Moduli of vector bundles and group action

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July 13, 2006

These notes are still in preliminary form and they are intended only for distribution to the participants to the 24rd Autumn School in Algebraic Geometry to be held in Wykno (Poland), September 9-16, 2001
We thank the organizers of the school and all the participants for their criticism that will help to improve these notes. Acknowledge EC Grant HTN **** 2000-00099 "Eager" Polosh KBN SPVB-M

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Chapter 1

Introduction

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Chapter 2

Short introduction to algebraic group actions

The geometric invariant theory, developped by Mumford in [MFK], is studying the action of algebraic groups on algebraic varieties. Its main goal is to construct and to describe quotient varieties in a sense weaker than an orbit space.

When the group G, acting on the variety X, is reductive (it means G is affine and all its representations are semi-simple) the algebra of invariants of X is finitely generated (Nagata's theorem). Thanks to this result we can define the quotient of X (in fact of an open set of X) by G. We will give a characterization of these quotient morphisms, and a useful criterion (Hilbert-Mumford criterion) which will give us a practical way to describe the orbits.

After this short and not exhaustive introduction we will give many applications in chapter 3 and in the rest of this book.

2.1 Algebraic groups and their actions

A closed subgroup G of $\mathbf{GL}(n, \mathbb{C})$ is called a linear group. It is a smooth algebraic variety, moreover the multiplication map $G \times G \to G$ and the inverse map $G \to G$ are algebraic, that is G is a algebraic group. A homomorphism

$$f : G \to \mathbf{GL}(N, \mathbb{C})$$

is called an N-dimensional linear representation of G. Obviously, a linear group is affine.

2.1. Theorem. *[Bor], Prop. I.10 Every affine algebraic group is isomorphic to a linear group.*

Let G an algebraic group and X an algebraic variety (we always omit 'over the field $\mathbb{C}$')

2.2. Definition. *An algebraic action of G on X is a morphism of algebraic varieties*

$$\sigma : G \times X \to X$$

satisfying the following properties $\sigma(g, (\sigma(h, x)) = \sigma(gh, x)$ and $\sigma(e, x) = x$.

For an action σ of G on X we will denote $\sigma(g, x) = g.x$. The orbit of $x \in X$ is the set

$$G.x = \{g.x \mid g \in G\}$$

and the stabilizer of x is

$$G_x := \{g \in G \mid g.x = x\}$$

We denote X/G the orbit space and $\pi : X \to X/G$ the quotient.

When G acts on two varieties X and Y a map $f : X \to Y$ is G-equivariant if

$$f(g.x) = g.f(x)$$

for all $g \in G$ and all $x \in X$. In particular f is invariant if $f(g.x) = f(x)$ i.e. if f is constant on the orbits. Any invariant map factorizes by the quotient map $X \to X/G$.

In general the quotient map is not as nice as expected. For example consider the action of $\mathbb{C}^*$ on $\mathbb{C}^2$ by multiplication

$$t.(x, y) = (tx, ty)$$

The orbits are the origin and the vector subspaces of dimension one (lines) without the origin. Then the closure of every orbit contains the origin. Thus there is no structure of separated topological space on X/G for which the quotient map is continuous. But the quotient $(X \setminus \{0\})/G$ exists, it is the projective line.

2.3. Proposition. *[Bor]I.8 Let G an algebraic group acting on an algebraic variety X.*
(i) Every orbit of G in X is open in its closure
(ii) The closure of every orbit contains the orbit and others orbits of smaller dimension; it contains at least one closed orbit.
(iii) for any x, $dim(G.x) = dim(G) - dim(G_x)$
(iv) For any $n \geq 0$, the set $\{x \in X \mid dim(G.x) \leq n\}$ is closed in X.

Let X be an algebraic variety and $A = \mathcal{O}(X)$ the algebra of morphisms $X \to \mathbb{C}$. An action σ of G on X induces an action of G on A given by

$$\sigma(g, f)(x) = g.f(x) = f(g^{-1}.x) \text{ for } f \in A \ g \in G \text{ and } x \in X$$

We denote also this action by $f \mapsto g^*(f)$.

Example. (binary forms) For any integer d let V_d be the vector space of homogeneous polynomials of degree d in the variables x, y. Then $G = SL(2)$ acts on points written as column vectors by left multiplication and acts on V_d

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} .f(x, y) = f(dx - by, -cx + ay)$$

Of course every $f \in V_d$ can be written $f(x, y) = \prod_{i=1}^{d}(b_i x - a_i y)$. If f is not zero the points (a_i, b_i) of the projective line determine f. Note that the points $g.(a_i, b_i)$ determine $g.f$. The stabilizer G_f acts on the roots by permutation. If the roots of f are distincts and if $d \geq 3$ then G_f is finite and $\dim(G.f) = \dim(SL(2)) = 3$.

For $f \in V_d$ we define the discriminant of f by the formula

$$\Delta(f) = \prod_{1 \leq i < j \leq d} (a_i b_j - a_j b_i)^2$$

The map $\Delta : V_d \to \mathbb{C}$ is a polynomial map G-invariant. If $\Delta(f) = 0$ then f has a multiple (at least double) root.
If $f \in V_d$ has only simple roots and if $d \geq 2$ the orbit $G.f$ is closed in V_d. In fact we have

$$\overline{G.f} \subset \{\phi \in V_d \mid \Delta(\phi) = \Delta(f)\}$$

This last set is closed in V_d and it contains only orbit of maximal dimension (remind that the stabilizer of a polynomial with simple roots is finite, this argument works for $d \geq 3$ but the reader can supply a easier argument in the case $d = 2$). We conclude with the proposition 2.3 (ii).

2.2 Quotients

All the proofs of the result of this section can be found in [LP0].
We begin by the notion of categorical quotient which is the weakest. Then we will define the good quotient and the geometric quotient which is the natural notion of orbit space.

2.4. Definition. *A categorical quotient of X by the action of G is a pair (Y, π) given by an algebraic variety Y and a morphism $\pi : X \to Y$ which satisfy the following properties*
(i) the morphism π is G-invariant.
(ii) the pair (Y, π) is universal for (i). That means that for any G-invariant morphism $f : X \to Z$ there exist an unique morphism $\phi : Y \to Z$ such that $f = \phi \circ \pi$

Example. Let M_n be the vector space of $n \times n$ matrices with coefficients in $\mathbb{C}$. The linear group $\mathbf{GL}(n, \mathbb{C})$ acts on M_n by conjugation. Consider the morphism

$$\pi : M_n \to \mathbb{C}^n$$

which associates to a matrix M the n-coefficients of its characteristic polynomial. Then the couple $(\mathbb{C}^n, \pi)$ is a categorical quotient of M_n.

2.1. Exercise. *Prove this claim. When $n = 2$ describe the orbits and the fibers. Do they coincide?*

2.5. Definition. *A good quotient of X by the action of G is a pair (Y, π) of an algebraic variety Y and a G-invariant morphism π which satisfy the following conditions*
(i) the morphism π is affine and surjective.
(ii) the canonical morphism of sheaves $\mathcal{O}_Y \to \pi_*(\mathcal{O}_X)^G$ is an isomorphism.
(iii) The image by π of a G-invariant subset is a closed subset of Y.
(iv) The morphism π separates the G-invariant disjoint closed subsets of X.

2.6. Proposition. *Let $\pi : X \to Y$ be a good quotient of X by G. Then we have the following properties*
(i) The topology of Y is the quotient topology.
(ii) The pair (Y, π) is a categorical quotient.
(iii) In each fiber of π there is one and only one closed orbit.

The third assertion implies that the underlying set of Y can be identified with the set of closed orbits.

2.7. Definition. *Let X be an algebraic variety with an action of G. A good quotient*

$$\pi : X \to Y$$

is called a geometric quotient if the orbits are closed.

Example. Let $U \subset M_n$ be the open subset of matrices whith distinct eigenvalues. Then there exist a geometric quotient of U by $\mathbf{GL}(n, \mathbb{C})$ (conjugation). This quotient is the open set of $\mathbb{C}^n$ of points $(c_1, \cdots, c_n)$ such that the discriminant of the polynomial

$$t^n + \sum_{i \geq 1} c_i t^{n-i}$$

is different from zero. It is easy to verify that the orbits are closed. We will see next that it is a good quotient.

Let σ be an action of G on X. For $x \in X$ we denote by $\sigma_x : G \to X$ the map which sends G onto $G.x$. If this map is proper, the orbit of x is closed in X.

2.8. Proposition. *Let $\pi : X \to Y$ be a good quotient of X by the action σ of G. Let $U \subset X$ be the set of points such that the morphism σ_x is proper. Then*
(i) the set U is open in X
(ii) this open set is the inverse image by π of an open set $V \subset Y$, and the induced morphism $U \to V$ is a geometric quotient.

2.9. Proposition. *The map σ_x is proper iff the two following conditions are filled*
(i) the orbit of x is closed in X
(ii) the stabilizer G_x of x is finite.

2.2.1 Quotient of an affine variety

2.10. Theorem. *Let G be a linear reductive group acting on an affine algebraic variety $X = SpecA$. Then the algebra A^G is finitely generated. Moreover the morphism*

$$\pi : X = SpecA \to Y = SpecA^G$$

induced by the inclusion $A^G \subset A$ is a good quotient.

Example. Consider the action of $\mathbf{GL}(n, \mathbb{C})$ on M_n by conjugation. By Theorem 2.10 there exists a good quotient. This good quotient is a categorical quotient, and by uniqueness property, this good quotient is the quotient

$$M_n \to \mathbb{C}^n$$

described before.

2.2.2 Quotient of a projective variety

Let $\phi : B \to A$ be a morphism of graded finitely generated algebras such that the graded piece of degree 0 is $\mathbb{C}$. Then the inverse image of the vertex O of the cone $\mathrm{Spec}(B)$ by the induced morphism $\mathrm{Spec}(A) \to \mathrm{Spec}(B)$ is a closed $\mathbb{C}^*$-invariant subvariety of $\mathrm{Spec}(A)$, corresponding to a closed subvariety of Proj(A), called the center of ϕ, and denoted by $C(\phi)$. This subvariety is associated to the ideal of A generated by $\phi(B_+)$. Thus we obtain a morphism of algebraic varieties

$$\mathrm{Proj}(A) \setminus C(\phi) \to \mathrm{Proj}(B)$$

Let G be a reductive group which acts linearly on $\mathbb{C}^{n+1}$. Let X be a G-invariant subvariety, defined by a homogeneous ideal I. Consider the algebra A of homogeneous polynomials on $\mathbb{C}^{n+1}$, and the quotient $R = A/I$. This is a graded algebra such that $R_0 = \mathbb{C}$. The variety X can be identified to Proj(R). Consider the inclusion

$$i : R^G \hookrightarrow R$$

2.11. Definition. *A point of X is unstable for the action of G if it belongs to the center of i.*

Let $X^{ss} = X \setminus C(i)$. We remark that $x \in X^{ss}$ if and only if there is an G-invariant homogeneous polynomial $P \in R$ of degree ≥ 1 such that $P(x) \neq 0$.

2.12. Theorem. *The canonical morphism $\pi : X^{ss} \to Proj(R^G)$ is a good quotient.*

2.13. Definition. *A point $x \in X$ is called semi-stable for the action of G if there is an G-invariant homogeneous polynomial $P \in R$ of degree ≥ 1 such that $P(x) \neq 0$.*

We denote by X^{ss} the set of semi-stable points of X. We denote by $\hat{X} = \mathrm{Spec} R$ the affine cone of X and we denote by $\hat{x}$ a representative of $x \in X$. So the above definition means that

$$x \in X^{ss} \Leftrightarrow O \notin \overline{G.\hat{x}}$$

This leads to the following definition

2.14. Definition. *A point $x \in X$ is semi-stable if and only if $O \notin \overline{G.\hat{x}}$. It is stable if $G.\hat{x}$ is closed and if $G_{\hat{x}}$ is finite.*

As before we denote by σ the action of G on X.

2.15. Proposition. *Let G an affine algebraic group which acts on X. Then the orbit morphism*

$$\sigma_x : G \to X$$

is proper if and only if x is a stable point.

By the proposition 2.8 we know that the set of stable points X^s is open. It is the inverse image of an open set Y^s and the map $X^s \to Y^s$ is a geometric quotient.

2.2.3 The Hilbert-Mumford criterion.

To compute the semi-stable (stable) points of a linear action of a group G on a variety X we have a very useful numerical criterion, called the *Hilbert-Mumford criterion.*

2.16. Definition. *A group morphism $\mathbf{G}_m = \mathbb{C}^* \to G$ is called an one parameter subgroup of G.*

2.17. Proposition. *Let V a G-module. Then $v \in V$ is semi-stable (resp.stable) under the action of G if and only if $v \in V$ is semi-stable (resp. stable) under the induced action of every one parameter subgroup.*

Let $\lambda : \mathbb{C}^* \to G$ a one parameter subgroup of G and V be a n-dimensional representation of G. Then in convenient coordinates we have

$$\lambda(t) = \begin{pmatrix} t^{r_1} & & 0 \\ & \ddots & \\ 0 & & t^{r_n} \end{pmatrix} \text{ with } r_1 \geq \cdots \geq r_n$$

Let $\mu(\lambda, v) = max_{i|v_i \neq 0}\{-r_i\}$. Assume now that G acts linearly on $\mathbb{P}^n$ and let $X \subset \mathbb{P}^n$ be a closed G-invariant subvariety. We can show that $\mu(\lambda, x) = \mu(\lambda, \hat{x})$ for every non zero representative of x. Then we observe that

- $\mu(\lambda, x) > 0 \Leftrightarrow \mathrm{Lim}_{t \to 0}\lambda(t).\hat{x}$ does not exist.
- $\mu(\lambda, x) \geq 0 \Leftrightarrow \mathrm{Lim}_{t \to 0}\lambda(t).\hat{x} \neq 0$ if the limit exists.

Hence a consequence of Proposition 2.17 is

2.18. Theorem. (Hilbert-Mumford criterion) *Assume that G acts linearly on $\mathbb{P}^n$ and let $X \subset \mathbb{P}^n$ a closed G-invariant subvariety. A point $x \in X$ is semi-stable (resp. stable) if and only if $\mu(\lambda, x) \geq 0$ (resp. $\mu(\lambda, x) > 0$) for every one parameter subgroup λ.*

Chapter 3

Applications: binary forms, hypersurfaces

3.1 Binary forms

3.1.1 Generalities and binary forms of degree ≤ 3

Let U be a vector space of dimension 2. $\mathbb{P}(U)$ is a projective line $\mathbb{P}^1$. We have a natural isomorphism $S^n\mathbb{P}^1 \simeq \mathbb{P}^n$ which now we recall. The symmetric group $\mathfrak{S}_n$ acts on the ring $\mathbb{C}[x_0, \dots, x_n]$ by permuting the variables. A symmetric polynomial in the n variables $x_1, \dots x_n$ is a polynomial which is invariant for this action.
Let $\prod_{j=1}^n (1 + x_i t) = \sum_{j=0}^n E_j t^j$ so that $E_0 = 1$, $E_1 = \sum_{j=1}^n x_j$, ..., $E_n = \prod_{j=1}^n x_j$. The E_j are called the elementary symmetric polynomials.
Consider the variety $S^n\mathbb{P}^1$ of effective divisors of degree n in $\mathbb{P}^1$.
To such a divisor

$$D = \sum n_x x, \text{ where } \sum n_x = n \text{ and } x \in \mathbb{P}^1$$

we can associate a homogeneous polynomial $P \in S^n U$, unique modulo invertible scalars, vanishing in x with multiplicity n_x. This shows that $S^n\mathbb{P}^1 \simeq \mathbb{P}(S^n U) \simeq \mathbb{P}^n$. Consider the group $SL(U)$ and its natural action on $S^n U$:

$$g.P(x) = P(g^{-1}.x)$$

Then by Theorem 2.12 we have a good quotient of the open set of semi-stable points :

$$\pi : (\mathbb{P}^n)^{ss} \to Y$$

3.1. Proposition. *A divisor $D = \sum n_x x$ gives a semi-stable (resp. stable) point in $\mathbb{P}^n$ for the action of $SL(U)$ if and only if for any $x \in \mathbb{P}^1$ we have $n_x \leq \frac{n}{2}$, (resp. $n_x < \frac{n}{2}$)*

Proof. Let $\lambda : \mathbb{C}^* \to SL(U)$ a one parameter subgroup of $SL(U)$. We have

$$\lambda(t) = \begin{pmatrix} t^a & 0 \\ 0 & t^b \end{pmatrix}$$

with $a+b=0$. We can assume that $a<b$. A basis of $S^nU^\vee$ is given by the monomials X^iT^j with $i+j=n$ so we have $P=\sum_{i=0,\cdots,n} a_iX^iT^{n-i}$. Then the action of λ on $S^nU^\vee$ is the following

$$\lambda(t).P = \sum_{i=0,\cdots,n} a_i t^{-ai-b(n-i)} X^iT^{n-i}$$

We have

$$\mu(\lambda,P) = Max\{ia+b(n-i), a_i\neq 0\} = Max\{a(2i-n), a_i \neq 0\}$$

but $a<0$ so $\mu(\lambda,P)=(-a)(n-2Min\{i,a_i\neq 0\})$. By the Hilbert-Mumford criterion 2.18 we know that P is unstable if and only if there exists such a one parameter subgroup with $\mu(\lambda,P)<0$. This is equivalent to $a_i=0$ for $i\leq \frac{n}{2}$. That means that the point $(0,1)$ is a root with multiplicity $\geq \frac{n}{2}$ of the polynomial P. Conversely if P has one root of multiplicity $\geq \frac{n}{2}$ we can assume that this root is $(0,1)$. Then by choosing $a=-1$ and $b=1$ we prove that P is unstable (i.e. the point D is unstable). The method is exactly the same to find the semi-stable points. $\square$

By this way we found the semi-stable points but we did not describe the quotient nor the morphism $\pi:(\mathbb{P}^n)^{ss}\to Y$. In the following cases we will do it.

• For $\mathbf{n=1}$ (one point on $\mathbb{P}^1$) all points are unstable. There exists a categorical quotient (which is a point). The action is transitive and there is no $SL(U)$-invariant open subset with a good quotient.

• For $\mathbf{n=2}$, (two points on $\mathbb{P}^1$) the open set of semi-stable points is the open set of divisors $D=x+y$ with $x\neq y$. This open set $(\mathbb{P}^2)^{ss}$ is the complementary of a conic in the plane. Of course we can identify the conic with the discriminant of

$$P=a_0X^2+2a_1XT+a_2T^2$$

The polynomial $a_1^2-a_0a_2$ is $SL(U)$-invariant. The good quotient is just one point because there is only one orbit (i.e. $SL(U)$ is 2-transitive). We can also remark that the algebra of invariants is $\mathbb{C}[S^2U^\vee]^{SL(U)}=\mathbb{C}[a_1^2-a_0a_2]$.

• For $\mathbf{n=3}$ the semi-stable points correspond to the homogeneous polynomials with three distincts roots. If $\Delta=0$ is the equation of the discriminant of the generic binary cubic $a_0X^3+a_1X^2T+a_2XT^2+a_3T^3$ we have $(\mathbb{P}^3)^{ss}=\mathbb{P}^3\setminus\{\Delta=0\}$, where the set of points satisfying $\Delta=0$ is a quartic surface in $\mathbb{P}^3$. There is a nice interpretation of this surface : it is the surface given by the tangent of the normal rational cubic image of $\mathbb{P}^1$ by the Veronese imbedding. The semi-stable points coincide with the stable points. Since $SL(U)$ acts transitively on the set of triplets in $\mathbb{P}^1$, the quotient is, once again, a point. We can also describe the algebra of the invariants $\mathbb{C}[S^3U^\vee]^{SL(U)}=\mathbb{C}[\Delta]$.

For $\mathbf{n=4}$, the situation becomes more complicated (then more funny). Before to consider this case we prefer to recall some facts about the cross ratio.

3.1.2 Cross Ratio

Let z_1, z_2, z_3 and z_4 be four distinct points in $\mathbb{P}^1$. Let $\sigma \in \mathrm{PGL}(2, \mathbb{C})$ be the homography such that $\sigma.(z_1, z_2, z_3) = (0, \infty, 1)$. The remaining point will be sent to the point $\lambda = \sigma.z_4$ where λ is the cross-ratio

$$\lambda(z_1, z_2, z_3, z_4) = \frac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_4)(z_2 - z_3)}$$

Remark that $\lambda(\infty, 0, 1, z) = z$. By its construction the cross ratio is $SL(2)$-invariant on the ordered 4-uples of points. Permuting the four points has the effect of changing the cross-ratio from λ to either $1 - \lambda$, $\frac{1}{\lambda}$, $\frac{1}{1-\lambda}$, $\frac{\lambda-1}{\lambda}$ or $\frac{\lambda}{\lambda-1}$. A quick way to see that is through the Plücker relation

$$(z_1 - z_2)(z_3 - z_4) - (z_1 - z_3)(z_2 - z_4) + (z_1 - z_4)(z_2 - z_3) = 0$$

Thus, two (not ordered) 4-uples can be carried into each other if and only if the subsets

$$\{\lambda, 1 - \lambda, \frac{1}{\lambda}, \frac{1}{1 - \lambda}, \frac{\lambda - 1}{\lambda}, \frac{\lambda}{\lambda - 1}\} \subset \mathbb{C} \setminus \{0, 1\} \tag{3.1}$$

coincide. To characterize when this is the case, we introduce the celebrated j-function

$$j(\lambda) = 2^8 \frac{(\lambda^2 - \lambda + 1)^3}{\lambda^2(\lambda - 1)^2} \tag{3.2}$$

3.1. Exercise. *Show that two subsets* $\{\infty, 0, 1, \lambda\}$ *and* $\{\infty, 0, 1, \lambda'\}$ *are* $SL(2)$*-equivalent if and only if* $j(\lambda) = j(\lambda')$.

It follows from the previous exercise that we have one orbit (in $\mathbb{P}(S^4U)$) for each value of $j \in \mathbb{C}$.

3.1.3 Binary forms of degree ≥ 4

For $\mathbf{n = 4}$, the stable points correspond by Proposition 3.1 to homogeneous polynomials of degree 4 without multiple root, i.e. outside the threefold in $\mathbb{P}^4$ defined by the discriminant. There are exactly two orbits of strictly semi-stable points :

- $D_1 = (X^2T^2)$ i.e. 2 double zeroes.
- $D_2 = X^2(X + T)(X - T)$ i.e. 1 double zero.

We can see (exercise 3.4) that the first one is of dimension 2 (2 points on $\mathbb{P}^1$), the second one is of dimension 3 (3 points on $\mathbb{P}^1$) and that the first orbit is closed and contained in the closure of the second orbit. Geometrically D_2 is the union of osculating 2-planes to the quartic rational curve, D_1 is the union of intersection points of two osculating 2-planes.

Let U be the open set in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ of 4-uples of distincts points. We have just seen that $U/\mathfrak{S}_4 = (\mathbb{P}^4)^s$. Consider the map

$$U \to \mathbb{P}^1 \setminus \{\infty, 0, 1\}$$

which associates to (x, y, z, t) the cross-ratio $[x, y, z, t]$. It gives an action of $\mathfrak{S}_4$ on $\mathbb{P}^1 \setminus \{\infty, 0, 1\}$. The orbit of λ according to this action was described in (3.1). The normal subgroup $\mathbb{Z}_2 \times \mathbb{Z}_2$ of $\mathfrak{S}_4$ acts trivially on $\mathbb{P}^1 \setminus \{\infty, 0, 1\}$ and we obtain an action of the finite group isomorphic to $\mathfrak{S}_3$ generated by the automorphism $z \mapsto \frac{1}{z}$ and $z \mapsto 1 - z$. This action has a geometric quotient, given by the map $j : \mathbb{P}^1 \setminus \{\infty, 0, 1\} \to \mathbb{C}$ (see (3.2)). This invariant is the usual invariant for elliptic curves (Hasse's invariant). This is a geometric quotient by the action of $SL(2)$ (fibres and orbits coincide), and the action of $\mathfrak{S}_4$ commutes with the action of $SL(2)$. We obtain a $SL(2)$-invariant morphism from $(\mathbb{P}^4)^s$ to $\mathbb{C}$ by composition of the cross ratio with j, which can be extended in a map $\pi : (\mathbb{P}^4)^{ss} \to \mathbb{P}^1$ by sending the strictly semi-stable orbit on ∞. We deduce that the morphism $(\mathbb{P}^4)^{ss}/SL(2) \to \mathbb{P}^1$ is birational and it is a isomorphism.

