a filtration of $M_H$ by the subvarieties $W_H^k(r; c_1, \cdots, c_s)$.

Any vector bundle $E$ on $X$ has $n+1$ cohomological groups and one is forced to look for a multigraded filtration of $M_H$ by the sets $\{E \in M_H | h^i(E) \geq k_i\}$.

Under our cohomological assumptions,

$$\dim H^0(E) - \dim H^1(E) = \chi(E) = \chi(r; c_1, \cdots, c_s).$$

Hence, it makes sense to consider only the filtration of the moduli space $M_H$ by the dimension of the space of global sections.

Instanton bundles on $\mathbb{P}^{2n+1}$, Schwarzenberger and Steiner bundles on $\mathbb{P}^n$, Steiner and Spinor bundles on $Q_n \subset \mathbb{P}^{n+1}$ and many others satisfy these cohomological conditions.

## COROLLARY:

Let $X$ be a smooth projective surface and assume that

$$c_1H \geq rK_XH.$$

Then, for any $k \geq 0$, there exists a determinantal variety $W_H^k(r;c_1,c_2)$ such that

$$Supp(W_H^k(r;c_1,c_2)) = \{E \in M_H | h^0(E) \geq k\}.$$

Moreover, each non-empty irreducible component of $W_H^k(r;c_1,c_2)$ has dimension greater or equal to

$$\rho_H^k(r;c_1,c_2) = \dim(M_H) - k(k - r(1+P_a(X)) + \frac{c_1K_X}{2} - \frac{c_1^2}{2} + c_2).$$

## DEFINITION:

The variety $W^k = W^k_H(r; c_1, \cdots, c_s)$ is called the $k$-**Brill-Noether locus** of $M_H$ and

$$\rho^k = \rho^k_H(r; c_1, \cdots, c_s) := \dim M_H - k(k - \chi(r; c_1, \cdots, c_s))$$

is called the **generalized Brill-Noether number.**

Remark: When $X$ is a smooth projective curve and we consider the moduli space $Pic^d(X)$ of degree $d$ line bundles on $X$, then we recover the classical Brill-Noether loci and the generalized Brill-Noether number is the classical Brill-Noether number $\rho = \rho(g, r, d) = g - (r + 1)(g - d + r)$

Remark:

The Brill-Noether locus $W^k$ has dimension greater or equal to $\rho^k$ and the number $\rho^k$ is also called the expected dimension of the corresponding Brill-Noether locus.

QUESTION:

Whether the dimension of $W^k_H(r; c_1, \cdots, c_s)$ and its expected dimension coincide provided $W^k_H(r; c_1, \cdots, c_s) \neq \emptyset$

## QUESTION:

- Whether $\rho^k < 0 \Rightarrow W^k = \emptyset$ ?
- Whether $\rho^k \geq 0 \Rightarrow W^k \neq \emptyset$ ?
- Whether $\rho^k \geq 0$ and $W^k \neq \emptyset$ implies

$$\rho^k = \rho^k_H(r; c_1, \cdots, c_s) = \dim W^k_H(r; c_1, \cdots, c_s) \quad ?$$

If we deal with varieties of higher dimension, a great number of different situations and pathologies appear and this makes this new theory and emerging field of interest.

## EXAMPLE:

Let $X = \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3$ and denote by $l_1$, $l_2$ the generators of $Pic(X)$. For any $n \geq 2$ we fix the ample line bundle

$$L = l_1 + nl_2.$$

For any $k$, $1 \leq k \leq n$, such that $8n - 3 < k(k-1)$

$$W_L^k(2; (2n-1)l_2, 2n) \subset M_{X,L}(2; (2n-1)l_2, 2n)$$

is non-empty and the generalized Brill-Noether number,

$$\rho_L^k(2; (2n-1)l_2, 2n) = 8n - 3 - k(k-1),$$

is negative.

# Brill-Noether theory on Hirzebruch surfaces

- For any integer $e \geq 0$, let $X_e \cong \mathbb{P}(\mathcal{E}) = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-e))$ be a non singular, Hirzebruch surface.
- We denote by $C_0$ and $F$ the standard basis of $Pic(X_e) \cong \mathbb{Z}^2$ such that $C_0^2 = -e$, $F^2 = 0$ and $C_0 F = 1$.
- In this basis $-K_{X_e} = 2C_0 + (e+2)F$ and a divisor $L = aC_0 + bF$ on $X_e$ is ample if, and only if, it is very ample, if and only if, $a > 0$ and $b > ae$, and that $D = a' C_0 + b' F$ is effective if and only if $a' \geq 0$ and $b' \geq 0$.

Our plan is to explain you results concerning:

- Non-emptiness of Brill-Noether loci of stable vector bundles on $X_e$.
- Emptiness of Brill-Noether loci of stable vector bundles on $X_e$.