3.2. Exercise. *(i) Show that the closure in $\mathbb{P}^4$ of the orbit $\pi^{-1}(z)$ is a threefold of degree 6 except $\pi^{-1}(1728)$ ($z = j(-1)$) which is a threefold of degree 3 (harmonic) and $\pi^{-1}(0)$ ($z = j(w)$, w root of $w^2 - w + 1$) which is a threefold of degree 2 (anarmonic).*
(ii) By considering the threefold of lines bisecant to the rational quartic give the equation of the Zariski closure of $\pi^{-1}(1728)$.

Hint : $\det \begin{pmatrix} X_0 & X_1 & X_2 \\ X_1 & X_2 & X_3 \\ X_2 & X_3 & X_4 \end{pmatrix} = 0$ is the wanted equation.

3.2. Remark. *The algebraic counterpart of the exercise 3.2 is that $f = a_0x^4 + 4a_1x^3y + 6a_2x^2y^2 + 4a_3xy^3 + a_4y^4$ is the sum of two 4-powers if and only if*

$$J := \det \begin{vmatrix} a_0 & a_1 & a_2 \\ a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \end{vmatrix} = 0$$

3.3. Exercise. *(i) Given generic $t_1, t_2 \in \mathbb{C}$, the equation $(x - t_1)(x - t_2) = \lambda_1(x - t_1)^2 + \lambda_2(x - t_2)^2$ has no solution in the unknowns λ_i, because for $x = t_i$ you get $\lambda_i = 0$.*
(ii) Given any $t_1, t_2, t_3 \in \mathbb{C}$ prove that the equation $(x - t_1)(x - t_2)(x - t_3) = \lambda_1(x - t_1)^3 + \lambda_2(x - t_2)^3 + \lambda_3(x - t_3)^3$ has a solution in the unknowns λ_i.

The correct generalization of the exercise 3.3 is the exercise 3.5 (iii).

Let U be a complex vector space of dimension 2 and consider the projective space $\mathbb{P}(S^nU)$ of hyperplanes in S^nU. The rational normal curve C_n is described by $< u^{\otimes n} >$, where $u \in U^*$.
S^nU is the space of homogeneous polynomials of degree n and C_n corresponds to the polynomials with a single root of multiplicity n.
In the projective space $\mathbb{P}(S^nU)$ it is customary to identify the class $[f]$ of a polynomial with the polynomial itself.

3.3. Definition. *Given a smooth point x of a curve C, by the implicit function theorem there is a C^∞-map $f : U \to \mathbb{P}^n$ from an open set $U \subset \mathbb{C}$ containing the origin parametrizing locally the curve. Let us assume that $f(0) = x$, the k-th osculating space at x is the span of the points $f(0)$, $f^{(1)}(0)$, ..., $f^{(k)}(0)$ and it is defined when these points are independent. A point where the k-th osculating space is not defined for some $k < n$ is called inflectionary point. Any curve not contained in a hyperplane has only finitely many inflectionary points.*

3.4. Proposition. *The i-th osculating space at the point $< u^{\otimes n} >$ corresponds to the polynomials which are divisible by $< u^{\otimes n-i} >$ and we denote it by $T^i_{u^n}$. In particular T^1_p is the tangent line at p and T^{n-1}_p is the osculating hyperplane at p.*

Proof. By the group action it is enough to compute the spaces in a neighborhood of x^n which correspond to $t = 0$ in the parametrization $(1, t, t^2, \dots, t^n)$. Then the space where the first i derivatives vanish at zero consists of points with the last $n-i$ coordinates vanish. These points correspond to polynomials which are multiple of x^{n-i}. $\square$

We underline that in this correspondence, every polynomial f in the line $< g, h >$ joining the polynomials g and h is a multiple of $GCD(g, h)$.

3.4. Exercise. *In $\mathbb{P}^4$ for every pair of points x^4 and y^4 in the rational normal curve C, show that the corresponding osculating planes meet in the point x^2y^2. Prove that the locus filled by this point is a orbit S. Prove that its closure is $S \cup C$, and it is a smooth surface of degree 4, which is isomorphic to the projective plane, and it is called the Veronese surface in $\mathbb{P}^4$. In fact it is the projection of the Veronese surface in $\mathbb{P}^5$ through a point. A theorem of Severi in 1901 (written at the age of 22) shows that it is the unique surface in $\mathbb{P}^5$ which projects from a point to a smooth surface in $\mathbb{P}^4$.*

3.5. Remark. *Zak has classified all varieties of dimension $\frac{2}{3}(n-1)$ in $\mathbb{P}^n$ that are not linearly normal, that is that are projection from a external point of a variety in a higher dimensional projective space. There are 4 such examples, and the Veronese surface of the previous exercise is the first one. For details see [LVdV]. Zak proof has been recently simplified by Chaput. There is a fascinating link of these 4 varieties with the 4 real division algebras $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, $\mathbb{O}$ [Chap].*

3.5. Exercise. *A generic hyperplane $H \subset \mathbb{P}(S^nU)$ meets the rational normal curve C in n distinct points $x_1, \dots, x_n$. We get a function $H \mapsto \cap_{i=1}^n T^{n-1}_{x_i} C$.*
(i) Prove that this function extends to a morphism $\mathbb{P}(S^nU)^* \to \mathbb{P}(S^nU)$ in the following way, if $H \cap C$ is $\sum_{i=1}^m n_i x_i$ then H goes to $\cap_{i=1}^m T^{n-i}_{x_i} C$.
(ii) For $n = 1$ the corresponding map $U \to U^*$ is given by the contraction by $\wedge^2 U^*$. For general n the linear map $S^nU \to S^nU^*$ is the n-symmetric power of this one.
(iii) Prove that the morphism $\mathbb{P}(S^nU)^* \to \mathbb{P}(S^nU)$ of part (i) is a linear projective transformation which is symmetric if n is even and skew-symmetric if n is odd. In particular $\cap_{i=1}^n T_{x_i} C \in < x_1, \dots, x_n >$ if n is odd, while in the case n even the locus $\{H \in \mathbb{P}(S^nU)^* | H \supset T^{n-1}_p C \quad \forall p \in H \cap C\}$ is a smooth quadric. Formally when $H \cap C$ contains points with multiplicities i, one has to take the corresponding osculating space of order $n - i$.
(iv) Prove that the inverse function $\mathbb{P}(S^nU) \to \mathbb{P}(S^nU)^*$ comes from the function

$$f \mapsto H = \{\textit{span of roots of } f \ \ \textit{counted with their multiplicity}\}$$

The previous exercise has the following interpretation in terms of representations: $\wedge^2(S^nU)$ contains a summand of rank 1 iff n is odd, $S^2(S^nU)$ contains a summand of rank 1 iff n is even . In particular every rational normal curve of even degree determines a unique smooth quadric containing it.

3.6. Exercise. Castelnuovo, 1891 *From the Veronese surface S in $\mathbb{P}^4$ (see 3.4 it is possible to reconstruct the rational normal quartic C. In fact $\forall x \in S$, the trisecants to S passing through x lie in a plane π_x, called singular plane. Then C can be obtained in one of the two following ways*
i) $C = \{x \in S | \dim \pi_x \cap T_x S \geq 1\}$ (in general $\pi_x \cap T_x S = \{x\}$)
ii) π_x meets S in x and in a conic C_x. Then $C = \{x \in S | x \in C_x\}$.

The Hessian of a polynomial $f(x, y) \in S^n(U)$ is by definition

$$H(x, y) = f_{xx} f_{yy} - f_{xy}^2 \in S^{2n-4}U$$

3.7. Exercise. *(i) Prove that $H(x, y) = 0$ gives a system of quadrics which defines as scheme the rational normal curve C. For $n \leq 3$ these quadrics generate the homogeneous ideal of quadrics containing C, but for $n \geq 4$ they are too few.*
(ii) Prove that the quadrics given by the 2×2 minors of the matrix

$$\begin{bmatrix} a_0 & a_1 & \cdots & a_{n-1} \\ a_1 & a_2 & \cdots & a_n \end{bmatrix}$$

generate the homogeneous ideal of C.

3.6. Lemma. *The hypersurface $R(a_0, a_1, \ldots, a_n) = 0$ given by the discriminant of $f(x, y) = \sum_{i=0}^n a_i \binom{n}{i} x^i y^{n-i}$ corresponds to the union of $T_{u^n}^{n-2} \simeq \mathbb{P}^{n-2}$.*

Proof. Every T^{n-2} contains polynomials with a double root. Conversely if f has a double root it lies in a T^{n-2}. $\square$

This lemma has the following generalization

3.8. Exercise. *Prove that the equations of the varieties $\cup_{u \in U} T_{u^n}^{n-i}$ give necessary and sufficient in order that $f \in S^n U$ has a root of multiplicity i. Compute explicitly the variety of polynomial of fourth degree with a double root.*

3.9. Exercise. Enriques-Fano *Consider the closure of the orbit of $f \in \mathbb{P}(S^n U)$ under the action of $SL(U)$, call it X_f. Prove that*
i) X_f is never a point.
ii) X_f is a curve iff it is the rational normal curve (of degree n) and $f(x, y) = x^n$ (in a suitable system of coordinates).
iii) X_f is a surface iff $f(x, y) = x^i y^{n-i}$ for some $0 < i < n$ (in a suitable system of coordinates). The degree of X_f is $2i(n-i)$ if $2i \neq n$ and i^2 if $2i = n$. This case is the only one where X_f is smooth (projection of the quadratic Veronese embedding)
iv) In the case not covered by ii) and iii) X_f is a threefold of degree $\frac{n(n-1)(n-2)}{|\Gamma_f|}$ where $\Gamma_f = \{g \in PGL(2) | g \cdot f = f\}$. When $n = 4$ and the 4 roots of f are distinct then Γ_f is $\mathbf{Z}_2 \oplus \mathbf{Z}_2$ with the only two exceptions of the harmonic case ($f = x^4 + y^4$) where Γ_f is dihedral of order 8, which is the simmetry group of a square, and of the anarmonic case ($f = x^4 + xy^3$) described in the next remark.

3.7. Remark. *The item iv) of the previous exercise has been analyzed by [Aluffi-Faber]. They prove that the threefold X_f is smooth only in the four following cases*
i) $\mathbf{n=3}$ $X_f = \mathbb{P}^3$ $\quad f = x^3+y^3$ *(Fano of index* 4*) and* Γ_f *is the dihedral group of order* 6 *corresponding to the isometries of a regular triangle.*
ii) $\mathbf{n=4}$ $X_f = Q_3$*(smooth quadric in* $\mathbb{P}^4$ $\quad f = x(x^3+y^3)$ *(anarmonic) (Fano of index* 3*) and* Γ_f *is the tetrahedral group of order* 12 *isomorphic to* A_4*. Remark that we have a central extension*

$$1 \to \mathbf{Z}_2 \to A_4 \xrightarrow{f} \mathfrak{S}_3 \to 1$$

(in fact the centrum of A_4 *is given by* $\mathbf{Z}_2 \times \mathbf{Z}_2$*) where* f *acts by permutation of the three medians which join the medium points of the opposite edges of the tetrahedron.* A_4 *can be seen as binary dihedral corresponding to the regular triangle.*
iii) $\mathbf{n=6}$ $X_f = Gr(\mathbb{P}^1, \mathbb{P}^4) \cap \mathbb{P}^6$ $\quad f = xy(x^4-y^4)$ *(Fano of index* 2*) and* Γ_f *is the octahedral group of order* 24 *isomorphic to* $\mathfrak{S}_4$*)*
iv) $\mathbf{n=12}$ X_f *has deg* 22 *in* $\mathbb{P}^{12}$ $\quad f = xy(x^{10}+11x^5y^5 - xy^{10})$ *(Fano of index* 1 *found by Mori, the genus is* 12 *and* $\Gamma_f = A_5$*=icosahedral group of order* 60*)*
The example iv) escaped classical list of Fano threefolds. One of its more intriguing properties is that its Hilbert scheme of lines is not reduced. Examples i)-iv) give compactifications of $\mathbb{C}^3$*, see also [Mukai].*

***pictures? Say more about finite subgroups of $SL(2)$ and regular polyhedra.

3.8. Remark. *By pullback with the* $2:1$ *covering* $SL(2) \longrightarrow PGL(2)$ *we get the so called binary polyhedral groups.*

3.2 Hypersurfaces, ternary forms

3.10. Exercise. *Consider the action of* $SL(V)$ *(change of variables) on the space* $\mathbb{P}(S^2V)$ *of quadrics in* $\mathbb{P}(V)$*. Find a natural invariant and give its equation. Give the number of orbits and show that only one orbit is closed. By studying the stabilizer of semi-stable point show that there is no stable point for* $n \geq 3$*.*

We consider now a vector space V of dimension 3 over $\mathbb{C}$ and we consider the action of $SL(V)$ on the space of cubic forms $S^3V^\vee = \mathbb{C}[x,y,z]_3$, by change of coordinates. Let $f \in S^3V^\vee$ be a cubic curve in $\mathbb{P}^2 = \mathbb{P}(V)$. Let λ a one parameter subgroup of $SL(V)$. We can write it

$$\lambda(t) = \begin{pmatrix} t^a & 0 & 0 \\ 0 & t^b & 0 \\ 0 & 0 & t^c \end{pmatrix}$$

where a, b, c are integers such that $a+b+c=0$ and $a \geq b \geq c$. We deduce that the vector space of cubic forms f such that $\mathrm{Lim}_{t\to 0}\lambda(t).f = 0$ is contained in the vector space generated by $x^3, x^2y, xy^2, y^3, x^2z$ (these two vector spaces are equal when $(a,b,c) = (2,1,-3)$). Then every unstable f can be written with convenient coordinates in the following form :

$$f(x,y,z) = ax^3 + bx^2y + cxy^2 + dy^3 + ex^2z$$

So we can understand what is a unstable plane curve of degree 3. By computing the partial derivatives we see that $(0,0,1)$ is a singular point, but not an ordinary singular point. More precisely a cubic curve is unstable if it has a triple point or a cusp. This happens in the following three cases. **pictures

• If $de \neq 0$ then the curve is a cuspidal curve (the cusp is the point $(0,0,1)$)

• If $d = 0$ and $ce \neq 0$ then the curve is the union of the conic $ax^2 + bxy + cy^2 + exz = 0$ and its tangent $x = 0$.

• If $c = d = 0$ or if $e = 0$ then the curve is the union of three lines with a common point.

This proves that the cubic curves which are smooth (i.e. outside the discriminant locus) or with an ordinary singular point (nodal curves) are semi-stable. We have a good quotient

$$\mathbb{P}(S^3V^\vee)^{ss} \to \mathbb{P}(S^3V^\vee)^{ss}/SL(V)$$

Since the dimension of $\mathbb{P}(S^3V^\vee)^{ss}$ is 9 the quotient is a normal unirational curve so it is $\mathbb{P}^1$. Remember that each fiber contains a closed orbit. When the cubic is smooth its stabilizer is finite, so its orbit is 8 dimensional and moreover the orbit is closed because if not you could find in the closure a smaller one. **this is false!**

When the cubic is nodal we have three cases :

• irreducible nodal curve, $xyz + y^3 + x^3 = 0$

• union of an irreducible conic and a line cutting the conic in two distincts points, $xyz + y^3 = 0$

• three non concurrent lines, $xyz = 0$

The cases 2 and 3 correspond to strictly semi-stable curves because the dimension of their stabilizer is ≥ 1, for example the one parameter subgroup $\lambda(t) = (t, 1, t^{-1})$ stabilizes the two curves. We can also see that the orbit of $xyz = 0$ is of dimension 6 and it is contained in the closure of the orbit (dimension 7) of $xyz + \epsilon y^3 = 0$. This latter is also contained in the closure of the orbit $xyz + \epsilon y^3 + \zeta x^3$. These three orbits are in the same fiber, the closed orbit is the smallest one. Since there is only one closed semi-stable but not stable orbit, namely the set of three non concurrent lines, we obtain

$$\mathbb{P}(S^3V^\vee)^s/SL(V) \simeq \mathbb{A}^1$$

3.11. Exercise. *Describe the unstable orbits, compute their dimension.*

3.12. Exercise. Clebsch quartics, 1865 *S^4V is a direct summand of $S^2(S^2V)$. In coordinates this means that any $f \in S^4V$ can be written as*

$$f(x,y,z) = W^t \cdot C_f \cdot W \tag{3.3}$$

where $W = (x^2, 2xy, 2xz, y^2, 2yz, z^2)$ and C_f is a (symmetric) 6×6 matrix.
(i) show that there are infinitely many symmetric matrices $\tilde{C}_f$ such that (3.3) holds
(ii) show that among the symmetric matrices $\tilde{C}_f$ such that (3.3) holds there is a unique one C_f characterized by the property $C_{(px+qy+rz)^4} = V \cdot V^t$ where $V = (p^2, pq, pr, q^2, qr, r^2)$. Prove that the rank of C_f is $SL(V)$-invariant.
(iii) write explicitly the first entries of C_f in terms of the coefficients of f.
(iv) prove that $rkC_f = 1$ if and only f is the fourth power of a linear polynomial.

(v) prove that $\det C_f = 0$ *if and only if* f *is the sum of five fourth powers. This shows that the* 5*-secant variety of the quartic Veronese embedding of* $\mathbb{P}^2$ *is a hypersurface of degree* 6 *in* $\mathbb{P}(S^4V) = \mathbb{P}^{14}$*. Remark that the freshman (and wrong!) numerical postulation gives that every* f *is the sum of five fourth powers. In fact* $4 + 5 \cdot 2 = 14$*. Alexander and Hirschowitz [AH] classified all the few special cases analogous to this one(see [IK]).*
(vi) prove that $xz(xz + y^2)$ *is not the sum of five* 4*-powers.*

3.13. Exercise. *(Dolgachev) Prove that any non-singular hypersurface of degree* d *in* $\mathbb{P}^n$ *is a semi-stable point for the linear action of* $SL(n+1)$ *on* $\mathbb{P}^n$*. If* $d \geq n+1$*, prove that any non singular hypersurface of degree* d *is a stable point under the action of* $SL(n+1)$ *(because the group of automorphism is finite in this case).*

3.3 Action of $\mathbf{PGL}(U) = \mathbf{PGL}(2)$ on $\mathbb{P}(S^2(S^2U))$

***make pictures of the several cases, orbits and their relation**
Consider the following imbeddings

$$\mathbb{P}(U) \overset{v}{\hookrightarrow} \mathbb{P}(S^2U) \overset{i}{\hookrightarrow} \mathbb{P}(S^2(S^2U))$$

We denote by C_2 the conic $C_2 = v(\mathbb{P}(U))$ and by C_4 the quartic curve $i(C_2)$. The space $\mathbb{P}(S^2(S^2U))$ is the space of conic curves of $\mathbb{P}(S^2U)$. The canonical decomposition

$$S^2S^2U = S^4U \oplus \mathbb{C}$$

shows that there is a fixed point in this space which is the conic C_2 and also an invariant hyperplane $\mathbb{P}(S^4U)$ which is generated by the quartic rational normal curve C_4. This curve is the curve of double lines tangent to C_2. The invariant varieties D_1 and D_2 defined at the beginning of the subsection 3.1.3. can be interpreted as the locus of intersection of respectivly the bitangent and tangent conics to C_2 with the hyperplane generated by C_4 (Observe that the unstable points of $\mathbb{P}^4$ are the points of the rational curve (C_4) and of the surface of tangent lines to the rational quartic. We can see this curve and this surface as the intersection of the hyperplane with the surface of surosculating conics and with the threefold of osculating conics.)

Now since a general conic C meets C_2 in four distinct points, the orbit $\mathrm{PGL}(U).C$ is linked to the cross-ratio of these four points. In the space of conics there is a natural invariant which is the set of degenerated conics. As we have seen before this is a cubic hypersurface, since in the pencil (C, C_2) there is exactly three degenerated conics. We would like to study the action of $\mathrm{PGL}(U)$ on this invariant hypersurface.

3.3.1 Action of $\mathbf{PGL}(2)$ on the degenerated conics

Let C a degenerated conic meeting C_2 in four distinct points. Two fourtuples of points on C_2 have same cross-ratio λ, $\frac{1}{\lambda}$, $1-\lambda$,$\frac{1}{1-\lambda}$, $\frac{\lambda}{\lambda-1}$ and $\frac{\lambda-1}{\lambda}$ if and only if they are equivalent under $\mathrm{SL}_2(\mathbb{C})$. Then we deduce that two degenerated conics associated to the cross ratio $\{\lambda, \frac{1}{\lambda}\}$ with $\lambda \in \mathbb{C}-\{0,1\}$ live in the same $\mathrm{SL}_2(\mathbb{C})$-orbit (Indeed if we have $[z_1, z_2, z_3, z_4] = \lambda$ then $[z_2, z_1, z_3, z_4] = \frac{1}{\lambda}$). Let C_λ a representative of this orbit. The function $c(\lambda) = (\frac{\lambda+1}{\lambda-1})^2$

is invariant by the transformation $\lambda \mapsto \frac{1}{\lambda}$. We associate, by this way, the complex number $c(\lambda) \neq 1$ to the conic C_λ. We denote by $\Omega_{c(\lambda)}$ the $\mathrm{SL}_2(\mathbb{C})$-orbit of C_λ.

The complex number $u = 0$ if and only if the cross-ratio of the four points on C_2 is -1. In that case the two lines of C are armonically conjugated i.e. C belongs to the pencil of conics (double line) (l^2, d^2) where l and d are tangent to C_2. This remark proves that $\bar{\Omega}_0 = \mathrm{Sec}(i(C_2))$.

Chapter 4

Poncelet porism

4.1 Fregier's involution and Pascal theorem

Let $D \subset \mathbb{P}^2$ be a smooth conic and x be a point in $\mathbb{P}^2$ such that $x \notin D$. A general line passing through x meets D in two distinct points. The homography $u \in \mathrm{Aut}(D) = \mathrm{PGL}(2, \mathbb{C})$ which permute these two points is an involution called Fregier's involution. The point x is called the center of the involution, and the two tangents to D coming from x give the fixed points of u.

4.1. Exercise. *Show that every involution on D is a Fregier's involution.*

4.1. Proposition. *Let u and v be two involutions with distincts fixed points. Then, (uv) is involutive if and only if the two fixed ponts of u and the two fixed ponts of v form an armonic 4-ple on D.*

Proof. Let (x_1, x_2) be the fixed points of u and (y_1, y_2) be the fixed points of v.

Assume that uv is involutive. Since $uv = vu$ we can see that v (resp. u) exchanges the fixed points of u (resp. of v). The cross ratio does not change by an homography (3.1.2) then we have $[x_1, x_2, y_1, y_2] = [u(x_1), u(x_2), u(y_1), u(y_2)]$. By the remark above it means that $[x_1, x_2, y_1, y_2] = [x_1, x_2, y_2, y_1]$, and this equality implies that $[x_1, x_2, y_1, y_2] = -1$.

Conversely, since $[x_1, x_2, y_1, y_2] = -1$ we have the following relations

$$[x_1, x_2, y_1, y_2] = [x_1, x_2, y_2, y_1] = [x_2, x_1, y_1, y_2]$$

This means that v (resp. u) exchanges the fixed points of u (resp. of v) because an involution is defined by its fixed points. Now we have $uv(x_1) = x_2, uv(x_2) = x_1, uv(y_1) = y_2, uv(y_2) = y_1$ and these four points are all fixed points for $(uv)^2$. Since an homography with three fixed points is the identity, the proposition is proved. □

4.2. Proposition. *Let u, v, and w be three involutions with distincts fixed points, x_u, x_v and x_w be their respective centers. Then, $(uvw)^2 = id_D \Leftrightarrow x_u, x_v$ and x_w are aligned.*

Proof. Assume that the three centers are aligned and let's call L this line. The line L is not tangent to D because the three involutions do not have a common fixed point. Then let $\{x, y\} = L \cap D$. We verify that $uvw(x) = y$ and $uvw(x) = y$. Let $z \in D$ be a fixed point of uvw. Now the three points x, y and z are fixed points for uvw. It means that $(uvw)^2 = id_D$.

Conversely, assume that uvw is involutive. Let $x \in D$ such that $v(x) = w(x)$. Then x is a fixed point of the two homographies vw and wv. Of course $v(x)$ is also a fixed point for vw because $vw(v(x)) = vw(w(x)) = v(x)$. We have found two fixed points for vw. We want now to prove that $u(x)$ is also a fixed point to vw. Assume for a while that $u(x) \neq x$. By hypothesis we have $(uvw)(uvw)(x) = x$. Since $vw(x) = x$ we find $uvwu(x) = x$, then $u^2vwu(x) = u(x)$ or $vw(u(x)) = u(x)$. Since $u(x) \neq x$ and $vw \neq id_D$ this proves that $u(x) = v(x) = w(x)$ i.e. that x_u, x_v and x_w are aligned.

It remains to verify that $u(x) \neq x$. An involution is defined by its two fixed points, hence uvw and u cannot have the same fixed points. So if $u(x) = x$, we have $u(v(x)) = z$ with $z \neq v(x)$. But $uvw(w(x)) = uv(x) = z$ and $wvu(w(x)) = wv(z) \neq z$ which contradicts $uvw = wvu$. □

4.3. Corollary. *(Pascal's theorem) Let $p_1, p_2, p_3, q_3, q_2, q_1$ be six (ordered) points on a smooth conic D. Let $x_{ij}, i < j$ the point of intersection of the two lines joining p_i to q_j and p_j to q_i. Then the three points x_{12}, x_{13} and x_{23} are aligned.*

Proof. We denote by, u the involution defined by x_{12}, v the one defined by x_{23} and w the last one defined by x_{13}. Then by following lines you verify that

$$(uvw)(p_1) = q_1, (uvw)(q_1) = p_1.$$

Let z a fixed point of uvw. Then z, p_1, q_1 are fixed points of $(uvw)^2$. Since an element of $\mathrm{PGL}(2, \mathbb{C})$ which posses more than three fixed point is the identity, we have proved that $(uvw)^2 = id_D$. The result now follows from Proposition 4.2. □

4.2 Poncelet porism

4.4. Definition. *1) A true n-gone is the union of n-distinct lines. A true n-gone has $\binom{n}{2}$ vertices.*
2) A true n-gone is circumscribed to a smooth conic D if all its lines are tangent to D.
3) A true n-gone is inscribed into a conic C (even a singular one) if at least n of its vertices belong to C.
4) A conic C (even singular) is n-circumscribed to a smooth conic D if there exist a true n-gone circumscribed to D and inscribed in C.
5) A conic C (even singular) is strictly n-circumscribed to a smooth conic D if C is n-circumscribed to D and C is not m-circumscribed to D with $m < n$.

When the two conics C and D are smooth then we will sometimes write that C is n-inscribed into D when the dual conic C^* is n-circumscribed to the dual conic D^*. We will say that a homography $f \in \mathrm{Aut}D$ is of order n if and only if $f^n = id_D$ and $f^{n-1} \neq id_D$.

4.2.1 Smooth case

We will begin this section by a short review of old results which is an average of the papers [BB], [BKOR], [GH1].

Let C and D two smooth conics such that C meets D in four distinct points and C is n-circumscribed to D. Poncelet has showed the following theorem (called 'grand théorème de Poncelet')

4.5. Theorem. *If C is n-circumscribed to D then any general point of C is a vertex of a true n-gone inscribed in C and circumscribed to D.*

Proof. (from Griffiths and Harris's proof) Consider the incidence curve $E \subset C \times D^*$, where $E = \{(x, l), x \in l\}$. This curve is a smooth elliptic curve .

4.2. Exercise. *Prove it.*

Then one can define two involutions on E. Indeed let $(x, l) \in C \times D^*$. The line l cuts C in an other point x'. Let l' be the second tangent to D from x. Then we have the following involutions :

$$E \xrightarrow{i_1} E, \quad (x, l) \mapsto (x', l)$$

$$E \xrightarrow{i_2} E, \quad (x, l) \mapsto (x, l')$$

Let $\mathfrak{o}$ be the origin of E for the group law $+$. Then there exists $a \in E$ and $b \in E$ such that $i_1(z) = -z + a$ and $i_2(z) = -z + b$. It follows that the product $i_2 i_1$ is a translation on E, more precisely $i_2 i_1(z) = z + (b - a)$. Then the polygone closes after n steps if and only if $n.(b - a) = \mathfrak{o}$. It means that C is n-circumscribed to D if and only if $(b - a)$ is a n-torsion point on E. This does not depend on the choice of the beginning vertex, but only on the conics C and D. $\square$

Remark. If we begin the construction from a vertex x by drawing the second tangent to D then $(b - a)$ becomes $(a - b)$ which does not change its nature (it is still a n torsion point).

Cayley showed that the set of n-circumscribed conics to D is an hypersurface in $\mathbb{P}^5 = \mathbb{P}(H^0(O_{\mathbb{P}^2}(2)))$. This is well explained in a modern way by Griffiths and Harris. We will denote this hypersurface by $\mathcal{C}_n$. Of course $\mathcal{C}_n$ is not irreducible in general. In the set of n-circumscribed conics you will find the r-circumscribed conics with $r \geq 3$ and $r \mid n$ (draw an example for the case $n = 6$, $r = 3$). Thus this justifies to introduce an other notation M_n for strictly n-circumscribed conics to D.

We explain now briefly how to compute the degrees of these hypersurfaces, following the article of Barth and Bauer.

We denote by $T(n)$ the number of n-primitive torsion points of E. Barth and Bauer show that the the line (in the conic projective space) $C_{\lambda,\mu}$ contains $\frac{T(n)}{4}$ conics *strictly* n-circumscribed to D ([BB], prop **). The point p is determined by one conic in the pencil generated by C and D. The same conic gives the point $-p$. Then in the pencil we find $\frac{T(n)}{2}$ conics inscribed in C (remind that C is fixed and the ramification conic moves in the pencil). To find the number the degree of n-circumscribed conics we have to dualize. Since the Gauss map for conics $\mathbb{P}^5 - - \rightarrow \mathbb{P}^{5*}$ is a quadratic map (defined by the 2-minors of the generic symmetric 3×3-matrix) the image of the line is a conic which meets the hypersurface M_n along $\frac{T(n)}{2}$ smooth points. It follows that $\deg M_n = \frac{T(n)}{4}$.

It results immediatly from their proof that M_n is reduced and that the degree of $\mathcal{C}_n$ is the sum of the degrees of the hypersurfaces M_r for $3 \leq r < n$ and $r \mid n$. Since the number of n-torsion points on E is n^2, the degree of $\mathcal{C}_n$ is $\frac{n^2-4}{4}$ when n is even (we remove the 2-torsion points), $\frac{n^2-1}{4}$ when n is odd (we remove the origine).

Remark. If $\Gamma \in M_n \cap M_m$ for $m \neq n$ then Γ is a degenerated conic. This is an immediate consequence of the theorem 4.5. It follows that $\mathcal{C}_n = \bigcup_{r\geq 3, r|n} M_r$ is reduced.

The cases of tangency, in other words, when the conic is tangent, bitangent, an osculating or an surosculating to D, are studied (for smooth conics) in [BKOR] 7.14 page 329-331. The authors show that a smooth conic C tangent or bitangent to D can be n-circumscribed to D but that it is never the case if C osculates or surosculates D. We give now two different proofs for the cases of simple tangency and bitangency. In the part concerned with jumping conics (see Section 8.7, prop. 8.19) we will prove, by an original argument, that a smooth osculating (and also surosculating) to D is never n-circumscribed to D.

• When C is tangent to D, i.e. the intersection consists of three distinct points, the curve E is a rational cubic curve with a ordinary double point denoted by (x_0, l_0). The involutions i_1 and i_2 extend to the non-singular model $\tilde{E}$. The fixed points of $\tilde{i}_1$ and of $\tilde{i}_2$ are distinct points. The fixed points of $\tilde{t}$ are the preimages of (x_0, l_0). We have Aut($\tilde{E}$)=Aut($\mathbb{P}_1$)=$SL_2(\mathbb{C})$. Since $SL_2(\mathbb{C})$ acts transitively on the set of three points of $\mathbb{P}_1$ we may assume that $(1,0)$, $(0,1)$ are the two fixed point of $\tilde{i}_1$ and that $(1,1)$ is a fixed point of $\tilde{i}_2$. Then we get

$$M(\tilde{i}_1) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix},\ M(\tilde{i}_2) = \begin{pmatrix} z & \epsilon i - z \\ z + \epsilon i & -z \end{pmatrix}$$

where $\epsilon = 1, -1$ and $z \in \mathbb{C}$. The other fixed point of $\tilde{i}_2$ is $(\epsilon i - z, \epsilon i + z)$. Then the matrix of $\tilde{t}$ is $M(\tilde{t}) = \begin{pmatrix} zi & zi+\epsilon \\ zi-\epsilon & zi \end{pmatrix}$. Since the eigenvalues of this matrix are $(z+\sqrt{z^2+1})i$ and $(z-\sqrt{z^2+1})i$, the homography $\tilde{t}$ is of order n when z verifies

$$(z+\sqrt{z^2+1})^n = (z-\sqrt{z^2+1})^n$$

or in other words when $u = \frac{z+\sqrt{z^2+1}}{z-\sqrt{z^2+1}}$ is n-root of unity.

4.3. Exercise. *By studying the incidence curve E, like above, try to prove that a smooth osculating or surosculating conic to D is never n-circumscribed to D.*

• Let C be a smooth bitangent conic to D. Without lost of generality we can assume that the points $C \cap D$ are cyclic points. In the real plane C and D are concentric circles. We can choose the equations such that C and D are given respectively by :

$$X^2 + Y^2 = R^2 \text{ and } X^2 + Y^2 = 1$$

where R is a real number greater than 1. Then C is n-circumscribed to D if the angulus between the two intersection points with C of a tangent line to D is $\frac{2k\pi}{n}$ with $k = 1, \cdots n-1$ i.e. if $R = \sqrt{\frac{1}{1-sin^2(k\pi/n)}}$.

4.2.2 Singular case****verify the commentaries, add picture!!and add a general grand thm de poncelet, including the singular case.

We explained above which smooth conics are n-circumscribed to a fixed one called D. Now we want to describe the locus of singular conics n-circumscribed to D. We will do it when they are 4 distinct points of intersection with D. Moreover we will consider only the case of singular conics $2n$-circumscribed to D. In fact, intuitively it is clear that a singular conic which intersects D in 4 points cannot be $2n+1$-circumscribed (because the two lines should play the same role) and it is also clear by drawing a picture that if the singular conic meets D in three points then there is a convergent proccess which show that any singular tangent conic is n-circumscribed for all $n > 2$. We will give a precise meaning and also a proof of these facts in Section 8.7 with the help of vector bundle techniques.

4.6. Proposition. *Let u and v two involutions on a smooth conic D, x_u and x_v the respective centers. Then the followings are equivalent*
1) the product uv is of order n.
2) the singular conic $x_u^* \cup x_v^*$ is strictly $2n$-circumscribed to D^*

Proof. If uv is of order n then the result is clear. Indeed let $x \in D$, then the points $v(x)$, $uv(x)$, $vuv(x)$, $\cdots$, $v(uv)^{n-1}(x)$ and $uv^n(x) = x$ are the vertices of an inscribed $2n$-gone into D. The dual $2n$-gone (its lines are the tangent lines to D^* which correspond to the above vertices) is circumscribed to D^* and all its vertices belong to $x_u^* \cup x_v^*$.

On the other hand, the existence of a $2n$-gone circumscribed to D^* implies the existence of a $2n$-gone inscribed into D. Let x one of its vertices. By following its sides we have $(uv)^n(x) = x$. Since $2n \geq 3$ we deduce that the product uv is of order n.

We have seen in Proposition 4.6 that a degenerated conic, say $l \cup d$, where l and d are lines such that $l \notin C_2^*$ and $d \notin C_2^*$, is $2n$-circumscribed to C_2 if and only if the product uv, of the homomorphism u and v with center l^* and d^* in C_2^*, is of order n. Since clearly these conditions are preserved by any element of $\mathrm{SL}(2,\mathbb{C})$ we would like to find the complex numbers which characterize the correspondings orbits Ω_η. This is the object of the following proposition. We denote by $\mathfrak{P}_m$ the m-primitive roots of unity.

4.7. Proposition. *Let l and d two lines of $\mathbb{P}^2$ with $l^*, d^* \notin C_2^*$, u (resp. v) the Frégier's involution on C_2^* defined by the point l^* (resp d^*). The following conditions are equivalent ($n \geq 2$) :*
i) $C = l \cup d$ is $2n$-circumscribed to C_2.
ii) the product uv is of order n
iii) $l \cup d \in \bigcup_{z \in \mathfrak{P}_{2n}} \Omega_{(\frac{1+z^2}{2z})^2}$

**** ie it depends only on the data of l, d and C_2 so it is the Poncelet Thm for singular conics **** **Proof.** We have already proved the first equivalence $i) \Leftrightarrow ii)$ in Proposition 4.6. We will prove now $ii) \Leftrightarrow iii)$.

First of all we need to prove that $l \cup d$ meets C_2 in four distinct points. Otherwise u and v defined by the points l^* and d^* have a common fixed point since we have $l \cap d \in C_2$. Moreover this common fixed point is the unique fixed point of uv. Then uv is a translation, i.e. could not be of order n. It follows that there exist $\eta \in \mathbb{C}$ such that $l \cup d \in \Omega_\eta$. Let z a complex number such that $\eta = (\frac{1+z^2}{2z})^2$. After the identification $C_2^* \simeq \mathbb{P}^1$, we can assume that $l \cap C_2 = \{(1,i),(1,-i)\}$ and $(1,iz) \in d \cap C_2$. Since the invariant associated to $l \cup d$ (see 3.3.1) is $c(\lambda) = (\frac{1+z^2}{2z})^2$ the cross-ratio of the four points $(l \cup d) \cap C_2$ is $\lambda = (\frac{1-z}{1+z})^2$ or $\lambda = (\frac{1+z}{1-z})^2$. Then the second fixed point of v is $(1,-iz)$. Thus the involutions are

$$u = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \text{ et } v = \begin{pmatrix} 0 & -\frac{1}{z} \\ z & 0 \end{pmatrix}$$

We see that the product uv is of order n if and only if $z \in \mathfrak{P}_{2n}$. $\square$

Remark If $|z| = 1$ then $(\frac{1-z}{1+z})^2 \in \mathbb{R}$, hence the four points in the complex plane lie in a circle. ***picture jean-francoise***

Chapter 5

Some remarks about bundles on $\mathbb{P}^n$

5.1 The theorem of Segre-Grothendieck

We refer to [OSS] and to the appendix 10.2 for the definition of vector bundle and spanned vector bundle.

5.1. Example. *On $\mathbb{P}^n = \mathbb{P}(V)$ with homogeneous coordinates $(x_0, \ldots, x_n)$ we have the line bundles $\mathcal{O}(t)$ that on the standard covering given by $U_i = \{x | x_i \neq 0\}$ have transition functions $g_{ij} = (\frac{x_j}{x_i})^t$. Then for $t \geq 0$ $\mathcal{O}(-t) = \mathcal{O}(-1)^{\otimes t}$ and $\mathcal{O}(t) = (\mathcal{O}(-1)^*)^{\otimes t}$. All the line bundles on $\mathbb{P}^n$ are isomorphic to $\mathcal{O}(t)$ for some integer t.*

If F is a coherent sheaf, it is usual to denote $F \otimes \mathcal{O}(t)$ by $F(t)$. For $t \geq 0$ the space $H^0(\mathbb{P}^n, \mathcal{O}(t))$ consists of all homogeneous polynomials in $(x_0, \ldots, x_n)$ of degree t, or in equivalent way $H^0(\mathbb{P}^n, \mathcal{O}(t)) \simeq S^t V$. All the intermediate cohomology of $\mathcal{O}(t)$ is zero, that is

$$H^i(\mathbb{P}^n, \mathcal{O}(t)) = 0 \quad \text{for } 0 < i < n \quad \forall t \in \mathbb{Z}$$

The zero loci of sections of $\mathcal{O}(t)$ are exactly the hypersurfaces of degree t. The zero loci of a general section of $\mathcal{O}(n_1) \oplus \ldots \oplus \mathcal{O}(n_k)$ is called a complete intersection.
It is important to underline that $Hom(\mathcal{O}(a), \mathcal{O}(b)) \simeq S^{b-a}V$ that is sheaf morphisms $\mathcal{O}(a) \to \mathcal{O}(b)$ are given in coordinates by homogeneous polynomials of degree $b - a$. In general a morphism $\oplus \mathcal{O}(a_i) \to \oplus \mathcal{O}(b_j)$ is represented by a matrix whose entries are homogeneous polynomials. As a particular case, note that any isomorphism $\mathcal{O}(a)^k \to \mathcal{O}(a)^k$ is represented by a invertible $k \times k$ matrix of constants.
On $\mathbb{P}(V)$ there is the natural action of $SL(V)$. $SL(V)$ is the universal covering of the automorphism group of $\mathbb{P}(V)$ which is $PGL(V)$. If E is a bundle over $\mathbb{P}(V)$, for any $g \in SL(V)$ we can consider the bundle g^*E.

5.2. Definition. *The group of symmetry of a bundle E in $\mathbb{P}(V)$ is its stabilizer for the $SL(V)$-action and it is denoted by $Sym(E)$. In formula*

$$Sym(E) := \{g \in SL(V) | g^*E \simeq E\}$$

5.3. Definition. *A bundle E is called homogeneous if $Sym(E) = SL(V)$. It can be shown that it is equivalent to the existence of a action of $SL(V)$ over E which lifts the natural action on $\mathbb{P}(V)$.*

5.4. Theorem. Segre-Grothendieck *Let E be a bundle on $\mathbb{P}^1$. Then E splits as the direct sum $E = \oplus \mathcal{O}(a_i)$ for some integers a_i*

Proof. The proof is by induction on the rank of E. Up to tensor E with a line bundle, we can assume that $H^0(E) \neq 0$, $H^0(E(-1)) = 0$. Then any nonzero $s \in H^0(E)$ does not vanish anywhere. In fact if $s(x) = 0$ for some $x \in \mathbb{P}^1$, pick $t \in H^0(\mathbb{P}^1, \mathcal{O}(1))$ such that $t(x) = 0$, then s/t is a nonzero section of $E(-1)$. It follows that we have an exact sequence

$$0 \longrightarrow \mathcal{O} \longrightarrow E \longrightarrow F \longrightarrow 0 \tag{5.1}$$

where F is a bundle which splits by the inductive hypothesis. Let $F = \oplus \mathcal{O}(k_i)$. The assumption $H^0(E(-1)) = 0$ and the vanishing $H^1(\mathcal{O}(-1)) = 0$ imply that $k_i \leq 0 \quad \forall i$. At this point the standard proof by Grauert and Remmert tells us that $Ext^1(F, \mathcal{O}) = H^1(F^*)$ vanishes and then the sequence splits. Due to the importance of this theorem, we offer the alternative argument of Grothendieck, which is near to the original Segre construction (although Grothendieck was not aware of it!) and does not use the property of Ext^1 of classifying extensions (although essentially it reproves it in this special case). For more historical informations, see [GO].
Apply $Hom(-, \mathcal{O})$ (i.e. dualize) to the sequence (5.1)

$$0 \longrightarrow Hom(F, \mathcal{O}) = F^* \xrightarrow{f} Hom(E, \mathcal{O}) = E^* \xrightarrow{g} Hom(\mathcal{O}, \mathcal{O}) = \mathcal{O} \longrightarrow 0 \tag{5.2}$$

The cohomology sequence associated to (5.2) is

$$H^0(E^*) \xrightarrow{H^0(g)} \mathbf{C} \longrightarrow H^1(F^*) = 0$$

where for any $s \in H^0(E^*)$ we have $H^0(g)(s) = g \cdot s$. Since $H^0(g)$ is surjective, there exists s such that $g \cdot s$ is the identity, this implies that (5.2) splits so that $E = \mathcal{O} \oplus F$ as we wanted.

□

5.1. Exercise. *Prove that the decomposition of the previous theorem is unique.*

5.2. Exercise. *Let X be a variety such that $Pic(X) = \mathbb{Z}$. Let E be a bundle(or a torsion free sheaf) of rank r on X. Let $k_0 = \min\{k | h^0(E(k)) \neq 0\}$. Prove that any nonzero $s \in H^0(E(k_0))$ vanishes in codimension at least two and at most r (it is possible that the zero locus is empty).*

5.5. Corollary. *Every bundle on $\mathbb{P}^1$ is homogeneous, i.e. it is $SL(2)$-invariant.*

The above theorem is an essential tool to study vector bundles on higher dimensional projective spaces. If E is a bundle on $\mathbb{P}^n$ and L is a line then $E_{|L}$ splits as $\oplus \mathcal{O}(a_i(L))$ with well determined integers $a_i(L)$. When L changes the integers $a_i(L)$ can change. By semicontinuity properties it is easy to check (see [OSS]) that for generic L the integers $a_i(L)$ are more balanced than for special L. The lines such that their splitting is not the generic one are called jumping lines (see chapter 8 on Barth morphism). When the integers $a_i(L)$ are the same for any line L we say that E is uniform. In particular every homogeneous bundle is uniform. The converse is not true (see exercise 5.6).

5.6. Example. *In the next chapter of this book Schwarzenberger bundles are defined. They appear on* $\mathbb{P}^n$ *in sequences*

$$0 \longrightarrow \mathcal{O}^k \longrightarrow \mathcal{O}^{k+n}(1) \longrightarrow E \longrightarrow 0$$

The Schwarzenberger bundle E *on* $\mathbb{P}^2$ *defined from the exact sequence*

$$0 \longrightarrow \mathcal{O}^2 \longrightarrow \mathcal{O}^4(1) \longrightarrow E \longrightarrow 0$$

splits as $\mathcal{O}(1) \oplus \mathcal{O}(3)$ *on lines in a conic* $C \subset \mathbb{P}^{2*}$ *and as* $\mathcal{O}(2) \oplus \mathcal{O}(2)$ *on lines in* $\mathbb{P}^{2*} \setminus C$

5.2 The Euler sequence and the tangent bundle

Let $\mathbb{P}^n = \mathbb{P}(V)$ be the projective space of one dimensional vector subspaces of the $(n+1)$-dimensional vector space V^*. Consider the incidence variety $W = \{(v, x) \in V^* \times \mathbb{P}^n | v \in x\}$ We have a fibration $W \to \mathbb{P}^n$ whose fibers are isomorphic to $\mathbb{C}$. Hence W is a line bundle on $\mathbb{P}^n$.