## Non-emptiness of some Brill-Noether loci of stable vector bundles on $X_e$

We fix our attention into the following moduli space

$$M_H(r; C_0 - xF, c_2)$$

where $x$, $r$, $c_2$ are integers with $x > 0$, $r \geq 2$ and $c_2 \gg 0$ and we have fixed the ample divisor

$$H := C_0 + (x + e + 1)F.$$

### Remark:

Under the assumptions $x > 0$ and $c_2 \gg 0$, we have

$$\rho_H^k(r; C_0 - xF, c_2) < 0 \quad \text{for} \quad k \geq r - 1.$$

So, we will study the Brill-Noether loci $W_H^k(r; C_0 - xF, c_2)$ of the Brill-Noether stratification of $M_H(r; C_0 - xF, c_2)$ for $k$ in the range $1 \leq k \leq r - 2$

**THEOREM:**

Let $X_e$ be a Hirzebruch surface. Then for any integer $k$, $1 \leq k \leq r-2$, and $c_2$ such that $c_2 \equiv 0 \mod (r-k-1)$ there exists an irreducible component of the Brill-Noether loci

$$W_H^k(r; C_0 - xF, c_2)$$

of the moduli space $M_H(r; C_0 - xF, c_2)$ which has the expected dimension $\rho_H^k(r; C_0 - xF, c_2)$

Idea:

We will construct a family $\mathcal{F}$ of rank $r$ vector bundles $E$ given by a non-trivial extension

$$0 \to \mathcal{O}_{X_e}^k \to E \to G \to 0$$

where $G$ sits in an open subset $\mathcal{H}^0(r-k; C_0 - xF, c_2) \subset M_H(r-k; C_0 - xF, c_2)$.

Then check that $E$ is stable and compute the dimension of $\mathcal{F}$.

**Definition:**

Let $X_e$ be a Hirzebruch surface. A coherent sheaf $E$ on $X_e$ is said to be prioritary if it is torsion free and if

$$\mathrm{Ext}^2(E, E(-F)) = 0.$$

The following Lemma is the key point in order to be able to use prioritary sheaves to study moduli spaces of stable vector bundles.

**Lemma:**

Let $X_e$ be a Hirzebruch surface and let $H$ be an ample divisor on $X_e$. Then any $H$-semistable, torsion free sheaf $E$ on $X_e$ is prioritary.

## Proposition:

Let $X_e$ be a Hirzebruch surface and let $H$ be an ample divisor on $X_e$. Fix integers $\alpha$, $x$ and $c$ with $\alpha, x > 0$ and $c < 0$. If $c \ll 0$, then a generic vector bundle $E \in M_H(\alpha + 1; C_0 - xF, -\alpha c)$ sits in an exact sequence of the following type

$$0 \to \mathcal{O}_{X_e}(C_0 - (x - \alpha c)F) \to E \to \mathcal{O}_{X_e}(-cF)^{\alpha} \to 0. \qquad (1)$$

THEOREM:

Let $X_e$ be a Hirzebruch surface. Then for any integer $k$, $1 \leq k \leq r-2$ such that $c_2 \equiv 0 \mod (r-k-1)$ there exists an irreducible component of the Brill-Noether loci

$$W^k_H(r; C_0 - xF, c_2)$$

of the moduli space $M_H(r; C_0 - xF, c_2)$ which has the expected dimension $\rho^k_H(r; C_0 - xF, c_2)$

# Sketch of the Proof.

Fix $k$ an integer, $1 \leq k \leq r-2$, such that $(r-k-1)$ divides $c_2$ and denote by $f$ the negative integer such that $c_2 = -(r-k-1)f$.
Let $\mathcal{F}$ be the irreducible family of rank $r$ vector bundles $E$ given by a non-trivial extension

$$0 \to \mathcal{O}_{X_e}^k \to E \to G \to 0 \tag{2}$$

where $G$ sits in the open subset $\mathcal{H}^0(r-k; C_0 - xF, c_2) \subset M_H(r-k; C_0 - xF, c_2)$ given above.
First of all notice that any $E \in \mathcal{F}$ is a rank $r$ vector bundle with Chern classes

$$(c_1(E), c_2(E)) = (C_0 - xF, -(r-k-1)f) = (C_0 - xF, c_2)$$

and by construction $h^0(E) \geq k$.

Claim: $E$ is $H$-stable.

**Proof of the Claim:** We proceed by induction on $k$.

In case $k = 1$, $E$ is given by a non-trivial extension

$$0 \to \mathcal{O}_{X_e} \to E \xrightarrow{\sigma} G \to 0$$

where $G$ is a rank $r - 1$, $H$-stable vector bundle. Checking the definition of stability we wee that $E$ is $H$ stable.