5.3. Exercise. *Prove that* W *is isomorphic to* $\mathcal{O}(-1)$.

Hint compute the transition functions.

We get an exact sequence

$$0 \longrightarrow \mathcal{O}(-1) \longrightarrow \mathcal{O} \otimes V^* \longrightarrow Q \longrightarrow 0$$

where Q is called the quotient bundle and has rank n.
The main basic result of the theory is the following

5.7. Theorem. $Q(1) \simeq T\mathbb{P}^n$

Proof. $GL(V^*)$ acts on $\mathbb{P}^n$. Let $x \in \mathbb{P}^n$. We have the natural map $GL(V^*) \to \mathbb{P}^n$ given by $g \mapsto gx$. The derivative computed in the origin is the surjective linear map $End(V) \to T_x\mathbb{P}^n$. Its kernel is $\{g \in End(V) | g(v) \subset < v >\}$ Hence

$$T_x\mathbb{P}^n \simeq End(V)/\{g | g(v) \subset < v >\} \simeq Hom(< v >, V/ < v >)$$

so that $T\mathbb{P}^n \simeq Hom(\mathcal{O}(-1), Q) \simeq Q(1)$. □

5.4. Exercise. *Prove the isomorphism needed in the proof, that is if* v *is a nonzero vector in* V, *the natural map* $End(V) \to Hom(< v >, V/ < v >)$ *is surjective with kernel equal to* $\{g | g(v) \subset < v >\}$

Tensoring by $\mathcal{O}(1)$ we get the Euler sequence

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}(1) \otimes V \longrightarrow T\mathbb{P}^n \longrightarrow 0 \tag{5.3}$$

It follows from the Euler sequence that $H^0T\mathbb{P}^n \simeq \mathfrak{sl}(V)$. $\mathfrak{sl}(V)$ can be interpreted as the space of $(n+1)$-matrices of trace zero. Every $A \in \mathfrak{sl}(V)$ induces $A_{|<v>}: < v > \to V/ < v >$ so that the section vanishes in $< v >$ if and only if v is a eigenvector of A. Since the generic matrix has $(n+1)$ distinct eigenvectors we get

5.8. Theorem. *The generic section of $T\mathbb{P}^n$ vanishes in $n+1$ points.*

5.9. Corollary.

$$T\mathbb{P}^1 = \mathcal{O}(2)$$

5.10. Theorem. *The tangent bundle on $\mathbb{P}^n$ splits on any line as $\mathcal{O}(1)^{n-1} \oplus \mathcal{O}(2)$.*

First proof Let l be a line. By Theorem 5.4 $T\mathbb{P}^n_{|l} \simeq \oplus_{i=1}^n \mathcal{O}(a_i)$ with $\sum_{i=1}^n a_i = c_1(T\mathbb{P}^n) = n+1$. By the Euler sequence we see that $T\mathbb{P}^n(-1)$ is globally generated, hence $a_i \geq 1$, which concludes the proof.

Second proof Let $\mathbb{P}^{n-1} = H$ be a hyperplane in $\mathbb{P}^n$. Then $Q_{\mathbb{P}^n|\mathbb{P}^{n-1}} = Q_{\mathbb{P}^{n-1}} \oplus \mathcal{O}$. Then apply Corollary 5.9.

5.5. Exercise. *Let $f\colon \mathbb{P}^2 \to \mathbb{P}^5$ be the Veronese embedding. Prove that $f^*T\mathbb{P}^5$ is homogeneous.*

5.6. Exercise. ** Let $f\colon \mathbb{P}^2 \to \mathbb{P}^4$ be obtained as a smooth projection of the Veronese embedding. Prove that $f^*T\mathbb{P}^4$ is uniform but not homogeneous.*
Hint: restrict $f^*T\mathbb{P}^4$ to conics

5.11. Remark. *For a vector space V of dimension n we denote $detV := \wedge^n V$. We recall that any linear map $\Phi \in Hom(V,W)$ between vector spaces of the same dimension induces the map $det\Phi \in Hom(detV, detW)$. If A and B are vector spaces of dimension a and b respectively, then there are canonical isomorphisms:*

$$det(A \otimes B) \simeq (detA)^{\otimes b} \otimes (detB)^{\otimes a} \qquad det(S^k A) \simeq (detA)^{\otimes \binom{a+k-1}{a}}$$

$$\wedge^k A \simeq \wedge^{a-k} A^* \otimes (detA)$$

The above isomorphisms hold also if A and B are replaced by vector bundles over a variety X.

5.7. Exercise. *We denote $\Omega^p = \wedge^p \Omega^1$. Prove that there is the following exact sequence*

$$0 \to \Omega^p(p) \to \wedge^p V^* \otimes \mathcal{O} \to \Omega^{p-1}(p) \to 0$$

Deduce that $H^0(\Omega^p(p+1)) = \wedge^p V^*$ Hint: see the appendix 10.3.

The basic exact sequence on the grassmannian

We consider the grassmannian $G = Gr(\mathbb{C}^{k+1}, V^*) = Gr(\mathbb{P}^k, \mathbb{P}(V^*))$, see the appendix. Consider the incidence variety $W = \{(v,x) \in V^* \times G | v \in x\}$. We have a fibration $W \to G$ whose fibers are isomorphic to $\mathbb{C}^{k+1}$. Hence W is a vector bundle on G of rank $k+1$ which is called the universal bundle. We get the exact sequence

$$0 \longrightarrow U \longrightarrow \mathcal{O} \otimes V^* \longrightarrow Q \longrightarrow 0$$

where the quotient bundle Q has rank $n-k$. Repeating word by word the arguments of Theorem 5.7 we get

5.12. Theorem. $TG \simeq Hom(U,Q) \simeq U^* \otimes Q$

Again we have $H^0(G) = \mathfrak{sl}(V)$. The section corresponding to the matrix $A \in \mathfrak{sl}(V)$ vanishes on the linear spaces $\mathbb{C}^{k+1}$ which are invariant by A. If A is generic with $n+1$ distinct eigenspaces, the $\mathbb{C}^{k+1}$ invariant are exactly those which are spanned by $k+1$ among these eigenspaces. We have proved the following

5.13. Theorem. *The generic section of $TGr(\mathbb{C}^{k+1}, V^*)$ vanishes at $\binom{n+1}{k+1}$ points*

Let E be a bundle of rank r on X. $\mathbb{P}(E) \xrightarrow{p} X$ is the projective bundle with fiber isomorphic to $\mathbb{P}^{r-1}$ (see [Ha] II.7). There are a canonical line bundle called $O_{\mathbb{P}(E)}(1)$ which restricts to every fiber as $\mathcal{O}(1)$ and a relative Euler sequence

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}(1) \otimes p^*E^* \longrightarrow T_{rel} \longrightarrow 0$$

It can be proved that $H^*(\mathbb{P}(E), \mathbb{Z})$ is generated by $h = O_{\mathbb{P}(E)}(1)$ and by $p^*H^*(X, \mathbb{Z})$ with the only relation

$$h^r - p^*c_1(E)h^{r-1} + p * c_2(E)h^{r-2} + \ldots (-1)^r c_r(E) = 0$$

It is possible to take this relation (the decomposition of h^r) as the definition of Chern classes of E, $c_i(E) \in H^i(X, \mathbb{Z})$. The Chern classes are well defined even in the Chow ring $A(X) = \oplus A_i(X)$. The above relation is called Wu-Chern equation .
The Wu-Chern equation can be reformulated as $c_r\,(p^*E^* \otimes \mathcal{O}(1)) = 0$ which follows from the Whitney formula (5.5) that we will see in a while.
When E is spanned, also the line bundle $O_{\mathbb{P}(E)}(1)$ is spanned and this gives as usual a map $\mathbb{P}(E) \xrightarrow{\phi} \mathbb{P}(H^0(E))$ where the fibers in $\mathbb{P}(E)$ are mapped by ϕ to linear spaces. When $O_{\mathbb{P}(E)}(1)$ is very ample then $\mathbb{P}(E)$ is classically called a scroll.
Remark that when $E \to F$ is a surjective map between bundles then it is induced an imbedding $\mathbb{P}(F) \to \mathbb{P}(E)$ which takes fibers to fibers.

5.8. Exercise. *The flag manifold $F(0,1,2) \subset \mathbb{P}^2 \times \mathbb{P}^{2*}$ consists of pairs (p, l) where $p \in \mathbb{P}^2$ is a point, $l \in \mathbb{P}^{2*}$ is a line and $p \in l$. Prove that it is a hyperplane section of $PP^2 \times \mathbb{P}^{2*}$ and it is isomorphic to the projective bundle $\mathbb{P}(T\mathbb{P}^2)$.*

5.3 Geometrical definition of Chern classes

There are several equivalent definitions of the Chern classes of a vector bundle E. The analytic definitions via the curvature is the more useful to prove formulas about the Chern classes. In the spirit of this book we sketch the geometrical definition of Chern classes of degeneracy loci that involves the map Φ_E in the grassmannian.
Let E be a spanned vector bundle of rank r over X. We denote by $s_1, \ldots, s_{r-p+1}$ $r-p+1$ generic sections of E. The subvariety

$$\{x \in X | s_1(x), \ldots, s_{r-p+1}(x) \text{ are lin. dep.}\} \tag{5.4}$$

has codimension p and its homology class in $H_{2n-2p}(X, \mathbb{Z})$ does not depend on the sections (it is easy to check that even the rational equivalence class in the Chow ring is well defined).

5.14. Definition. *The Chern classes $c_p(E) \in H^{2p}(X, \mathbb{Z})$ of a spanned vector bundle E are defined as the Poincaré dual of the class in (5.4).*

If $p = r$ in (5.4) we get the zero locus of a generic section of E.
If $p = 1$ in (5.4) we get that $c_1(E) = c_1(\det\ E)$, furthermore c_1 of a line bundle associated to a divisor D is the class of D itself.

5.9. Exercise. *Prove that* $c_1(E) = c_1(\wedge^r E)$

5.10. Exercise. *Prove that if* E *has rank* 2 *then* $E \simeq E^* \otimes c_1(E)$. *Hint: use the exercise 10.7.*

When E is not spanned there are two ways to supply the definition of Chern classes. The first one (as in [GH]) is to consider convenient C^∞ sections, in fact the Chern classes are C^∞-invariant. The second one is to tensor E with some ample line bundle L in order to get $E \otimes L$ spanned and then use the formula

$$c_k(E \otimes L) = \sum_{i=0}^{k} \binom{r-i}{k-i} c_i(E) c_1(L)^{k-i}$$

(of course one has to check that this definition is well posed!)
In particular we will use often $c_1(E \otimes L) = c_1(E) + rc_1(L)$
The Chern polynomial is the formal expression

$$c_E(t) := c_0(E) + c_1(E)t + c_2(E)t^2 + \dots$$

In the case $X = \mathbb{P}^n$ we have $c_i(E) \in \mathbb{Z}$ and $c_E(t) \in \mathbb{Z}[t]/t^{n+1}$. If

$$0 \to E \to F \to G \to 0$$

is an exact sequence of vector bundles, the *Whitney formula* is

$$c_E(t) c_G(t) = c_F(t) \tag{5.5}$$

In particular

$$c_1(F) = c_1(E) + c_1(G)$$
$$c_2(F) = c_2(E) + c_1(E)c_1(G) + c_2(G)$$

5.11. Exercise. *An instanton bundle on* $\mathbb{P}^3$ *is defined as* $E = Kerb/Ima$ *where*

$$\emptyset(-1)^k \xrightarrow{a} \mathcal{O}^{k+2} \xrightarrow{b} \mathcal{O}(1)^k$$

satisfies b *surjective,* a *injective and* $b \cdot a = 0$ *(such a complex is called a monad). Prove that*

$$c(E) = \frac{1}{(1-t^2)^k}$$

An example where $c_{codim Z}(\mathcal{O}_Z) = degZ$ even if Z is in a Segre product*.
We recall from [GH] the following basic

5.15. Theorem. Gauss-Bonnet *For any compact complex variety of dimension* n

$$\chi(X, \mathbb{Z}) = c_n(TX)$$

5.16. Corollary.

$$\chi(\mathbb{P}^n) = n+1$$

$$\chi(Gr(\mathbb{C}^{k+1}, V^*)) = \binom{n+1}{k+1}$$

Proof. Apply the theorems 5.8 and 5.13. $\square$

The Thom-Porteous formula allows to compute the homology class (and even the class in the Chow ring) of the degeneracy locus of a map between two vector bundles. This is defined as follows. Let $E \xrightarrow{\phi} F$ be a sheaf map between vector bundles of rank e and f. The k-degeneracy locus is $D_k(\phi) := \{x \in X | rk(\phi_x) \leq k\}$. We have

$$codim D_k(\phi) \leq (e-k)(f-k) \tag{5.6}$$

and $codim D_k(\phi) \leq (e-k)(f-k)$ in the generic case. Assume that $codim D_k(\phi) = (e-k)(f-k)$, then the Thom-Porteous formula is

$$[D_k(\phi)] = \det\left(c_{f-k+j-i}(F-E)_{1 \leq i,j \leq e-k}\right) \tag{5.7}$$

where $c_i(E-F)$ is the i-th coefficient in the expansion of the quotient c_E/c_F and we pose $c_i = 0$ if $i < 0$.

5.3.1 The splitting principle

For practical computations of Chern classes it is useful the so called *splitting principle*. It says that for a given bundle E over X there exists a variety Y and a morphism $p\colon Y \to X$ such that p^*E has a filtration whose quotients are line bundles L_i and moreover $p^*\colon H^*(X,\mathbb{Z}) \to H^*(Y,\mathbb{Z})$ is injective. Hence $p^*(c(E)) = c(p^*(E)) = \prod_i(1 + c_1(L_i))$. By the injectivity of p^* one can factor formally in a convenient ring extension of $H^*(X,\mathbb{Z})$ as $c(E) = \prod(1 + x_i(E))$ and compute with $x_i(E)$ as if they were the first Chern classes of line bundles L_i.
We describe this procedure with an example. Let E be a bundle of rank two. We want to compute the Chern classes of the symmetric power $S^3(E)$ by means of $c_i(E)$. Split formally $c_1(E) = a+b$, $c_2(E) = ab$. Then

$$c_1(S^3E) = 3a + (2a+b) + (a+2b) + 3b = 6(a+b)$$

$$c_2(S^3E) = 3a(2a+b) + 3a(a+2b) + 9ab + (2a+b)(a+2b) + (2a+b)(3b) + (a+2b)(3b) = 11(a+b)^2 + 10ab$$

$$c_3(S^3E) = 3a(2a+b)(a+2b) + 3a(2a+b)(3b) + 3a(a+2b)(3b) + (2a+b)(a+2b)(3b) = 6(a+b)^3 + 30(a+b)ab$$

$$c_4(S^3E) = 3a(2a+b)(a+2b)(3b) = 18(a+b)^2(ab) + 9(ab)^2$$

The formulas obtained

$$c_1(S^3E) = 6c_1(E)$$

$$c_2(S^3E) = 11c_1^2(E) + 10c_2(E)$$

$$c_3(S^3E) = 6c_1^3(E) + 30c_1(E)c_2(E)$$

$$c_4(S^3E) = 18c_1^2(E)c_2(E) + 9c_2^2(E)$$

are valid for any bundle of rank two E by the splitting principle.
These computations extend in a straightforward way to coherent sheaves, because any coherent sheaf has a locally free resolution.

5.12. Exercise. *Compute $c_i(S^2E)$ for a bundle E of rank 2 or 3 by means of $c_i(E)$.*

**$c_1(S^kE) = \sum_{i=0}^{k}(k-i)\binom{r+i-2}{i}c_1(E)$
**quote Lascoux

5.13. Exercise. *Compute the Chern polynomial of a bundle E on $\mathbb{P}^n$ with the resolution*

$$0 \longrightarrow \mathcal{O}^s \longrightarrow \mathcal{O}(1)^t \longrightarrow E \longrightarrow 0$$

Chapter 6

Steiner Bundles on $\mathbb{P}^n$

6.1 Incidence varieties, duality and vector bundles.

Let $X \subset \mathbb{P}^N$ a projective variety, we denote by $X^\vee$, and called it the dual variety, the closure in $(\mathbb{P}^N)^\vee$ of the set of hyperplanes in $\mathbb{P}^N$ containing one (in a smooth point) tangent space of X. More precisely,if T_xX is the (projective) tangent space of X in a smooth point x, and X^{sm} the open subset of smooth points in X, we have

$$X^\vee := \overline{\{H \in \mathbb{P}^\vee \mid \exists x \in X^{sm}, T_xX \subset H\}}$$

When X is the product of two varieties, $X = X_1 \times X_2$ the tangent space in a point (x_1, x_2) is the projective space generated by $(T_{x_1}X_1) \times \{x_2\}$ and $\{x_1\} \times (T_{x_2}X_2)$.

6.1.1 Generalities

We begin with the classical incidence variety point-hyperplane in $\mathbb{P}^N$

$$\begin{array}{ccc} \mathbb{F} & \xrightarrow{q} & \mathbb{P}^{N\vee} \\ {\scriptstyle p}\big\downarrow & & \\ \mathbb{P}^N & & \end{array}$$

Let X be a non degenerated smooth subvariety of $\mathbb{P}^N$, and $\mathbb{P}W \subset \mathbb{P}^{N\vee}$ a linear subvariety in the dual space. We denote by $\mathfrak{X} \subset \mathbb{F}$ the following variety $p^{-1}X \cap q^{-1}\mathbb{P}W$. We have the restricted diagram

$$\begin{array}{ccc} \mathfrak{X} & \xrightarrow{q} & \mathbb{P}W \\ {\scriptstyle p}\big\downarrow & & \\ X & & \end{array}$$

The resolution of $\mathfrak{X}$ as a subscheme of $X \times \mathbb{P}W$ is

$$0 \longrightarrow O_{X\times\mathbb{P}W}(-1,-1) \longrightarrow O_{X\times\mathbb{P}W} \longrightarrow O_{\mathfrak{X}} \longrightarrow 0$$

In fact $\mathfrak{X}$ is an hyperplane section of $X \times \mathbb{P}W$ in $\mathbb{P}(V \otimes W)$ where $\mathbb{P}(V) = \mathbb{P}^N$. So we identify $\mathfrak{X}$ with a linear form in $\mathbb{P}(V \otimes W)$. The fiber over a point of $\mathbb{P}W$ is the hyperplane section of X by the corresponding hyperplane. So when X is not linear the variety $\mathfrak{X}$ is never a projective vector bundle over $\mathbb{P}W$. The fiber over $x \in X$ is an hyperplane section of $\mathbb{P}W$ by the hyperplane $x^\vee$. Then we have

6.1. Theorem. *The following conditions are equivalent :*
i) $\mathfrak{X}$ *is a projective bundle over* X
ii) $X \cap (\mathbb{P}W)^\vee = \emptyset$
iii) $\dim\ (\mathbb{P}W) \geq \dim\ (X)$ *and* $\mathfrak{X} \notin (X \times \mathbb{P}W)^\vee$

Proof. $\mathfrak{X}$ is a projective bundle over X if and only if for all $x \in X$ the fibers are projective spaces of the same dimension (by Proposition 10.9). Since $p^{-1}(x) \simeq x^\vee \cap \mathbb{P}W$ it means that for all $x \in X$ the projective space $x^\vee \cap \mathbb{P}W$ is an hyperplane of $\mathbb{P}W$ and not all the ambient space. In other words there is no $x \in X$ such that $\mathbb{P}W \subset x^\vee$ or equivalently $x \in (\mathbb{P}W)^\vee$. This proves $i) \Leftrightarrow ii)$.

We assume that $X \cap (\mathbb{P}W)^\vee = \emptyset$ then it is clear that $\dim\ (\mathbb{P}W) \geq \dim\ (X)$. We denote by Φ the linear map $V \otimes W \to \mathbb{C}$ or the map $V \to W^*$ corresponding to $\mathfrak{X}$. The hypothesis means that for all $x \in X$ the linear form $\Phi(x) : W \to \mathbb{C}$ is not everywhere zero. This proves that the hyperplane $\Phi = 0$ is not tangent to $X \times \mathbb{P}W$.

Conversely, if there exists $x \in X \cap (\mathbb{P}W)^\vee$ and $\dim\ (\mathbb{P}W) \geq \dim\ (X)$ then we have to find $z \in \mathbb{P}W$ such that the kernel of the linear form $\Phi(z) : V \to \mathbb{C}$ contains $\mathbb{P}(T_xX)$. Let $r = \dim X$ and $(x_0, \cdots, x_r)$ a basis of T_xX. Since $\Phi(\sum \lambda_i x_i) = \Phi(x) = 0$ the vector subspace $\bigcap_{i=0}^{i=r} \ker \Phi(x_i) \subset W$ contains a non zero vector z. □

We assume now that $\dim X \leq \dim \mathbb{P}W$ and that $\mathfrak{X} = \mathbb{P}(\mathcal{S})$ is a projective bundle over X, then we have :

$$0 \longrightarrow O_X(-1) \xrightarrow{\Phi} W \otimes O_X \longrightarrow \mathcal{S} \longrightarrow 0$$

We have seen in the previous theorem that a general Φ gives a vector bundle on X, which is equivalent to say that for all $x \in X$, $\Phi(x) \neq 0$. General, here, means outside the closed set $(X \times \mathbb{P}W)^\vee$. In the following proposition we give the codimension of this set and also its degree, according to the degree of X.

6.2. Proposition. *Let* $r = \dim X$ *and* $\dim \mathbb{P}W = r + k, k \geq 0$*, then*
(i) $\operatorname{codim}(X \times \mathbb{P}W)^\vee = k + 1$
(ii) $\deg(X \times \mathbb{P}W)^\vee = \deg(X \times \mathbb{P}^{k+1})$

Proof. Let $(\Phi_0, \cdots, \Phi_{k+1})$ be $k+2$ general linear forms on $V \otimes W$. We consider the map :

$$W \otimes O_{\mathbb{P}^{k+1} \times X} \xrightarrow{\sum X_i \Phi_i} O_{\mathbb{P}^{k+1} \times X}(1,1)$$

Since the codimension of $\mathbb{P}(W)^\vee$ is exactly $r + k + 1$, it meets the variety $\mathbb{P}^{k+1} \times X$ along a finite scheme of lenght equal to the degree of $\mathbb{P}^{k+1} \times X$. Let $(a_0, \cdots, a_{k+1}; x)$ an intersection point. The linear form $\sum a_i \Phi_i$ vanishes identically on the point $x \in X$, so by the previous theorem it is tangent to $X \times \mathbb{P}W$. □

6.1.2 Application to Segre varieties : Steiner bundles

Before applying the previous construction to the Segre varieties we define a particular case of vector bundles on projective spaces, called Steiner bundles.

The **Steiner bundle**, say S, on $\mathbb{P}^n$ are one of the simplest case of vector bundles in the sense that they are defined by a short exact sequence like the following :

$$0 \to O_{\mathbb{P}^n}^{m+1}(-1) \longrightarrow O_{\mathbb{P}^n}^{m+k} \longrightarrow S \to 0$$

By (5.6), to have a bundle it is necessary that $k \geq n+1$. Then the rank of a Steiner bundle on $\mathbb{P}^n$ is always greater than n. Then we will replace $m+k$ by $m+n+1+k$ so that $k \geq 0$.

Example : The tangent bundle over $\mathbb{P}^n$ is a Steiner bundle.

Let n, m, k be three integers such that $k \geq 0$ and $1 \leq n \leq m$. We assume that $k \geq 0$ in order to obtain a vector bundle on the Segre variety (see 6.1 iii)) $\mathbb{P}^n \times \mathbb{P}^m \hookrightarrow \mathbb{P}^{(n+1)(m+1)-1}$. The incidence diagram is

$$\begin{array}{ccc} \mathfrak{X} & \xrightarrow{q} & \mathbb{P}^{n+m+k} \\ {\scriptstyle p}\big\downarrow & & \\ \mathbb{P}^n \times \mathbb{P}^m & & \end{array}$$

The variety $\mathfrak{X} \subset \mathbb{P}^n \times \mathbb{P}^m \times \mathbb{P}^{n+m+k}$ is an hyperplane section of $\mathbb{P}^n \times \mathbb{P}^m \times \mathbb{P}^{n+m+k}$ by an hyperplane of $\mathbb{P}^{(n+1)(m+1)(n+m+k)-1}$ defined by a trilinear form $\Phi = \sum_{i,j,k} a_{i,j,k} X_i Y_j Z_k$.

We fix some notations. Let V, I, W three vector spaces such that $\mathbb{P}V = \mathbb{P}^n, \mathbb{P}I = \mathbb{P}^m$ and $\mathbb{P}W = \mathbb{P}^{n+m+k}$. The linear form Φ could be written $\Phi : V^* \otimes I^* \otimes W^* \to \mathbb{C}$ or $\Phi : V^* \otimes I^* \to W$. So we have the following exact sequences

$$0 \longrightarrow O_{\mathbb{P}V \times \mathbb{P}I \times \mathbb{P}W}(-1,-1,-1) \xrightarrow{\Phi} O_{\mathbb{P}V \times \mathbb{P}I \times \mathbb{P}W} \longrightarrow \mathfrak{X} \longrightarrow 0$$

$$0 \longrightarrow O_{\mathbb{P}V \times \mathbb{P}I}(-1,-1) \xrightarrow{\Phi} W \otimes O_{\mathbb{P}V \times \mathbb{P}I} \longrightarrow \mathcal{S} \longrightarrow 0$$

$$0 \longrightarrow I^* \otimes O_{\mathbb{P}V}(-1) \xrightarrow{\Phi} W \otimes O_{\mathbb{P}V} \longrightarrow S_V \longrightarrow 0$$

$$0 \longrightarrow V^* \otimes O_{\mathbb{P}I}(-1) \xrightarrow{\Phi} W \otimes O_{\mathbb{P}I} \longrightarrow S_I \longrightarrow 0$$

The different expressions of Φ are respectively

$$\Phi = \sum_{i,j,k} a_{i,j,k} X_i Y_j Z_k$$

$$\Phi = \left(\sum_{i=0,j=0}^{n,m} a_{i,j,0} X_i \otimes Y_j, \cdots, \sum_{i=0,j=0}^{n,m} a_{i,j,n+m+k} X_i \otimes Y_j \right)$$

$$\Phi = \begin{pmatrix} \sum_{i=0}^n a_{i,0,0} X_i & \cdots & \sum_{i=0}^n a_{i,0,n+m+k} X_i \\ \sum_{i=0}^n a_{i,1,0} X_i & \cdots & \sum_{i=0}^n a_{i,1,n+m+k} X_i \\ \cdots & \cdots & \cdots \\ \sum_{i=0}^n a_{i,m,0} X_i & \cdots & \sum_{i=0}^n a_{i,m,n+m+k} X_i \end{pmatrix}$$

$$\Phi = \begin{pmatrix} \sum_{j=0}^m a_{0,j,0}Y_j & \cdots & \sum_{j=0}^m a_{0,j,n+m+k}Y_j \\ \sum_{j=0}^m a_{1,j,0}Y_j & \cdots & \sum_{j=0}^m a_{1,j,n+m+k}Y_j \\ \cdots & \cdots & \cdots \\ \sum_{j=0}^m a_{n,j,0}Y_j & \cdots & \sum_{j=0}^m a_{n,j,n+m+k}Y_j \end{pmatrix}$$

We write again the theorem 6.1 in this particular case

6.3. Theorem. *Let V, I, W and Φ be given as before. The following conditions are equivalent*
1) For all non zero vectors $x \in V^*$ and $y \in I^*$, $\Phi(x \otimes y) \neq 0$.
2) $(\mathbb{P}W)^\vee \cap \mathbb{P}V \times \mathbb{P}I = \emptyset$
3) $\mathcal{S}$ is a vector bundle (of rank $n+m+k$) over $\mathbb{P}V \times \mathbb{P}I$.
4) S_V is a Steiner bundle (of rank $n+k$) over $\mathbb{P}V$.
5) S_I is a Steiner bundle (of rank $m+k$) over $\mathbb{P}I$.
6) $\Phi \notin (\mathbb{P}V \times \mathbb{P}I \times \mathbb{P}W)^\vee$

Proof. The proof of this theorem is essentially the same than the one given to prove the theorem 6.1. But according to the importance of Steiner bundles in this text we prefer to repeat the proof. We choose here to write $\Phi : V^* \otimes I^* \to W$. Let $x \in V^*$ and $y \in I^*$, then $\Phi(x \otimes y)$ is a linear form on W, i.e

$$\Phi(x \otimes y) : W^* \to \mathbb{C}, z \mapsto \Phi(x \otimes y)(z)$$

The sets $H_{x,y} = \{z \in W^*, \Phi(x \otimes y)(z) = 0\}$ are hyperplanes in W^* if $\Phi(x \otimes y) \neq 0$ and are equal to W^* if $\Phi(x \otimes y) = 0$. Then it is clear that 1), 2) and 3) are equivalent.

Let $\{y_1, \cdots, y_{m+1}\}$ be a basis of I^* and $\{x_1, \cdots, x_{n+1}\}$ be a basis of V^*. For a fixed $x \in V^*$ and a fixed $y \in I^*$ we define the sets

$$H_x = \cap_{i=1}^{m+1} H_{x,y_i},\ H_y = \cap_{j=1}^{n+1} H_{x_j,y}$$

The expected dimension for H_x is $n+k$. If the dimension is $n+k$ for all $x \in V^*$ we obtain a vector bundle of rank $n+k$ over $\mathbb{P}V$. Assume that $\dim_{\mathbb{C}} H_x > n+k$, then the $m+1$ hyperplanes are not linearly independent, or at least one of them is not an hyperplane. In both cases we have a non zero family of complex numbers such that $\sum a_i \Phi(x \otimes y_i) = 0$, then $\Phi(x \otimes \sum a_i y_i) = 0$, which proves the equivalence between 3) and 4). The same holds for 3) and 5).

By hypothesis $\Phi \in \mathbb{P}(V \otimes I \otimes W)$. The point Φ belongs to $(\mathbb{P}V \times \mathbb{P}I \times \mathbb{P}W)^\vee$ if and only if the hyperplane Φ in $\mathbb{P}(V^* \otimes I^* \otimes W^*)$ contains a tangent space in a point $x_0 \otimes y_0 \otimes z_0$. But the tangent space to a product is just the space generated by the tangent spaces of each component of the product. In our case, it is the projective space generated by $\{x_0 \otimes y_0\} \times \mathbb{P}(W^*)$, $\{x_0 \otimes z_0\} \times \mathbb{P}(I^*)$ and $\{y_0 \otimes z_0\} \times \mathbb{P}(V^*)$. In other words, $\Phi \in (\mathbb{P}V \times \mathbb{P}I \times \mathbb{P}W)^\vee$ if and only if it exists $x_0 \otimes y_0 \otimes z_0 \in V^* \otimes I^* \otimes W^*$ such that

$$\forall y \in I^*, \forall x \in V^*, \forall z \in W^*, \Phi(x_0 \otimes y)(z_0) = \Phi(x \otimes y_0)(z_0) = \Phi(x_0 \otimes y_0)(z) = 0$$

Now to prove the equivalence between 1) and 6) we just need to show that if $\Phi(x_0 \otimes y_0)(z) = 0$ for all $z \in W^*$ then there exists $z_0 \in W^*$ such that

$$\forall y \in I, \forall x \in V, \Phi(x_0 \otimes y)(z_0) = \Phi(x \otimes y_0)(z_0) = 0$$

But we have already seen that $\Phi(x_0 \otimes y_0) = 0$ is equivalent to $\dim_{\mathbb{C}} H_{x_0} > n + k$ and $\dim_{\mathbb{C}} H_{y_0} > m + k$, then, since $k \geq 0$, the intersection $H_{x_0} \cap H_{y_0}$ contains a non zero vector z_0. $\square$

6.4. Proposition. *Let $k \geq 0$,*
(i) $\operatorname{codim}(\mathbb{P}^n \times \mathbb{P}^m \times \mathbb{P}^{n+m+k})^\vee = k + 1$
(ii) $\deg(\mathbb{P}^n \times \mathbb{P}^m \times \mathbb{P}^{n+m+k})^\vee = \frac{(n+m+k+1)!}{(k+1)!n!m!}$

Proof. It is just a reformulation of the proposition 6.2. $\square$

This proposition can be generalized to more than three factors, as we will see in Theorem 9.20.

When $k = 0$, the dual variety is an hypersurface and its equation is given by the hyperdeterminant (see chapter 9).

The following theorem says that any morphism between Steiner bundles is induced by a morphism between the corresponding short exact sequences.

6.5. Theorem. *Let E, F be Steiner bundles on $\mathbb{P}(V)$ appearing in the following sequences*

$$0 \longrightarrow I \otimes \mathcal{O}(-1) \longrightarrow W \otimes \mathcal{O} \longrightarrow E \longrightarrow 0$$

$$0 \longrightarrow I \otimes \mathcal{O}(-1) \longrightarrow W \otimes \mathcal{O} \longrightarrow F \longrightarrow 0$$

where $\dim V = n + 1$, $\dim I = m + 1$, $\dim W = n + m + k + 1$. *For every $f \colon E \to F$ there are $a \in End(I)$, $b \in End(W)$ such that the following diagram commutes*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I \otimes \mathcal{O}(-1) & \longrightarrow & W \otimes \mathcal{O} & \xrightarrow{p} & E \longrightarrow & 0 \\ & & \downarrow a & & \downarrow b & & \downarrow f & \\ 0 & \longrightarrow & I \otimes \mathcal{O}(-1) & \longrightarrow & W \otimes \mathcal{O} & \longrightarrow & F \longrightarrow & 0 \end{array} \tag{6.1}$$

Proof. Applying $Hom(W \otimes \mathcal{O}, -)$ to the second row of (6.1) we have the exact sequence

$$Hom(W \otimes \mathcal{O}, W \otimes \mathcal{O}) = End(W) \to Hom(W \otimes \mathcal{O}, F) \to Ext^1(W, I \otimes \mathcal{O}(-1))$$

The composition $f \cdot p \in Hom(W \otimes \mathcal{O}, F)$ lifts to $End(W)$ because

$$Ext^1(W, I \otimes \mathcal{O}(-1)) = W^* \otimes I \otimes H^1(\mathcal{O}(-1)) = 0$$

Let b be a lifting, it makes commutative the right part of (6.1). Then the existence of a is trivial. $\square$

6.1.3 Some $SL(V)$ invariant Steiner bundles

Let V a $\mathbb{C}$ vector space such that $\dim_{\mathbb{C}} V = r + 1 \geq 2$. We denote by S_k the vector space $Sym^k V$, i.e. the k-symmetric power of V, and by v_k the image of $\mathbb{P}V$ by the Veronese map $\mathbb{P}(S_1) \hookrightarrow \mathbb{P}(S_k)$. We consider now the canonical maps, the multiplication of forms and the dual map δ which is the derivation,

$$S_n \otimes S_m \xrightarrow{\times} S_{n+m}, \quad S_{n+m} \xrightarrow{\delta} S_n \otimes S_m$$

Since these maps are $SL(V)$-equivariants we omit the dual sign for vector spaces. Also we want to define an order on the basis of S_k. Let $(x_0, \cdots, x_r)$ a basis of V. Since a form of S_k could be written $\prod_{i=0}^{i=r} x_i^{k_i}$ with $\sum k_i = k$, we choose the lexicographic order on partitions i.e.

$$(k_0, \cdots, k_r) \leq (l_0, \cdots, l_r) \Leftrightarrow \exists s \mid l_i = k_i \text{ for } i \leq s, \text{ and } k_{i+1} \leq l_{i+1}$$

We denote respectively by $X_{(n_0,\cdots,n_r)}$, $Y_{(m_0,\cdots,m_r)}$ and $Z_{(s_0,\cdots,s_r)}$ (s like sum) the coordinates on S_n, S_m and S_{n+m}.
Then in coordinates the multiplication map is given by the following trilinear form

$$\Phi = [\sum_{(s_k)|\sum s_k=n+m} [\sum_{(n_i)|\sum n_i=n} [\sum_{(m_j)|\sum m_j=m, n_k+m_k=s_k} X_{(n_0,\cdots,n_r)} Y_{(m_0,\cdots,m_r)} Z_{(s_0,\cdots,s_r)}]]]$$

If we denote by $p(r, n, m, (s))$ the number of couples of $r+1$-partitions of n and m such that their sum is equal to the $r+1$-partition of $n+m$ called (s) we have in a simple way

$$\Phi = \sum_{(s_k)|\sum s_k=n+m} p(r, n, m, (s)) X_{(n_0,\cdots,n_r)} Y_{(m_0,\cdots,m_r)} Z_{(s_0,\cdots,s_r)}$$

Remind the incidence diagramm

$$\begin{array}{ccc} \mathfrak{X} & \xrightarrow{q} & \mathbb{P}(S_{n+m}) \\ {\scriptstyle p}\downarrow & & \\ \mathbb{P}(S_n) \times \mathbb{P}(S_m) & & \end{array}$$

We remark that $p(r, n, m, (0, \cdots, n+m, \cdots, 0)) = 1$. This remark is sufficient to prove that the sheaf $\mathfrak{X}$ is a vector bundle over $\mathbb{P}(S_n) \times \mathbb{P}(S_m)$, indeed we have

6.6. Proposition. *The multiplication map $\times$ induces a vector bundle on $\mathbb{P}(S_n) \times \mathbb{P}(S_m)$.*

Proof. Let us call Φ this map. It is enough to prove that for all $x \otimes y \in \mathbb{P}(S_n) \times \mathbb{P}(S_m)$, $\Phi(x \otimes y) \neq 0$. Let $X_0, \cdots, X_r$ a basis of V. Since $\Phi^{-1}(X_i^{n+m}) = \{X_i^n \otimes X_i^m\}$, we have

$$X_i^{n+m}(x \otimes y) = 0 \Leftrightarrow X_i^n(x) = X_i^m(y) = 0.$$

$\square$

Let us introduce some notations. The multiplication map gives the following $SL(V)$-bundles on $\mathbb{P}(S_n) \times \mathbb{P}(S_m)$,$\mathbb{P}(S_m)$ and on $\mathbb{P}S_n$,

$$0 \longrightarrow O_{\mathbb{P}(S_n)\times\mathbb{P}(S_m)}(-1,-1) \longrightarrow S_{n+m} \otimes O_{\mathbb{P}S_n} \longrightarrow E_{n+m} \longrightarrow 0$$

$$0 \longrightarrow S_n \otimes O_{\mathbb{P}S_m}(-1) \longrightarrow S_{n+m} \otimes O_{\mathbb{P}S_n} \longrightarrow E_{m,n+m} \longrightarrow 0$$

$$0 \longrightarrow S_m \otimes O_{\mathbb{P}S_n}(-1) \longrightarrow S_{n+m} \otimes O_{\mathbb{P}S_n} \longrightarrow E_{n,n+m} \longrightarrow 0$$

The projective Steiner bundle $\mathbb{P}E_{n,n+m}$ over $\mathbb{P}(S_n)$ is imbedded in $\mathbb{P}(S_n) \times \mathbb{P}(S_{n+m})$ and it is defined by the s_m equations

$$\Phi = [\sum_{(n_i)|\sum n_i = n} X_{(n_0,\cdots,n_r)} Z_{(n_0+m_0,\cdots,n_r+m_r)}]_{(m_j)|\sum m_j = m} = 0$$

Over a general point $(x_{(n_0,\cdots,n_r)})_{(n_i)}$ the projective fiber is $\mathbb{P}^{s_{n+m}-s_m-1} \subset \mathbb{P}^{s_{n+m}}$ which is defined by the s_m equations

$$\Phi = [\sum_{(n_i)|\sum n_i = n} x_{(n_0,\cdots,n_r)} Z_{(n_0+m_0,\cdots,n_r+m_r)}]_{(m_j)|\sum m_j = m} = 0$$

6.7. Proposition. *Over a point $(x_0^{n_0} \cdots x_r^{n_r})_{(n_i)}$ the fiber is the set of hyperplanes containing the m-osculating space of v_{n+m} in the point $(x_0^{s_0} \cdots x_r^{s_r})_{(s_k)|\sum s_k = n+m}$*

Proof.

$$\partial^{(m_0,\cdots,m_r)}(x_0^{s_0} \cdots x_r^{s_r})_{(s_k)|\sum s_k=n+m} = (x_0^{s_0-m_0} \cdots x_r^{s_r-m_r})_{(s_k)|\sum s_k=n+m}$$

where $x_0^{s_0-m_0} \cdots x_r^{s_r-m_r} = 0$ if $s_k < m_k$. □

Let us consider the restriction of $E_{n,n+m}$ to the Veronese v_n, it gives

$$0 \longrightarrow S_m \otimes O_{\mathbb{P}S_1}(-n) \longrightarrow S_{n+m} \otimes O_{\mathbb{P}S_1} \longrightarrow \mathfrak{E}_{n,n+m} \longrightarrow 0$$

By the above proposition we can interpret the first arrow as the matrix of m-partial derivatives of S_{n+m}, and by the way to consider $\mathfrak{E}_{n,n+m}$ as the bundle of degree $n+m$ hypersurfaces in $\mathbb{P}S_1$ with a singular point of order $\geq m+1$. In other terms the fiber over a point x is $H^0(\mathfrak{m}_x^{m+1}(n+m))^*$

6.8. Remark. *The image of the projective bundle $\mathbb{P}\mathfrak{E}_{n,n+m} \subset \mathbb{P}S_1 \times \mathbb{P}S_{n+m}$ by the second projection is the m-osculating variety of v_{n+m}.*

One more time we consider the classical incidence variety and its restriction to v_n

$$\begin{array}{ccccc} \mathbb{F} & \xrightarrow{q} & \mathbb{P}(S_n)^\vee q^{-1}(v_n) & \xrightarrow{q} & v_n \\ {\scriptstyle p}\big\downarrow & & {\scriptstyle p}\big\downarrow & & \\ \mathbb{P}(S_n) & & \mathbb{P}(S_n) & & \end{array}$$

6.9. Proposition. $E_{n,n+m} = p_* q^* O_{v_n}(\frac{n+m}{n}) (= p_* q^* O_{\mathbb{P}S_1}(n+m))$

Proof. We have the following resolution of $q^{-1}(v_n)$,

$$0 \longrightarrow O_{\mathbb{P}S_n \times v_n}(-1,-1) \longrightarrow O_{\mathbb{P}S_n \times v_n} \longrightarrow O_{q^{-1}(v_n)} \longrightarrow 0$$

and via the isomorphism of sheaves $O_{v_n}(1) = O_{\mathbb{P}S_1}(n)$ it becomes

$$0 \longrightarrow O_{\mathbb{P}S_n \times \mathbb{P}S_1}(-1,-n) \longrightarrow O_{\mathbb{P}S_n \times \mathbb{P}S_1} \longrightarrow O_{q^{-1}(v_n)} \longrightarrow 0$$

We tensorize by $q^*O_{\mathbb{P}S_1}(n+m)$ and take the direct image of the exact sequence on $\mathbb{P}S_n$ to obtain

$$0 \longrightarrow S_m \otimes O_{\mathbb{P}S_n}(-1) \longrightarrow S_{n+m} \otimes O_{\mathbb{P}S_n} \longrightarrow E_{n,n+m} \longrightarrow 0$$

$\square$

6.10. Example. *Let* (X_0, X_1, X_2) *be a basis of* V *and* $(X_0^2, X_0X_1, X_0X_2, X_1^2, X_1X_2, X_2^2)$ *be a basis of* S^2V. *Since* $S^2V = H^0(O_{\mathbb{P}V}(2)) = H^0(O_{\mathbb{P}S^2V}(1))$ *we introduce the notation* $Z_.$ *to have a basis given by linear forms on* $\mathbb{P}S^2V$ *instead of quadratic forms on* $\mathbb{P}V$, *it means that*

$$(X_0^2, X_0X_1, X_0X_2, X_1^2, X_1X_2, X_2^2) = (Z_0, Z_1, Z_2, Z_3, Z_4, Z_5)$$

Then over $\mathbb{P}V^*$ *the matrix associated to* Φ *is*

$$\begin{pmatrix} X_0 & X_1 & X_2 & 0 & 0 & 0 \\ 0 & X_0 & 0 & X_1 & X_2 & 0 \\ 0 & 0 & X_0 & 0 & X_1 & X_2 \end{pmatrix}$$

And the matrix associated to the map Φ *over* $\mathbb{P}S^2V$ *is*

$$\begin{pmatrix} Z_0 & Z_1 & Z_2 \\ Z_1 & Z_3 & Z_4 \\ Z_2 & Z_4 & Z_5 \end{pmatrix}$$

In $\mathbb{P}S^2V$ *the locus defined by the vansishing of the determinant of this matrix is the locus of singular conics.*

6.1. Exercise. *Consider the multiplication map* $S^2V \otimes S^2V \to S^4V$ *where* $\dim V = 3$, *and find (after choosing a basis) the matrix of the map*

$$S^2V \otimes O_{\mathbb{P}S^4V} \to S^2V^* \otimes O_{\mathbb{P}S^4V}(1)$$

Show that the degeneracy locus is the hypersurface of **Clebsch quartics**, *already seen in exercise 3.12.*

6.1.4 Schwarzenberger bundles

Let U be a two dimensional vector space over $\mathbb{C}$. According to the proposition 6.6, the multiplication map

$$S^mU \otimes S^nU \xrightarrow{\phi} S^{n+m}U$$

gives a bundle $E_{n,n+m}$ on $\mathbb{P}S^nU$. These bundles, introduced by Schwarzenberger in [Sch1], are now called **Schwarzenberger bundles**.