Now, we fix $k > 1$ and we consider the following commutative diagram of vector bundles

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{O}_{X_e} & \longrightarrow & \mathcal{O}_{X_e}^k & \longrightarrow & \mathcal{O}_{X_e}^{k-1} & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{O}_{X_e} & \longrightarrow & E & \longrightarrow & \overline{E} & \longrightarrow & 0. \\ & & & & \downarrow & & \downarrow & & \\ & & & & G & = & G & & \\ & & & & \downarrow & & \downarrow & & \\ & & & & 0 & & 0 & & \end{array}$$

By hypothesis of induction, $\overline{E}$ is $H$-stable and thus by the first case $k = 1$, $E$ is $H$-stable which concludes the proof of the Claim.

Hence, the family $\mathcal{F}$ defines a non-empty irreducible component of the Brill-Noether loci $W_H^k(r; C_0 - xF, c_2)$.

We compute its dimension and see that

$$\dim \mathcal{F} = \dim \mathcal{H}^0(r-k; C_0 - xF, c_2) + \dim Grass(k, h)$$

being $h = ext^1(G, \mathcal{O}_{X_e})$ and $Grass(k, h)$ the Grassmann variety of $k$-dimensional linear subspaces of $\mathrm{Ext}^1(G, \mathcal{O}_{X_e})$ coincides with $\rho_H^k(r; C_0 - xF, c_2)$

Therefore, $\mathcal{F}$ is an irreducible component of the Brill-Noether loci $W_H^k(r; C_0 - xF, c_2)$ of the moduli space $M_H(r; C_0 - xF, c_2)$ which has the expected dimension, namely, $\rho_H^k(r; C_0 - xF, c_2)$. □

**THEOREM:**

Let $X$ be a smooth algebraic surface, $H$ an ample divisor s.t. $K_X H \leq 0$ and $E$ a semistable rank $r \geq 2$ vector bundle. Set $a := \lceil \frac{(r^2-1)H^2}{2} \rceil$ or $a = 2H^2$ if $r = 2$. If $0 \leq c_1(E)H < arH^2 + rK_X H$, then

$$h^0(E) \leq r + \frac{ac_1(E)H}{2}.$$

**COROLLARY:**

Let $X$ be a smooth algebraic surface, $H$ an ample divisor on $X$ such that $K_X H \leq 0$. Let $r \geq 2$, $c_2 \gg 0$ and set $a := \lceil \frac{(r^2-1)H^2}{2} \rceil$ or $a = 2H^2$ if $r = 2$. Assume that $0 \leq c_1(E)H < arH^2 + rK_X H$, then

$$W_H^k(r; c_1, c_2) = \emptyset$$

# Sketch of the Proof of the Theorem.

Let $C \in |aH|$ be a general smooth connected curve.
Since

$$\frac{\binom{a+2}{2} - a - 1}{a} > deg(X) \cdot \text{máx}\{\frac{r^2 - 1}{4}, 1\},$$

by Flenners restriction theorem, $E|_C$ is a rank $r$ semistable vector bundle on $C$ of degree equal to $ac_1(E)H$.
On the other hand, by the adjunction formula

$$2g(C) - 2 = C(C + K_X) = aH(aH + K_X) = a^2H^2 + aHK_X.$$

Hence,

$$0 \leq \mu(E|_C) = \frac{ac_1(E)H}{r} \leq a^2H^2 + aHK_X = 2g(C) - 2$$

and therefore, applying Clifford's Theorem for semistable vector bundles on curves, we have

$$h^0(E|_C) \leq r + \frac{ac_1(E)H}{2}.$$

To finish, we only need to check that

$$h^0(E) \leq h^0(E|_C).$$

To this end, we tensor by $E$ the short exact sequence

$$0 \to \mathcal{O}_X(-C) \to \mathcal{O}_X \to \mathcal{O}_C \to 0$$

and taking cohomology we get

$$0 \to H^0(E(-C)) \to H^0(E) \to H^0(E|_C) \to \cdots . \tag{3}$$

If $H^0(E(-C)) \neq 0$, then $\mathcal{O}_X(C) \hookrightarrow E$ and since $E$ is semistable with respect to $H$ we have

$$CH = aH^2 \leq \frac{c_1(E)H}{r} < aH^2 + K_X H$$

which contradict the fact that $K_X H \leq 0$. Therefore, $H^0(E(-C)) = 0$ and from the exact sequence (3) we deduce $h^0(E) \leq h^0(E|_C)$ and we finish the proof.

COROLLARY:

Let $x$, $r$, $c_2$ with $x > 0$, $r \geq 2$ and $c_2 \gg 0$. Let $H := C_0 + (x + e + 1)F$ be an ample divisor on $X_e$ and set $a := \lceil \frac{(r^2-1)H^2}{2} \rceil$ or $a = 2H^2$ if $r = 2$. Then, for any $k > r + \frac{a}{2}$

$$W_H^k(r; C_0 - xF, c_2) = \emptyset.$$