In order to have a nice matrix description, we denote by (t_0, t_1) a basis of U, $(u_i = t_0^i t_1^{n-i})$, $(v_j = t_0^j t_1^{m-j})$, and $(x_l = t_0^l t_1^{n+m-l})$ the respective basis of S^nU, S^mU and $S^{n+m}U$. Then $\phi(u_i \otimes v_j) = x_{i+j}$ We have also the map

$$S^mU \longrightarrow S^nU^* \otimes S^{n+m}U$$

$$v_j \mapsto \sum_{i=0}^{i=n} u_i^* \otimes x_{i+j}$$

where (u_i^*) is the (dual) basis of S^nU^*. Let

$$\mathbb{P}(S^{n+m}U^*) = \mathbb{P}(\mathbf{C}[X_0, ..., X_{n+m}]), \quad \text{and} \quad \mathbb{P}(S^nU) = \mathbb{P}(\mathbf{C}[U_0, ..., U_n])$$

The representative matrix of the composed homomorphism

$$S^mU \longrightarrow S^nU^* \otimes S^{n+m}U \longrightarrow S^{n+m}U \otimes O_{\mathbb{P}(S^nU)}(1)$$

$$v_j \mapsto \sum_{i=0}^{i=n} u_i^* \otimes x_{i+j} \mapsto \sum_{i=0}^{i=n} x_{i+j}U_i$$

is the following $(m+1) \times (n+m+1)$-persymmetric matrix

$$M_m = \begin{pmatrix} 0 & 0 & \cdots & 0 & U_0 \\ \cdots & \cdots & \cdots & U_0 & U_1 \\ \cdots & 0 & \cdots & U_1 & U_2 \\ 0 & U_0 & \cdots & U_2 & . \\ U_0 & U_1 & \cdots & \cdots & U_n \\ U_1 & U_2 & \cdots & U_n & 0 \\ U_2 & \cdots & \cdots & 0 & \cdots \\ \cdots & U_n & \cdots & \cdots & \cdots \\ U_n & 0 & \cdots & 0 & 0 \end{pmatrix}$$

In the same way the representative matrix of the composed homomorphism

$$S^mU \longrightarrow S^nU^* \otimes S^{n+m}U \longrightarrow S^nU^* \otimes O_{\mathbb{P}(S^{n+m}U^*)}(1)$$

$$v_j \mapsto \sum_{i=0}^{i=n} u_i^* \otimes x_{i+j} \mapsto \sum_{i=0}^{i=n} u_i^* X_{i+j}$$

is the following $(m+1) \times (n+1)$-persymmetric matrix

$$N_n = \begin{pmatrix} X_0 & X_1 & X_2 & \cdots & X_m \\ X_1 & X_2 & \cdots & \cdots & X_{m+1} \\ X_2 & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & X_{n+m-1} \\ X_n & \cdots & \cdots & X_{n+m-1} & X_{n+m} \end{pmatrix}$$

The matrix N_n is the pull back by the natural (given by the Clebsch-Gordan decomposition) embedding

$$\phi : \mathbb{P}(S^{n+m}U) \hookrightarrow \mathbb{P}(S^mU \otimes S^nU)$$

of the $(m+1) \times (n+1)$-generic matrix. It is well known that (see ***) the zero locus defined by the maximal minors of N_n is exactly the scheme of the $n-1$-plane n-secant (we assume here that $n \leq m$) to the normal rational curve C_{n+m} defined by the two-minors. More generally the zero scheme defined by the i-st Fitting ideal is identified to the scheme of $(n-i-1)$-plane $(n-i)$-secant to the normal rational curve defined by the two-minors. We will denote these varieties V_{n-i}

6.2. Exercise. *Prove that V_n consists of the closure of the union of linear $(n-1)$-dimensional spaces which are n-secant to C_n. It is called the k-secant variety to C_n. The 2-secant variety is the usual secant variety.*

6.11. Theorem. $\mathbb{P}E_{n,n+m}$ *is the blowing up of* V_n *along* V_{n-1}

Proof. According to the matrix description above, $\mathbb{P}(E_{n,n+m})$ is embedded in the product $\mathbb{P}(S^nU)\times\mathbb{P}(S^{n+m}U)$ as a subvariety defined by the equations $((\sum_{i=0}^n U_iX_{i+j}=0)_{j=0,\cdots,m})$. We denote respectively by p and π the projection morphisms on $\mathbb{P}(S^{n+m}U)$ and $\mathbb{P}(S^nU)$. Then we will denote by $O_{\mathbb{P}(E_{n,n+m})}(a,b)$ the line bundle $p^*O_{\mathbb{P}(S^{n+m}U)}(a)\otimes\pi^*O_{\mathbb{P}(S^nU)}(b)$ The image of $\mathbb{P}(E_{n,n+m})$ by p is just V_n. The fiber over a general point $(u_0,\cdots,u_n)$ is just the $\mathbb{P}^{n-1}$ defined by the $m+1$ linear equations $((\sum_{i=0}^n u_iX_{i+j}=0)_{j=0,\cdots,m})$ in $\mathbb{P}(S^{n+m}U)$. These equations are also obtained by the product $(u_0,\cdots,u_n)N_n$. It means that this $\mathbb{P}^{n-1}$ belongs to V_n. So it is clear that the morphism $p:\mathbb{P}(E_{n,n+m})\to V_n$ is birationnal and isomorphic outside V_{n-1}. To show that V_{n-1} is the center of the blowing up we have to prove that $p^{-1}(V_{n-1})$ is a divisor in $\mathbb{P}(E_{n,n+m})$

More generaly we determine the class of $p^{-1}(V_i)$ in the Chow ring $\mathbb{A}(\mathbb{P}(E_{n,n+m}))$ of $\mathbb{P}(E_{n,n+m})$.

6.12. Proposition. *For any $i<n$ there is an injective homomorphism of vector bundles*

$$(i+1)O_{\mathbb{P}(E_{n,n+m})}(-1,0)\xrightarrow{\ \psi\ }(\pi^*E_{n,n+m-i})^*$$

such that $\mathbf{V}(\wedge^{i+1}\psi)=p^{-1}V_i$ and $[p^{-1}V_i]=c_{n-i}(coker\psi)$ in $\mathbb{A}(\mathbb{P}(E_{n,n+m}))$.

Proof. For any $i<n$ we have the following exact sequence on $\mathbb{P}(S^nU)$

$$0\longrightarrow S^{m-i}U(-l)\xrightarrow{M_{m-i}}S^{n+m-i}U\xrightarrow{(X_0,\cdots,X_{n+m-i})}E^k_{n+m-i}\longrightarrow 0$$

We dualize this exact sequence and look at it on $\mathbb{P}(E_{n,n+m})$

$$0\longrightarrow(\pi^*E_{n,n+m-i})^*\longrightarrow(S^{n+m-i}U)^*\xrightarrow{{}^tM_{m-i}}(S^{m-i}U)^*(0,1)\longrightarrow 0$$

We have also for any $i<n$ a map on $\mathbb{P}(S^{n+m}U)$

$$S^{n+m-i}U\xrightarrow{\ \phi\ }S^iU^*(1,0)$$

where $V_i=\mathbf{V}(\wedge^{i+1}\phi)\subset\mathbb{P}(S^{n+m}U)$ and ϕ is given by the following matrix

$$N_i=\begin{pmatrix} X_0 & X_1 & X_2 & \cdots & X_{n+m-i}\\ X_1 & X_2 & \cdots & \cdots & X_{n+m-i+1}\\ X_2 & \cdots & \cdots & \cdots & \cdots\\ \cdots & \cdots & \cdots & \cdots & \cdots\\ \cdots & \cdots & \cdots & \cdots & X_{n+m-1}\\ X_i & \cdots & \cdots & X_{n+m-1} & X_{n+m}\end{pmatrix}$$

As before we dualize this map and look at it on $\mathbb{P}(E_{n,n+m})$

$$S^iU(-1,0)\xrightarrow{\ N_i\ }(S^{n+m-i}U)^*$$

Since for any $i < n$ the product ${}^tM_{m-i}N_i$ is zero on $\mathbb{P}(E_{n,n+m})$ the composed homomorphim

$$S^iU(-1,0) \xrightarrow{N_i} (S^{n+m-i}U)^* \xrightarrow{{}^tM_{m-i}} (S^{m-i}U)^*(0,1)$$

is zero. Then there exists a non zero map

$$S^iU(-1,0) \xrightarrow{\psi} (\pi^*E_{n,n+m-i})^*.$$

By the snake lemma the cokernels of ψ and N_i are the same on $\mathbb{P}(E_{n,n+m})$. So we can deduce that $\mathbf{V}(\wedge^{i+1}\psi) = \mathbf{V}(\wedge^{i+1}N_i)$ and this last one is the inverse image $p^{-1}V_i$. Next we verify easily that $\dim p^{-1}V_i = n + i - 1$. The codimension (equal to $(n-l)$) is the one expected and we can apply the Thom-Porteous formula to conclude.

It results that $p^{-1}V_{n-1}$ is a divisor of $\mathbb{P}(E_{n,n+m})$ defined by the determinant of

$$S^{n-1}U(-1,0) \xrightarrow{\psi} (\pi^*E_{n,m-1})^*.$$

By a simple computation of first Chern classes we find that $p^{-1}V_{n-1}$ is defined by a non zero section of $O_{\mathbb{P}(E_{n,n+m})}(n, n-m-2)$ i.e. by one section of $(S^nE_{n,n+m})(n-m-2)$. □

6.13. Remark. *[Va1] On $\mathbb{P}^2$, the existence of a non zero section of $S^2(E^2_{n+2})(-n)$ characterizes Schwarzenberger bundle amomg the stable rank two vector bundles.*

6.1.5 Application of Schwarzenberger bundles to the rational normal curves

6.14. Theorem. *Let U be a 2-dimensional vector space. Let $C = \mathbb{P}(U) \subset \mathbb{P}(S^nU) = \mathbb{P}^n$ be the rational normal curve. Then $T\mathbb{P}^n_C \simeq S^{n-1}U \otimes \mathcal{O}(n+1)$ $N_{C,\mathbb{P}^n} \simeq S^{n-2}U \otimes \mathcal{O}(n+2)$*

Proof. The exact sequence defining the Schwarzenberger bundle $\mathcal{O}(-n)$ on $\mathbb{P}^1$ is

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^1}(-n) \longrightarrow \mathcal{O}_{\mathbb{P}^1} \otimes S^nU \longrightarrow \mathcal{O}_{\mathbb{P}^1}(1) \otimes S^{n-1}U \longrightarrow 0$$

The first isomorphism immediately follows. Then we get the sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^1}(2-n) \longrightarrow \mathcal{O}_{\mathbb{P}^1}(1) \otimes S^{n-1}U \longrightarrow N(-n) \longrightarrow 0$$

Tensoring by $\mathcal{O}_{\mathbb{P}^1}(-3)$

$$0 \longrightarrow \mathcal{O}(-n-1) \longrightarrow \mathcal{O}(-2) \otimes S^{n-1}U \longrightarrow N \otimes \mathcal{O}_{\mathbb{P}^1}(-n-3) \longrightarrow 0$$

so that

$$0 \longrightarrow H^0(N \otimes \mathcal{O}_{\mathbb{P}^1}(-n-3)) \longrightarrow S^{n-1}U \xrightarrow{f} S^{n-1}U \longrightarrow H^1(N \otimes \mathcal{O}_{\mathbb{P}^1}(-n-3)) \longrightarrow 0$$

By Schur lemma the map f is an isomorphism (why is not zero?), so that $N \otimes \mathcal{O}_{\mathbb{P}^1}(-n-3) = \mathcal{O}(-1) \otimes W$ for some representation W. Comparing with the first sequence we get $W \simeq S^{n-2}U$ as we wanted . □ □

6.15. Remark. *$H^0(N_{C,\mathbb{P}^n}) = S^{n+2}U \otimes S^{n-2}U = \mathfrak{sl}(S^nU)/\mathfrak{sl}(U)$. This is the tangent space at the Hilbert scheme $SL(S^nU)/SL(U)$ of rational normal curves.*

6.16. Theorem. $E_{n,n+m|C} = S^{n-1}(U) \otimes \mathcal{O}_C(\frac{m+1}{n}) = S^{n-1}(U) \otimes \mathcal{O}_{\mathbf{P}^1}(m+1)$

Proof The bundle $E_{n,n+m}$ over C can be interpretated as the bundle of $m+1$-osculating spaces to C (see prop. 6.7)). More precisely we have over $\mathbb{P}^1$

$$0 \longrightarrow S^kU \otimes \mathcal{O}_{\mathbb{P}^1}(-n) \longrightarrow S^{n+m}U \otimes \mathcal{O}_{\mathbb{P}^1} \longrightarrow E_{n,n+m\,\mathbb{P}^1} \longrightarrow 0$$

The fiber of this bundle over $x \in \mathbb{P}^1$ is

$$H^0(\mathfrak{m}_x^{m+1}(n+m)) = H^0(\mathcal{O}_{\mathbb{P}^1}(n-1)) = S^{n-1}U$$

(see also Proposition 3.4) which is independent from x. $\square$

6.1.6 Splitting of Schwarzenberger bundle on lines

Approach with the incidence variety

Let U a vector space on $\mathbb{C}$ of dimension 2. We denote by C_n (resp. C_n^*) the rational normal curve in $\mathbb{P}(S^nU)$ (resp. $\mathbb{P}(S^nU^*)$) image of $\mathbb{P}(U)$ (resp. $\mathbb{P}(U^*)$) by the Veronese morphism. Let X be the inverse image of C_n^* in the incidence variety (point-hyperplane). i.e.

$$X = \{(x, H) \mid x \in H, H \text{ osculates } C_n\}$$

We call pr_1 and pr_2 the projection maps from X to $\mathbb{P}(S^nU)$ and C_n^*. We recall that we can define Schwarzenberger's bundle on $\mathbb{P}(S^nU)$ as direct images of lines bundle on C_n^*, i.e $E_{n,m} = pr_{2*}pr_1^*O_{C_n^*}(\frac{m}{n})$ (see prop. 6.9).

6.17. Theorem. *([ST] prop. 2.18) Let $l \in \mathbb{P}(S^nU)$ be a general line. Let (q, ϵ) such that $m = qn + \epsilon$ with $0 \le \epsilon < n$. Then*

$$(E_{n,m})_{|l}(-q) = [S^{\epsilon}U \otimes O_l] \oplus [S^{n-(\epsilon+2)}U \otimes O_l(-1)]$$

Proof. Since l is general we have an isomorphism $pr_2^{-1}(l) \simeq C_n^*$

$$\begin{array}{ccc} pr_2^{-1}(l) & \xrightarrow{i} & X \\ \downarrow & & \downarrow \\ l & \longrightarrow & \mathbb{P}(S^nU) \end{array}$$

Thus we have

$$(E_{n,m})_{|l} = pr_{2*}(pr_1^*O_{C_n^*}(\frac{n}{k})_{|pr_2^{-1}(l)}) = pr_{2*}O_{pr_2^{-1}(l)}(\frac{m}{n}) = pr_{2*}[O_{pr_2^{-1}(l)}(q) \otimes O_{pr_2^{-1}(l)}(\frac{\epsilon}{n})]$$

Since on the rational curve $pr_2^{-1}(l) \simeq C_n^*$ there exists only one line bundle of degree nq, we have $pr_2^*O_l(q) = O_{pr_2^{-1}(l)}(q)$. Then by the projection formula we find

$$(E_{n,m})_{|l} = pr_{2*}O_{pr_2^{-1}(l)}(\frac{\epsilon}{n}) \otimes O_l(q)$$

Remind the resolution of $X \subset \mathbb{P}(S^nU) \times \mathbb{P}(U^*)$

$$0 \to O_{\mathbb{P}(S^nU)\times\mathbb{P}(U^*)}(-1,-n) \to O_{\mathbb{P}(S^nU)\times\mathbb{P}(U^*)} \to O_X \to 0$$

Over l the above exact sequence becomes

$$0 \to O_{l\times\mathbb{P}(U^*)}(-1,-n) \to O_{l\times\mathbb{P}(U^*)} \to O_{pr_2^{-1}(l)} \to 0$$

and after tensorization by $O_{\mathbb{P}(U^*)}(\epsilon)$ we obtain

$$0 \to O_{l\times\mathbb{P}(U^*)}(-1,\epsilon-n) \to O_{l\times\mathbb{P}(U^*)}(0,\epsilon) \to O_{pr_2^{-1}(l)}(\frac{\epsilon}{n}) \to 0$$

Now take the direct image by pr_2 of this exact sequence

$$0 \to H^0(O_{\mathbb{P}(U^*)}(\epsilon)) \otimes O_l \to pr_{2*}O_{pr_2^{-1}(l)}(\frac{\epsilon}{n}) \to H^1(O_{\mathbb{P}(U^*)}(\epsilon-n)) \otimes O_l(-1) \to 0$$

in other words

$$(E_{n,m})_{|l}(-q) = pr_{2*}O_{pr_2^{-1}(l)}(\frac{\epsilon}{n}) = [S^\epsilon U \otimes O_l] \oplus [S^{n-(\epsilon+2)}U \otimes O_l(-1)]$$

$\square$

6.1.7 Approach with representation theory

Now we give a second proof of the splitting of the Schwarzenberger bundle, considering the geometry of $\mathbb{P}(S^nU)$. The isomorphisms

$$S^n\mathbb{P}(U) \simeq \mathbb{P}(S^nU) \simeq \mathbb{P}(S^nU^*)$$

are geometrically described by

$$u_1^{\otimes a_1}\dots u_k^{\otimes a_k} \to \cap_i T_{u_i}^{n-a_i} \to <\dots, T_{u_i}^{a_i-1}, \dots>$$

where $\sum_{i=1}^k a_i = n$ Let us fix $r \geq n$, a rational normal curve

$$C_r \subset P^r \tag{6.2}$$

and an isomorphism

$$\mathbb{P}(U) \simeq C_r \tag{6.3}$$

then for any point of $\mathbb{P}(S^nU)$ we get n points with multiplicity of C_r, hence a natural morphism

$$i\colon \mathbb{P}(S^nU) \to Gr(\mathbb{P}^{n-1}, \mathbb{P}^r)$$

The Schwarzenberger bundle $E_{n,r}$ on $\mathbb{P}(S^nU)$ is isomorphic to $i^*\mathcal{U}^*$ where $\mathcal{U}$ is the universal bundle on the Grassmannian.
Since $\mathcal{U}$ is homogeneous, the isomorphism class of $E_{n,r}$ does not depend on the choices (6.2) and (6.3) .
We consider a line r through the points f and g corresponding to two polynomials S^nU^* without common factors. Let $k+1 = qn+\epsilon$ with $0 \leq \epsilon < n$.
We want to give a second proof of the theorem [ST]

Proof. Since $c_1(E_{n+k}{}^n) = k+1$ it is sufficient to prove that $E_{n+k}{}^n_{|r}$ is isomorphic to the direct sum of some copies of $\mathcal{O}(m)$ and $\mathcal{O}(m+1)$ for a certain integer m. Indeed $m \cdot (n-e) + (m+1) \cdot e = k+1$ implies $m = q$ and $e = \epsilon$. It is sufficient to prove that

$$h^0(r, E_{n+k}{}^n_{|r}(t)) \cdot h^1(r, E_{n+k}{}^n_{|r}(t)) = 0 \quad \forall t \in \mathbf{Z}$$

¿From the sequence of the previous theorem it follows that $h^0(r, E_{n+k}{}^n_{|r}(t)) = 0$ for $t < 0$. For $t \geq 0$, $H^0(r, E_{n+k}{}^n_{|r}(t))$ and $H^1(r, E_{n+k}{}^n_{|r}(t))$ are respectively kernel and cokernel of the linear map

$$H^0(r, S^{n+k}U \otimes \mathcal{O}(t)) \to H^0(r, S^k U \otimes \mathcal{O}(t+1))$$

Then it is sufficient to prove that the previous map has maximal rank for $t \geq 0$.
Let us consider the dual map

$$S^k U^* \otimes S^{t+1}(f,g) \to S^{n+k} U^* \otimes S^t(f,g) \tag{6.4}$$

which is described by

$$\alpha \otimes f^i g^{t+1-i} \mapsto \alpha f \otimes f^{i-1} g^{t+1-i} + \alpha g \otimes f^i g^{t-i}$$

The map (6.4) is the cohomology H^0-map associated to the sheaf morphism on $\mathbb{P}(U)$

$$\mathcal{O}(k) \otimes S^{t+1}(f,g) \to \mathcal{O}(n+k) \otimes S^t(f,g) \tag{6.5}$$

with matrix

$$\begin{bmatrix} f & g & & \\ & \ddots & \ddots & \\ & & f & g \end{bmatrix}$$

Since f and g have no common factors it follows that the previous matrix has maximal rank on every point of $\mathbb{P}(U)$. Then the sheaf morphism (6.5) on $\mathbb{P}(U)$ is surjective with kernel $\mathcal{O}(k - n(t+1))$, and the associated H^0-map (6.4) has maximal rank because for every $a \in \mathbb{Z}$ $h^0(\mathbb{P}(U), \mathcal{O}(a)) \cdot h^1(\mathbb{P}(U), \mathcal{O}(a)) = 0$. □

6.1.8 Splitting on any line

6.18. Corollary. *Let r be the line through f and g with deg $GCD(f,g) = j$. Let $k+1 = q'(n-j) + \epsilon'$ with $0 \leq \epsilon' < n-j$. Then*

$$E_{n+k}{}^n_{|r} \simeq \mathcal{O}^j \oplus \mathcal{O}(q')^{n-\epsilon'} \oplus \mathcal{O}(q'+1)^{\epsilon'}$$

Proof. Let $h = GCD(f,g)$. We remark that $E_{n+k}{}^n$ restricted to T_h^{n-j} has a trivial summand of rank j (this follows from the geometrical description because the n-ples of points on C_{n+k} have j fixed points) and on the complementary summand we can apply the theorem. □

6.3. Exercise. *(i) Prove that the loci in the Grassmannian of lines in $\mathbb{P}^n$ where the splitting of $E_{n+k}{}^n$ is the same are $SL(2)$-invariant.*
(ii) Prove that under the splitting in Corollary 6.18 we have

$$E_{n+k}{}^n_{|r} \simeq [\mathcal{O} \otimes \mathbb{C}^j] \oplus [\mathcal{O}(q') \otimes S^{n-\epsilon'-1}U] \oplus [\mathcal{O}(q'+1) \otimes S^{\epsilon'-1}U]$$

6.2 Characterization of Schwarzenberger bundles among Steiner via the symmetry group

In this part we give a very short proof (based on Clebsch-Gordon problem for $SL(2,\mathbb{C})$-modules) of the following result

A rank n Steiner bundle on $\mathbb{P}^n$ which is $SL(2,\mathbb{C})$ invariant is a Schwarzenberger bundle.

We denote by S_i the irreducible $SL(2,\mathbb{C})$-representations of degree i and by $(x^{i-k}y^k)_{k=0,\cdots,i}$ a basis of S_i.

6.19. Theorem. *Let V, I and W be three non trivial $SL(2,\mathbb{C})$-modules with dimension $n+1$, $m+1$ and $n+m+1$ and $\phi \in \mathbb{P}(V\otimes I\otimes W)$ an invariant hyperplane under $SL(2,\mathbb{C})$. Then,*

$$\phi \notin (\mathbb{P}(V)\times\mathbb{P}(I)\times\mathbb{P}(W))^{\vee} \Leftrightarrow \phi \text{ is the multiplication } S_n\otimes S_m \to S_{n+m}$$

Proof. When $\phi\in\mathbb{P}(S_n\otimes S_m\otimes S_{n+m})$ is just the multiplication $S_n\otimes S_m\to S_{n+m}$ we have seen that it corresponds to Schwarzenberger bundles.

Conversely, let $V=\oplus_{i\in I}(S_i\otimes U_i)$, $B=\oplus_{j\in J}(S_j\otimes V_j)$ where U_i, V_j are trivial $SL(2,\mathbb{C})$-representations of dimension n_i and m_j. Let $x^i\in S_i$, $x^j\in S_j$ be two highest weight vectors and $u\in U_i$, $v\in V_j$. Since $Det(\phi)\neq 0$, $\phi((x^i\otimes u)\otimes(x^j\otimes v))\neq 0$ and by $SL(2,\mathbb{C})$-invariance $\phi((x^i\otimes u)\otimes(x^j\otimes v)) = x^{i+j}\phi(u\otimes v)\in S_{i+j}\otimes W_{i+j}$. By hypothesis $\phi(u\otimes v)\neq 0$ for all $u\in U_i$ and $v\in V_j$ so, by Theorem 6.3, it implies that $\dim W_{i+j}\geq n_i+m_j-1$, and $S_{i+j}^{n_i+m_j-1}\subset C^*$.

Assume now that B contains at least two distinct irreducible representations. Let i_0 and j_0 the greatest integers in I and J. We consider the submodule B_1 such that $B_1\oplus S_{j_0}^{m_{j_0}}=B$. Then the restricted map $V\otimes B_1\to C^*$ is not surjective because the image is concentrated in the submodule C_1^* of C^* defined by $C_1^*\oplus S_{i_0+j_0}^{n_{i_0}+m_{j_0}-1}=C^*$. Now since

$$\dim_{\mathbb{C}}(W_1)<\dim_{\mathbb{C}}(V)+\dim_{\mathbb{C}}(I_1)-1$$

there exist $a\in V$, $b\in I_1\subset B$ such that $\phi(a\otimes b)=0$. A contradiction with the hypothesis $Det(\phi)\neq 0$.

So $V=S_i^{n_i}$, $I=S_j^{m_j}$ and $S_{i+j}^{n_i+m_j-1}\subset W^*$. Since $\dim_{\mathbb{C}}W=\dim_{\mathbb{C}}V+\dim_{\mathbb{C}}I-1$, we have $(i+1)n_i+(j+1)m_j-1=\dim_{\mathbb{C}}W\geq(i+j+1)(n_i+m_j-1)$ which is possible if and only if $n_i=m_j=1$ and $W=S_{i+j}$. $\square$

6.20. Corollary. *A rank n Steiner bundle on $\mathbb{P}^n$ which is $SL(2,\mathbb{C})$ invariant is a Schwarzenberger bundle.*

Proof. Let S a rank n Steiner bundle on $\mathbb{P}^n$, i.e S appears in an exact sequence

$$0 \longrightarrow S \longrightarrow W\otimes O_{\mathbb{P}(V)} \longrightarrow I^*\otimes O_{\mathbb{P}(V)}(1) \longrightarrow 0$$

where $\mathbb{P}(V)=\mathbb{P}^n$, $\mathbb{P}(I)=\mathbb{P}^m$ and $\mathbb{P}(W)=\mathbb{P}^{n+m}$. If $SL(2,\mathbb{C})$ acts on S the vector spaces V, I and W are $SL(2,\mathbb{C})$-modules since V is the basis, $I^*=H^1S(-1)$ and $W^*=H^0(S^*)$. If S is $SL(2,\mathbb{C})$-invariant the linear surjective map

$$V\otimes(H^1S(-1))^*\to H^0(S^*)$$

is $SL(2,\mathbb{C})$-invariant too. $\Box$

Remark. The proofs of the theorem and the proposition, given here, are still valid for more than three vector spaces when the format is the boundary format.

6.2.1 Cosections and Poncelet's curves

Let $s \in H^0(E_{n,n+m})$ be a non zero section. We want to describe the zero locus $Z(s)$ geometrically. Since we have the following resolution

$$0 \longrightarrow S^m U \otimes O_{\mathbb{P}S^nU}(-1) \longrightarrow S^{n+m}U \otimes O_{\mathbb{P}S^nU} \longrightarrow E_{n,n+m} \longrightarrow 0$$

we obtain $H^0(E_{n,n+m}) = S^{n+m}U$. Thus we can see that this section s correspond to an hyperplane $H_s \subset \mathbb{P}(S^{n+m}U)$ or to an effective divisor of degree $n+m$ on the rational curve C_{n+m}. The section s, or to be precise the cosection

$$(E_{n,n+m})^* \xrightarrow{s} \mathcal{O}_{\mathbb{P}(S^nU)} \longrightarrow \mathcal{O}_{Z(s)} \longrightarrow 0$$

induces a rational map $\mathbb{P}(S^nU) \longrightarrow \mathbb{P}((E_{n,n+m})^*)$ which is not defined over the zero-scheme $Z(s)$. First of all we remind that over a point $x \in \mathbb{P}(S^nU)$ we have

$$\pi^{-1}(x) = \{(\sum_{i=0}^{i=n} x_i X_{i+j} = 0)_{j=0,\cdots,m}\} = \mathbb{P}(E_{n,n+m}(x)) = \mathbb{P}^{n-1}$$

The rational map $\mathbb{P}(S^nU) \longrightarrow \mathbb{P}((E_{n,n+m})^*)$ sends a point $x \in \mathbb{P}(S^nU)$ onto $H_s \cap \pi^{-1}(x)$ which is in general a $\mathbb{P}^{n-2}$, i.e a point in $\mathbb{P}((E_{n,n+m})^*(x)) = \mathbb{P}^{n-1*}$. This map is not defined when $\pi^{-1}(x) \subset H_s$. The hyperplane H_s cuts the rational curve C_{n+m} along an effective divisor D_{n+m} of length $n+m$. The subschemes $D_n \subset D_{n+m}$ of length n (they are $\binom{n+m}{n}$ when D_{n+m} is smooth) generate the $(n-1)$-planes n secant to C_{n+m} which are contained in H_s. The corresponding point belongs to the zero-scheme $Z(s)$.

Remind that we have $E_{n,n+m} = pr_{1*}pr_2^*\mathcal{O}_{C_n^*}(\frac{n+m}{m})$ (prop. 6.9). Then to a section $s \in H^0(E_{n,n+m})$ we can associate a divisor D_{n+m} of degree $n+m$ on C_n^*. The zero-scheme $Z(s)$ is the set of points $x \in \mathbb{P}(S^nU)$ such that the divisor $x^* \cap C_n^*$ of degree n belongs to D_{n+m}. When D_{n+m} is smooth, we get $n+m$ osculating hyperplanes of C_n in $\mathbb{P}(S^nU)$. Every subset of n-osculating hyperplanes gives a point in $\mathbb{P}(S^nU)$. These points are the zero-scheme of the section s. We have proved the following proposition.

6.21. Proposition. *Let $s \in H^0(E_{n,n+m})$ be a non zero section and D_{n+m} be the corresponding effective divisor of degree $n+m$ on C_n^*. We denote by $Z(s)$ the zero-scheme of s. Then,*

$$x \in Z(s) \Leftrightarrow x^* \cap C_n^* \subset D_{n+m}(s)$$

More generally we have

6.22. Proposition. *Let $s \in H^0(E_{n,n+m})$ be a non zero section and D_{n+m} be the corresponding effective divisor of degree $n+m$ on C_n^*. We denote by $Z(s)$ the zero-scheme of s. Then,*

$$\mathcal{I}_{Z(s)} \subset \mathfrak{m}_x^{r+1} \Leftrightarrow (x^*)^{r+1} \cap C_n^* \subset D_{n+m}(s)$$

Proof. Assume $\mathcal{I}_{Z(s)} \subset \mathfrak{m}_x^{r+1}$. Let H_s the hyperplane corresponding to the section s and D_n the divisor on C_n^* corresponding to x. Then $< (r+1)D_n >^* \subset H_s$. This proves $(x^*)^{r+1} \cap C_n^* \subset D_{n+m}(s)$.

On the other hand, the inclusion $< (r+1)D_n >^* \subset H_s$ proves that the exceptional divisor $< D_n >^*$ of $\mathbb{P}(\mathcal{I}_{Z(s)})$ appears with multiplicity $(r+1)$, it means that $\mathcal{I}_{Z(s)} \subset \mathfrak{m}_x^{r+1}$.

In this short part we want to explain the link between two geometric objects, the pencils of section of Schwarzenberger bundles and the Poncelet curves.

To do such a link, it is necessary to describe more explicitly the zero-scheme of any (non zero) section $s \in H^0 E_{2,n}$.

A non zero section $s \in H^0 E_{2,n}$ gives a exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^2} \xrightarrow{s} E_{2,n} \longrightarrow \mathcal{I}_{Z(s)}(n-1) \longrightarrow 0$$

6.23. Lemma. *Let $s \in H^0 E_{2,n}$ be a non zero section and $Z(s)$ its zero-scheme. If L is a tangent line to C_2 secant to $Z(s)$, then L is $(n-1)$-secant to $Z(s)$.*

Proof. Since L is tangent to C_2, we get a surjective homomorphism $E_{2,n} \to O_L$ which proves that $(E_{2,n})_{|L} = O_L(n-1) \oplus O_L$. The exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^2} \xrightarrow{s} E_{2,n} \longrightarrow \mathcal{I}_{Z(s)}(n-1) \longrightarrow 0$$

induces a surjective map $(E_{2,n})_{|L} \to \mathcal{I}_{L\cap Z(s)/L}(n-1)$ (where $\mathcal{I}_{L\cap Z(s)/L}$ is the ideal sheaf of the scheme $L \cap Z(s)$ in L). If L meets $Z(s)$, it implies $\mathcal{I}_{L\cap Z(s)/L}(n-1) = O_L$, in other words L is $n-1$ secant to $Z(s)$. □

6.24. Proposition. *Let $s \in H^0 E_{2,n}$ be a non zero section and $Z(s)$ be its zero-scheme. Let $D_n(s) = \sum n_i L_i^\vee$ the corresponding effective divisor of degree n on C_2 :*
1) The support of $Z(s)$ consists in the set :
a) $x_{ij} = L_i \cap L_j$ if $i \neq j$
b) $x_{ii} = L_i \cap C^\vee$ if $n_i \geq 2$.
2) Let $Z(s) = \cup Z_{ij}$ where Z_{ij} is the subscheme of $Z(s)$ supported by x_{ij}. Then we have :
a) $Z_{ii} = Z(s_i)$ where $s_i \in H^0(E_{2,n_i})$ is the section corresponding to the effective divisor $D_{n_i} = n_i L_i^\vee$.
b) $Z_{ii} \cup Z_{ij} \cup Z_{jj} = Z(s_{ij})$ where $s_{ij} \in H^0(E_{2,n_i+n_j})$ is the section corresponding to the effective divisor $D_{n_i+n_j} = n_i L_i^\vee + n_j L_j^\vee$.

Proof. The point $x \in \mathbb{P}^2$ belongs to $Z(s)$ if and only if the line $x^\vee$ of $\mathbb{P}^{2\vee}$ is twosecant to $D_n(s)$ (see prop 6.22). But the lines which are two-secant to $D_n(s)$ are evidently the lines $x_{ij}^\vee$ joining the points $L_i^\vee$ and $L_j^\vee$ of C_2^* for $i \neq j$ and the lines tangent to C_2* in a point $L_i^\vee$ such that $2L_i^\vee \in D_n(s)$. It proves 1).

For the second part, we remark that Z_{ii} is the two-secant subscheme of the divisor $D_{n_i} = n_i L_i^\vee$ and that $Z_{ii} \cup Z_{ij} \cup Z_{jj}$ is the two-secant subscheme of the divisor $D_{n_i+n_j} = n_i L_i^\vee + n_j L_j^\vee$. These divisors correspond to the sections $s_i \in H^0(E_{2,n_i})$ and $s_{ij} \in H^0(E_{2,n_i+n_j})$. □

6.25. Remark. *We assume that $n > 2$. $D_n(s)$ is smooth if and only if $Z(s)$ is smooth, in that case $Z(s)$ consists of the vertices of the n-lines (distincts) tangent to C_2.*

More precisely we have :

6.26. Corollary. $\deg(O_{Z_{ii}}) = n_i(n_i - 1)/2$, $\deg(O_{Z_{ij}}) = n_i n_j$.
$\deg(O_{L_i \cap Z_{ii}}) = n_i - 1$, $\deg(O_{L_i \cap Z_{ij}}) = n_j$.

Proof. $\deg(O_{Z_{ii}}) = c_2(E_{2,n_i}) = n_i(n_i - 1)/2$.
$\deg(O_{Z_{ij}}) = c_2(E_{2,n_i+n_j}) - \deg(O_{Z_{ii}}) - \deg(O_{Z_{jj}}) = n_i n_j$.
The line L_i is $n_i - 1$-secant to Z_{ii} because it is a tangent to C_2 meeting the zero-scheme of the section $s_i \in H^0(E_{2,n_i})$ (Remark 6.23). In the same way, the line L_i is $(n_i + n_j - 1)$-secant to $Z(s_{ij})$. Since it is $n_i - 1$-secant to Z_{ii} it is n_j secant to Z_{ij}. □

The following definition of Poncelet curves is given by Trautmann ([Tr], def).

6.27. Definition. *A curve $S \subset \mathbb{P}^2$ of degree $(n-1)$ will be called Poncelet related to C_2 if there is a pencil Λ of effective degree-n divisors on C_2 such that for any two points of a divisor of Λ the tangents of C_2 in these points meet on S.*

The above definition does not consider the case where the pencil contains an effective divisor of degree $n > 2$ concentrated on a point. Let us consider a pencil Λ of effective divisors of degree n on $C_2^* \subset \mathbb{P}^{2*}$ such that for every divisor D_n of Λ, the scheme of the two-secant to D_n is contained in S. This is equivalent to the above definition when the general divisor of the pencil is smooth. Then we could define Poncelet's curves when the general divisor is not smooth. Thus we will use the following definition.

6.28. Definition. *A curve $S \subset \mathbb{P}^2$ of degree $(n-1)$ is Poncelet related to C_2 if and only if S is the determinant of a pencil of sections of de $E_{2,n}$.*

Thanks to this observation we give a short proof to the following classical theorem due to Darboux.

6.29. Theorem. *(Darboux) Let $S \subset \mathbb{P}^2$ a curve of degree $(n-1)$ and C a smooth conic of $\mathbb{P}^{2\vee}$. If there exists a effective divisor D_n, of degree n, on C such that the scheme of two-secant is contained in S, then S is a Poncelet related curve to C^*.*

Proof. The divisor D_n is defined by a section of $\mathcal{O}_C(\frac{n}{2})$. It induces a section $s \in H^0(E_n)$ (with $E_n = pr_{1*}pr_2^*\mathcal{O}_C(\frac{n}{2})$) such that the zero-scheme $Z(s)$ is the scheme of the two-secant to D_n :

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^2} \xrightarrow{s} E_{2,n} \longrightarrow \mathcal{I}_{Z(s)}(n-1) \longrightarrow 0$$

The curve S is a section of $\mathcal{I}_{Z(s)}(n-1)$. Then it exists a non-zero section t of $E_{2,n}$ such that $s \wedge t = 0$ is the equation of S. □

Chapter 7

First examples of moduli of bundles

7.1 Basic definitions and properties

We refer to [HuLe] for a detailed treatment of moduli spaces of bundles. Here we want only to mention the basic definitions.

7.1. Definition. *A flat family of bundles on X parametrized by S is given by a scheme F together with a flat morphism $F \to S \times X$. Two such families $F_1 \xrightarrow{p_1} S \times X$, $F_2 \xrightarrow{p_2} S \times X$ are called equivalent if there is $L \in Pic(S)$ such that $F_1 = \pi^* L \otimes F_2$, where π is the first projection.*

All bundles in a flat family have the same Chern classes and the same rank. From now on we fix Chern classes $c_i \in A^i(X)$ and rank $r \in \mathbb{N}$. When $X = \mathbb{P}^n$ the c_i can be considered as integers.
A moduli space for bundles with fixed c_i and r is a scheme that intuitively parametrizes all possible bundles E such that $c_i(E) = c_i$ and $rank(E) = r$. A problem arises because there are in general nontrivial flat families of arbitrary large dimension. This problem has been solved by considering only some bundles called stable bundles, that in turn are a special case of the stable (torsion free) sheaves. The following definition of stability was given by Mumford in order to satisfy in a certain setting the stability requirement of GIT. We state the definition for an arbitrary ample line bundle L on X, but the reader interested only to the case $X = \mathbb{P}^n$ can take $L = \mathcal{O}(1)$.

7.2. Definition. *A bundle E on X is called stable (resp. semistable) with respect to a ample line bundle L if for every subsheaf F with $0 < rank(F) < rank(E)$ we have*

$$\frac{c_1(F) \cdot L^{n-1}}{rank(F)} < (resp.\ \leq) \frac{c_1(E) \cdot L^{n-1}}{rank(E)}$$

The expression $\frac{c_1(E) \cdot L^{n-1}}{rank(E)}$ is called the slope of E and it is denoted by $\mu(E)$.

On $\mathbb{P}^n$ $c_1(E)$ is an integer and we have the simpler expression

$$\mu(E) = \frac{c_1(E)}{rk(E)}$$

If a subsheaf $F \subset E$ has $\mu(F) \geq \mu(E)$ we say that F destabilizes E.

7.1. Exercise. *Given the exact sequence*

$$0 \to E \to F \to G \to 0$$

prove that $\mu(F)$ is strictly included between $\mu(E)$ and $\mu(G)$ with the only exception $\mu(E) = \mu(F) = \mu(G)$.

7.2. Exercise. *Given a bundle E prove that $\mu(\wedge^k E) = \mu(S^k E) = k\mu(E)$. Hint: Use the splitting principle 5.3.1.*

7.3. Exercise. *Given a bundle E on $\mathbb{P}^n$ prove that for any integer t*

$$\mu(E(t)) = \mu(E) + t$$

7.3. Definition. *Let $Pic(X) = \mathbb{Z}$, so that c_1 can be considered as a integer. A bundle E on X is called normalized if $c_1(E) \in \{-(r-1), \ldots, -1, 0\}$ (this condition is satisfied by $E(t)$ for a unique t, we denote by E_{norm} this unique twist of E).*

7.4. Remark. *E is normalized if and only if $-1 < \mu(E) \leq 0$.*

The following criterion is useful in order to check the stability

7.5. Proposition. (Hoppe) *Let $Pic(X) = \mathbb{Z}$. Let E be a bundle of rank r. If $H^0\left(\left(\wedge^k(E)\right)_{norm}\right) = 0$ for $1 \leq k \leq r$ then E is stable.*

Sketch of proof Consider a subsheaf F of E of rank k. From $0 \to F \to E$ we get $0 \to (\wedge^k F)^{**} \to \wedge^k E$. Now $(\wedge^k F)^{**}$ is a line bundle (see [OSS]), hence $(\wedge^k F)^{**} = \mathcal{O}(k\mu(F))$. It follows that there is a section of $\wedge^k E(-k\mu(F))$, hence $\mu\left(\wedge^k E(-k\mu(F))\right) > 0$, that is $\mu(F) < \mu(E)$ as we wanted.

7.6. Remark. *The converse of the above proposition does not hold. A counterexample is given by the nullcorrelation bundle N (see Section 7.4) on $\mathbb{P}^3$ having $c_1(N) = 0$, $c_2(N) = 2$, which is stable but contains $\mathcal{O}$ as direct summand of $\wedge^2 N$ so that $h^0(\wedge^2 N) \neq 0$.*

An interesting application of the above proposition is the following

7.7. Proposition. *A Steiner bundle of rank n on $\mathbb{P}^n$ is stable.*

Proof. Consider a bundle E on $\mathbb{P}(V)$ appearing in the sequence

$$0 \longrightarrow I \otimes \mathcal{O}(-1) \longrightarrow W \otimes \mathcal{O} \longrightarrow E \longrightarrow 0$$

where $\dim V = n+1$, $\dim I = k$, $\dim W = n+k$. Since $\left(\wedge^k(E)\right)_{norm} = \left(\wedge^k(E)\right)(t)$ for some $t \leq 1$ then it is sufficient to prove that $h^0\left(\left(\wedge^k(E)\right)\right)(-1) = 0$. Consider the $q-th$ exterior power, twisted by $\mathcal{O}(-1)$

$$0 \longrightarrow S^q I \otimes \mathcal{O}(-q-1) \longrightarrow \ldots \longrightarrow I \otimes \wedge^{q-1} W \otimes \mathcal{O}(-2) \longrightarrow \wedge^q W \otimes \mathcal{O}(-1) \longrightarrow (\wedge^q E)(-1) \longrightarrow 0$$

Taking cohomology the result follows. $\square$

7.4. Exercise. *Prove that a stable normalized bundle satisfies $h^0(E) = 0$*

7.5. Exercise. *Prove that $E \oplus F$ is semistable if and only if E and F are both semistable with the same slope. Prove that stable bundles are always indecomposable.*

7.6. Exercise. *Prove that if E is stable then E^* is stable. Prove that E is stable if and only if $E \otimes L$ is stable for some line bundle L.*

7.8. Remark. *If E and F are stable then $E \otimes F$ is polystable, i.e. is the direct sum of stable bundles with the same slope. This fact is quite deep and can be proved by using Einstein metrics. Maruyama proved in [Mar1] that if E, F are semistable then $E \otimes F$ is semistable.*

7.9. Definition. *A bundle is called simple if $H^0(EndE) = \mathbb{C}$ that is if its only endomorphisms are homotheties.*

7.10. Theorem. *Stable bundles are simple*

Proof. Let $f\colon E \to E$. Fix a point $x \in X$, there there is an eigenvalue λ for $f_x\colon E_x \to E_x$. It follows that $f - \lambda I$ is not a isomorphism. Suppose that it is not zero. Then $\ker(f - \lambda I)$ and $im(f - \lambda I)$ have both rank strictly included in $[0, r]$. By the exercise 7.1 one of these destabilizes E. $\square$

Let us fix c_i and r
There is a basic functor $F_{c_i,r}$ from the category of schemes to the opposite (with arrows reversed) category of sets, namely (for simplicity we omit the suffix $\{c_i, r\}$).

$$F\colon Schemes \to (Sets)^0$$

$$S \mapsto \{equivalence\ classes\ of\ flat\ families\ of\ stable\ bundles\ over\ S\ with\ fixed\ c_i, r\}$$

7.11. Definition. *M is called a* **coarse moduli space** *if there is transformation of functors*

$$F(-) \to Hom(-, M)$$

such that
i) for every M' with a transformation of functors $F(-) \to Hom(-, M')$ there is a unique $\pi\colon M \to M'$ such that the following diagram commutes

$$\begin{tikzcd} F \arrow[r] \arrow[dr] & Hom(-,M) \arrow[d] \\ & Hom(-,M') \end{tikzcd}$$

ii) there is a biunivoc correspondence between reduced points of M and stable bundles on X with assigned c_i and r.

M satisfying the minimality condition i) is called to corepresent F.

7.12. Definition. *F is called represented by M if $F(-) = Hom(-, M)$. In this case M is called a* **fine moduli space**.

This implies (why?) that there exists a flat family P parametrized by M such that all other flat families are obtained by this one as pullback. P is called the Poincaré bundle. A fine moduli space is also a coarse moduli space.
Fine moduli spaces do not exist in general, but only in special cases. A famous theorem of Maruyama states that coarse moduli spaces always exist for projective X, and they are even compactified by a projective scheme $M \subset \overline{M}$ adding equivalence classes of semistable sheaves.
Moduli spaces of G-bundles
In general the transition functions of bundles take value in GL. If the bundle carry a symmetric nondegenerate bilinear form (i.e. $\omega \colon E \to E^*$ such that $\omega^* = \omega$ then we can consider transition functions that leave ω invariant, i.e. they lie in SO or in its universal covering $Spin$. E becomes a $Spin$-bundle in this way. When the transition functions take value in G we say that E is a G-bundle and G is called the structural group of the bundle.It makes sense to consider flat families of G-bundles where $G \subset GL$ and we have the analogous notion of moduli space parametrizing G-bundles.

The tangent space at $[E]$

The tangent space of the coarse moduli space M of G-bundles on X with assigned c_i and r is isomorphic to the cohomology group $H^1(adE)$. Here adE is the adjoint bundle defined by the adjoint representation $G \to adG$. In general $G = GL$ and correspondingly $adE = EndE$. In case $G = SL$ we have $adE = EndE/\mathcal{O}$. In case $G = Spin$ or $G = SO$ we have $adE = \wedge^2 E$. In case $G = Sp$ we have $adE = S^2 E$.
More precisely there is a Kuranishi map, coming from deformation theory

$$H^1(adE) \xrightarrow{k} H^2(adE)$$

such that the holomorphic germ of M at $[E]$ is defined by $k = 0$. It follows that if $H^2(adE) = 0$ then $[E]$ is a smooth point and the estimate $h^1(adE) - h^2(adE) \le \dim_E M \le h^1(adE)$.
If E is only a stable sheaf, it belongs to the Maruyama moduli space of GL-bundles and the tangent space at E is isomorphic to $Ext^1(E, E)$. Moreover there is a Kuranishi morphism $Ext^1(E, E) \to Ext^2(E, E)$ with the same properties as in the bundle case.

7.2 Minimal resolutions

For any torsion free sheaf E over $\mathbb{P}^n$ there is a **minimal resolution**

$$0 \to F_{n-1} \to \ldots \to F_1 \to F_0 \to E \to 0$$

where every F_i is a direct sum of line bundles. The minimal resolution remains exact when we perform the H^0-sequence

$$0 \to H^0(F_{n-1}(t)) \to \ldots \to H^0(F_1(t)) \to H^0(F_0(t)) \to H^0(E(t)) \to 0$$

for every $t \in \mathbb{Z}$. Moreover the minimality condition requires that no line bundle can be dropped in two adjacents F_i and F_{i-1}, this is equivalent to the fact that no constant entry appears in the matrices representing the morphisms of the resolution. These properties characterize uniquely the minimal resolution.

In particular on $\mathbb{P}^2$ every torsion free sheaf E has a minimal resolution

$$0 \to \oplus\mathcal{O}(-a_i) \to \oplus\mathcal{O}(-b_j) \to E \to 0$$

We consider the case $rkE = 2$. We assume

$$a_1 \leq \ldots \leq a_k$$
$$b_1 \leq \ldots \leq b_{k+2}$$

¿From [DM] or [1] it follows that the generic E the a_i, b_j can take at most three different values and that $\max_j\{b_j\} - \min_i\{a_i\} \leq 2$.
We want to give some bounds on a_i, b_j. We first need a technical lemma.

7.13. Lemma. *Let* $0 \leq b_1 \leq b_2$ *be integer numbers. Then*
(i) $b_1^2 + b_2^2 - 2b_2(b_1 + b_2 - 1) - 1 \leq 0$
Let $1 \leq b_1 \leq b_2$ *be integer numbers. Then*
(ii) $b_1^2 + b_2^2 - 2b_2(b_1 + b_2 - 1) \leq 0$
(iii) $b_1^2 + b_2^2 - 2b_2(b_1 + b_2 - 2) - 2 \leq 0$

Proof (i) and (ii) are straightforward looking at the corresponding hyperbolas in the plane. (iii) is trivial by the factorization

$$b_1^2 + b_2^2 - 2b_2(b_1 + b_2 - 2) - 2 = (b_1 - b_2(1 + \sqrt{2}) + \sqrt{2})(b_1 - b_2(1 - \sqrt{2}) - \sqrt{2})$$

□

7.14. Proposition. *(i)* $c_1(E) = \sum_{i=1}^k a_i - \sum_{i=1}^{k+2} b_i$
(ii) $2c_2(E) - c_1^2(E) = \sum_{i=1}^k a_i^2 - \sum_{i=1}^{k+2} b_i^2$
(iii) $b_{i+2} < a_i$ *for* $i = 1 \ldots k$

Proof (i) and (ii) are straightforward. (iii) is proved in [BS]. □

7.15. Proposition. *Let* $c_1(E) = 0$. *E is stable (resp. semistable) if and only if* $b_1 \geq 1$ *(resp.* $b_1 \geq 0$*).*
Let $c_1(E) = -1$. *E is stable if and only if it is semistable if and only if* $b_1 \geq 1$ *(resp.* $b_1 \geq 0$*).*

Proof $b_1 \geq 1$ is equivalent to $H^0(E) = 0$. $b_1 \geq 0$ is equivalent to $H^0(E(-1)) = 0$. □
The following propositions are taken from the thesis [D], with some improvements.

7.16. Proposition. *If E is normalized and semistable then*

$$b_{k+2} + \frac{k-1}{2} \leq c_2$$

If moreover E is stable and $c_1(E) = 0$ *then*

$$b_{k+2} + \frac{k}{2} \leq c_2$$

Proof Assume E semistable and $c_1(E)=0$. Then

$$2c_2 = -b_1^2 - b_2^2 + \sum_{i=1}^{k}(a_i^2 - b_{i+2}^2) = -b_1^2 - b_2^2 + \sum_{i=1}^{k}(a_i - b_{i+2})(a_i + b_{i+2} - 2b_2) + 2b_2(b_1 + b_2) \geq$$

$$\geq -b_1^2 - b_2^2 + \sum_{i=1}^{k}(a_i + b_{i+2} - 2b_2) + 2b_2(b_1 + b_2) \geq$$

(by using $a_i \geq b_{i+2} + 1$)

$$\geq -b_1^2 - b_2^2 + 2\sum_{i=1}^{k}(b_{i+2} - b_2) + k + 2b_2(b_1 + b_2) \geq$$

$$\geq -b_1^2 - b_2^2 + 2b_{k+2} - 2b_2 + k + 2b_2(b_1 + b_2) \geq 2b_{k+2} + k - 1$$

where in the last inequality we have used (i) of Proposition 7.13. The other cases are similar by using (ii) and (iii) of Proposition 7.13. $\square$

7.17. Proposition. *If E is normalized and semistable then*

$$a_k + \frac{k-3}{2} \leq c_2$$

If moreover E is stable and $c_1(E)=0$ then

$$a_k + \frac{k-2}{2} \leq c_2$$

Proof Assume E stable and $c_1(E)=0$.

$$2c_2 = -b_1^2 - b_2^2 + (a_k^2 - b_{k+2}^2) + \sum_{i=1}^{k-1}(a_i^2 - b_{i+2}^2) =$$

$$= -b_1^2 - b_2^2 + (a_k^2 - b_{k+2}^2) + \sum_{i=1}^{k-1}(a_i - b_{i+2})(a_i + b_{i+2} - 2b_2) + 2b_2(b_1 + b_2 + b_{k+2} - a_k) \geq$$

$$\geq -b_1^2 - b_2^2 + (a_k^2 - b_{k+2}^2) + \sum_{i=1}^{k-1}(a_i + b_{i+2} - 2b_2) + 2b_2(b_1 + b_2 + b_{k+2} - a_k) \geq$$

(by using $a_i \geq b_{i+2} + 1$)

$$\geq -b_1^2 - b_2^2 + (a_k^2 - b_{k+2}^2) + 2\sum_{i=1}^{k-1}(b_{i+2} - b_2) + (k-1) + 2b_2(b_1 + b_2 + b_{k+2} - a_k) \geq$$

$$\geq -b_1^2 - b_2^2 + (a_k - b_{k+2})(a_k + b_{k+2} - 2b_2) + (k-1) + 2b_2(b_1 + b_2)$$

If $a_k - b_{k+2} = 1$ we have the thesis from the previous proposition.

If $a_k - b_{k+2} \geq 2$ we get

$$2c_2 \geq -b_1^2 - b_2^2 + 2(a_k + b_{k+2} - 2b_2) + (k-1) + 2b_2(b_1 + b_2) \geq 2(a_k + b_{k+2} - b_2) + (k-1) \geq 2a_k + (k-1)$$

where we have used (ii) of Proposition 7.13. This is slightly stronger that we claimed. The other cases are similar by using (i) and (iii) of Proposition 7.13. □

Remark Given (c_1, c_2) of a normalized semistable torsion free sheaf of rank 2 on $\mathbb{P}^2$ there are only finitely many sequences of integers $a_1 \leq \ldots \leq a_k$, $b_1 \leq \ldots \leq b_{k+2}$ which satisfy the Proposition 7.14 and the inequalities of Proposition 7.15, Proposition 7.16 and Proposition 7.17. This is the first nontrivial case of what is called the boundedness theorem for semistable sheaves, proved first by Maruyama (see [HuLe]). We underline that without the semistability assumptions, it is possible to find infinitely many sequences of integers $a_1 \leq \ldots \leq a_k$, $b_1 \leq \ldots \leq b_{k+2}$ which satisfy only the Proposition 7.14.

Applications:

7.18. Theorem. *Let E be a torsion free sheaf of rank 2 over $\mathbf{P}^2$.*
If $(c_1, c_2) = (-1, 1)$ and E is semistable then there is a sequence

$$0 \longrightarrow \mathcal{O}(-2) \longrightarrow \mathcal{O}(-1)^3 \longrightarrow E \longrightarrow 0$$

If $(c_1, c_2) = (0, 2)$ and E is stable then there is a sequence

$$0 \longrightarrow \mathcal{O}(-2)^2 \longrightarrow \mathcal{O}(-1)^4 \longrightarrow E \longrightarrow 0$$

If $(c_1, c_2) = (0, 2)$ and E is strictly semistable then there is a sequence

$$0 \longrightarrow \mathcal{O}(-3) \longrightarrow \mathcal{O}(-2) \oplus \mathcal{O}(-1) \oplus \mathcal{O} \longrightarrow E \longrightarrow 0$$

If $(c_1, c_2) = (-1, 2)$ and E is (semi)stable then there is a sequence

$$0 \longrightarrow \mathcal{O}(-3) \longrightarrow \mathcal{O}(-2) \oplus \mathcal{O}(-1)^2 \longrightarrow E \longrightarrow 0$$

Proof $(c_1, c_2) = (-1, 1)$ and (semi)stable implies $1 \leq b_i \leq 1 - \frac{k-1}{2}$ hence $k = 1$, $b_1 = b_2 = b_3 = 1$ and $a_1 = -1 + \sum_{i=1}^3 b_i = 2$.

$(c_1, c_2) = (0, 2)$ and stable implies $1 \leq b_i \leq 2 - \frac{k}{2}$ hence $k \leq 2$, and $b_i = 1$. If $k = 1$ then $a_1 = 0 + \sum_{i=1}^3 b_i = 3$ and $a_1^2 - \sum_{i=1}^3 b_i^2 = 6$ which is a contradiction. Hence $k = 2$ and $a_1 + a_2 = 4$, $a_1^2 + a_2^2 = \sum_{i=1}^4 b_i^2 + 4 = 8$ which implies $a_1 = a_2 = 2$. If E is only semistable we have the weaker inequality $0 \leq b_i \leq 2 - \frac{k-1}{2}$ which gives $k \leq 5$. Moreover $1 \leq a_i \leq 2 - \frac{k-3}{2}$. If $a_1 = 1$ then $b_i = 0$ and $c_1(E) > 0$ which is a contradiction. Hence $a_1 \geq 2$ which gives $k \leq 3$. If $k = 3$ then $a_1 = a_2 = a_3 = 2$ which forces $\sum_{i=1}^5 b_i = 6$, $0 \leq b_i \leq 1$ which is a contradiction. If $k = 2$ then $a_1 = a_2 = 2$ and we find again the stable case. The last possibility is $k = 1$, hence $2 \leq a_1 \leq 3$ and $0 \leq b_i \leq 2$. The system $\sum_{i=1}^3 b_i = a_1$, $\sum_{i=1}^3 b_i^2 = a_1^2 - 4$ has the only solution $a_1 = 3$, $(b_1, b_2, b_3) = (0, 1, 2)$.

$(c_1, c_2) = (-1, 2)$ and (semi)stable implies $1 \leq b_i \leq 2 - \frac{k-1}{2}$ hence $k \leq 3$. If $k = 3$ then $b_i = 1$, $a_i \geq 2$ which contradicts $\sum_{i=1}^3 a_i = -1 + \sum_{i=1}^5 b_i = 4$. The same argument excludes the case $k = 2$. Hence $k = 1$ and $1 \le b_i \le 2$, $2 \le a_1 \le 3$. The system $\sum_{i=1}^{3} b_i = a_1 + 1$, $\sum_{i=1}^{3} b_i^2 = a_1^2 - 3$ has the only solution $a_1 = 3$, $(b_1, b_2, b_3) = (1, 1, 2)$. $\square$

Exercise With the notations above let $(c_1, c_2) = (0, 1)$ and E semistable. Prove that there is only the following minimal resolution

$$a_1 = 2, (b_1, b_2, b_3) = (0, 1, 1)$$

The following two exercises require more computations. They become straightforward by using a computer.
Exercise With the notations above let $(c_1, c_2) = (0, 3)$ and E semistable, prove first that $k \le 3$. Then the possible minimal resolutions are the following ones:

$$(a_1, a_2) = (3, 3)(b_1, b_2, b_3, b_4) = (0, 2, 2, 2)$$

$$(a_1, a_2) = (2, 3)(b_1, b_2, b_3, b_4) = (1, 1, 1, 2)(stable)$$

$$a_1 = 4, (b_1, b_2, b_3) = (0, 1, 3)$$

$$a_1 = 3, (b_1, b_2, b_3) = (1, 1, 1)(stable)$$

Exercise Let $(c_1, c_2) = (-1, 3)$ and E (semi)stable, prove first that $k \le 3$. Then the possible minimal resolutions are the following ones:

$$(a_1, a_2) = (3, 3)(b_1, b_2, b_3, b_4) = (1, 2, 2, 2)$$

$$a_1 = 4, (b_1, b_2, b_3) = (1, 1, 3)$$

7.3 Examples on $\mathbb{P}^2$

7.19. Theorem. *Moduli spaces of G-bundles over $\mathbb{P}^2$ are smooth.*

Proof. By Serre duality $h^2(EndE) = h^0(EndE(-3)) = 0$. $\square$

The Hirzebruch-Riemann-Roch theorem implies that for a rank 2 bundle on $\mathbb{P}^2$ we have

$$\chi(E) = \frac{1}{2}(c_1(E)^2 - 2c_2(E) + 3c_1(E) + 4) \tag{7.1}$$

$$\chi(EndE) = c_1(E)^2 - 4c_2(E) + 4$$

Hence if E is stable we get

$$h^1(EndE) = \dim_E M = -c_1^2(E) + 4c_2(E) - 3 \tag{7.2}$$

which is the dimension of the moduli space at E.
We will see the first examples of bundles of rank 2 on $\mathbb{P}^2$ and $\mathbb{P}^3$.

7.20. Theorem. *Let E be a stable rank 2 bundle on $\mathbb{P}^2$ with $c_1(E) = 0$. Then $c_2(E) \ge 2$.*

Proof. By the exercise 7.4 we have $h^0(E) = 0$. By Serre duality $h^2(E) = h^0(E^*(-3)) = h^0(E(-3)) = 0$. Hence $0 \leq h^1(E) = -\chi(E) =$ (by (7.1) $= \frac{1}{2}(2c_2(E) - 4)$ as we wanted. $\square$

7.7. Exercise. *Let E be a stable rank 2 bundle on $\mathbb{P}^2$ with $c_1(E) = -1$. Prove that $c_2(E) \geq 1$.*

We denote by $M_{\mathbb{P}^2}(c_1, c_2)$ the moduli space of stable 2-bundles on $\mathbb{P}^2$.
By the previous results $M_{\mathbb{P}^2}(0, c_2)$ is empty if $c_2 \leq 1$ and $M_{\mathbb{P}^2}(-1, c_2)$ is empty if $c_2 \leq 0$. Moreover $\dim M_{\mathbb{P}^2}(0, k) = 4k - 3$ for $k \geq 2$ and $\dim M_{\mathbb{P}^2}(-1, k) = 4k - 4$ for $k \geq 1$.

$\mathbf{M}_{\mathbb{P}^2}(\mathbf{-1}, \mathbf{1})$

7.21. Theorem. *$M_{\mathbb{P}^2}(-1, 1)$ is given by one point, it contains only the bundle $T\mathbb{P}^2(-2)$.*

We sketch two different proofs of the previous basic theorem. Although each of these two needs some tool not covered in these notes, we think they touch different interesting aspects of the theory that deserve to be deepened.
first proof By Proposition 7.5 $T\mathbb{P}^2(-2) \in M(-1, 1)$, which is then not empty. By (7.2) $\dim M(-1, 1) = 0$. It follows that $SL(3)$ acts trivially on $M(-1, 1)$, that is every $E \in M(-1, 1)$ is homogeneous. We have $\mathbb{P}^2 = SL(3)/P$ and it is now a standard fact that E comes from an irreducible representation of rank 2 of P. Since the semisimple part of P is $SL(2)$, there is only the standard representation which gives our bundle. $\square$
second proof By Theorem 7.18 $E \in M(-1, 1)$ appears in a sequence

$$0 \to \mathcal{O}(-2) \xrightarrow{A} \mathcal{O}(-1)^3 \to E \to 0$$

We have to prove that all bundles E appearing in the above sequence are isomorphic. In fact A is given in coordinates by (l_0, l_1, l_2) where $l_i = \sum_{j=0}^{2} a_{ij} x_j$. Now A is a constant rank map iff the three lines $\{l_i = 0\}$ have no common intersection, which means that the 3×3 matrix a_{ij} is nondegenerate. Since two nondegenerate matrices are equivalent by $GL(3)$-action the result follows. $\square$

7.22. Remark. *It is not known if the moduli space containing $T\mathbb{P}^n$ is a point for $n \geq 5$. This is true for $n \leq 4$.*

$\mathbf{M}_{\mathbb{P}^2}(\mathbf{-1}, \mathbf{2})$

We sketch now a proof that $M(-1, 2)$ is the projective space of symmetric matrices 3×3 of rank 2, that is $M(-1, 2)$ is isomorphic to $S^2\mathbb{P}^2 \setminus \Delta \simeq Sec(v_2) \setminus v_2$ where $v_2 := v_2(\mathbb{P}^2)$ is the Veronese surface in $\mathbb{P}^5$
You see that $SL(3)$ acts transitively over $M(-1, 2)$.
By Theorem 7.18 $E \in M(-1, 2)$ iff it appears in a sequence

$$0 \to \mathcal{O}(-3) \xrightarrow{f, q_1, q_2} \mathcal{O}(-2) \oplus \mathcal{O}(-1)^2 \to E \to 0$$

f is the equation of a line, q_i are conics. We want to associate a pair of distinct lines to such a bundle. This is accomplished by the following exercises.
$< q_1, q_2 >$ defines a 2-dimensional space in $H^0(f, \mathcal{O}(2)) = \mathbb{C}^3$. This is called classically a g_1^2 (linear series).

7.8. Exercise. *E is locally free iff the corresponding g^2_1 has no fixed points.*

7.9. Exercise. *$\{f = 0\}$ is the unique jumping line of E.*

7.10. Exercise. *The g^2_1 has two distinct double points P_1, P_2 on the line $\{f = 0\}$. E is uniquely determined by these two points.*

The isomorphism

$$M(-1, 2) \to S^2\mathbb{P}^2 \setminus \Delta$$

is given geometrically by

$$E \mapsto \{P_1, P_2\}$$

7.23. Remark. *The jumping lines of the second kind (see the next chapter 8) are exactly the lines in the two pencils through P_1 and P_2.*

7.24. Remark. *The closure $\overline{M(-1, 2)}$ is NOT $Sec(v_2)$ but it is the blow-up of $Sec(v_2)$ along v_2. In this way the variety obtained lies naturally in the variety of complete conics.*

$\mathbf{M}_{\mathbb{P}^2}(\mathbf{0}, \mathbf{2})$

7.25. Theorem. *$M(0, 2)$ is the projective space of nondegenerate conics, that is $M(0, 2)$ is isomorphic to $\mathbb{P}^5 \setminus V_3$, where V_3 is the determinantal hypersurface of degree 3.*

By Theorem 7.18 $E \in M(0, 2)$ iff it appears in a sequence

$$0 \to \mathcal{O}(-2)^2 \to \mathcal{O}(-1)^4 \to E \to 0$$

The morphism is represented by a 2×4 matrix with entries homogeneous linear polynomials in three variables, that is by a $2 \times 3 \times 4$ matrix. The hyperdeterminant of this matrix is nonzero (see the chapter 9 on hyperdeterminants). It is a basic fact that two such matrices are $GL(2)\times GL(3)\times GL(4)$-equivalent (see the Theorem 9.24 on multidimensional matrices). E is a Schwarzenberger bundle. The jumping lines fill a smooth conic in $\mathbb{P}^2$. By the geometrical construction of the Schwarzenberger bundles, the conic of jumping lines determines the bundle, (see the Proposition 6.9, where it is shown that the bundle can be reconstructed by the rational normal curve) proving the Theorem 7.25.
The Maruyama closure $\overline{M(0, 2)}$ is obtained by adding the semistable sheaves F which have the minimal resolution

$$0 \to \mathcal{O}(-3) \to \mathcal{O}(-2) \oplus \mathcal{O}(-1) \oplus \mathcal{O} \to F \to 0$$

Such F has a section vanishing on two points Z. The sheaf $\mathcal{O}\oplus\mathcal{I}_Z$ is also a semistable free torsion sheaf which is equivalent to F in $\overline{M(0, 2)}$. It can be proved that the sheaves added correspond exactly to the degenerate conics, that is $\overline{M(0, 2)} = \mathbb{P}^5$. $\overline{M(0, n)}$ is singular for $n \geq 3$.

7.26. Remark. *$\mathbb{P}^2$ is a toric variety, under the action of $T = \mathbb{C}^* \times \mathbb{C}^*$. There is a induced action of T over $\overline{M(c_1, c_2)}$. The fixed points of this action are the T-invariant sheaves. Hence from these informations it is possible to compute topological invariants of $\overline{M(c_1, c_2)}$. For example, suppose that T acts as $(t_1, t_2) \cdot (x_0, x_1, x_2) = (x_0, t_1x_1, t_2x_2)$. The T-invariant bundles with $c_1 = 0$, $c_2 = 2$ are*

$$0 \to \mathcal{O}(-3) \xrightarrow{x_0, x_1^2, x_2^3} \mathcal{O}(-2) \oplus \mathcal{O}(-1) \oplus \mathcal{O} \to F \to 0$$

and the other five bundles corresponding to the 6 permutations of $\{x_0, x_1, x_2\}$. In fact $\chi(\mathbb{P}^5, \mathbb{Z}) = 6$.

7.11. Exercise. *Compute χ of $M(-1, 2)$ and of $\overline{M(-1, 2)}$.*

Hint: The exceptional divisor in $\overline{M(-1, 2)}$ is a $\mathbb{P}^1$-bundle over a surface isomorphic to $\mathbb{P}^2$ and its χ is the same as $\chi(\mathbb{P}^1 \times \mathbb{P}^2)$.

7.4 The nullcorrelation bundle

Let $M_{\mathbb{P}^3}(0, 1)$ be the moduli space of stable rank 2 bundles over $\mathbb{P}^3$ with $c_1 = 0$, $c_2 = 1$.

7.27. Theorem. *$M_{\mathbb{P}^3}(0, 1)$ is the projective space of nondegenerate skewsymmetric matrices 4×4, that is $M_{\mathbb{P}^3}(0, 1)$ is isomorphic to $\mathbb{P}^5 \setminus Q_4$, where Q_4 is a smooth 4-dimensional quadric (Klein quadric).*

It can be proved by Beilinson theorem that every $E \in M_{\mathbb{P}^3}(0, 1)$ appears in a sequence

$$0 \to \mathcal{O}(-1) \to \Omega^1(1) \to E \to 0$$

E is called a nullcorrelation bundle. In fact $Hom(\mathcal{O}(-1), \Omega^1(1)) \simeq \wedge^2 V$. This can be proved by considering the second wedge power of the Euler sequence. The Theorem 7.27 is now proved by the following exercise.

7.12. Exercise. *i)Prove that $\mathcal{O}(-1) \xrightarrow{\omega} \Omega^1(1)$ with $\omega \in \wedge^2 V$ is a injective bundle map if and only if ω is nondegenerate. ii) Prove that ω_1, $\omega_2 \in \wedge^2 V$ define isomorphic bundles iff there is $t \in \mathbb{C}^*$ such that $\omega_1 = t\omega_2$.*

7.13. Exercise. *Prove that the restriction of a nullcorrelation bundle to a linear $\mathbb{P}^2 \subset \mathbb{P}^3$ is never stable.*

7.28. Remark. *The above properties characterizes the nullcorrelation bundle among all stable 2-bundles on $\mathbb{P}^3$ ([Ba1]). In [Co1] were characterized the stable bundles on $\mathbb{P}^3$ that become unstable on a family of planes of dimension at least 2.*

You see that $SL(4)$ acts transitively over $M_{\mathbb{P}^3}(0, 1)$. Every nullcorrelation bundle is symplectic, moreover it is also $Sp(4)$-invariant.
The geometrical interpretation of nullcorrelation bundles is the following. A skew symmetric nondegenerate matrix 4×4 J induces a morphism $J \colon \mathbb{P}^3 \longrightarrow \mathbb{P}^{3\vee}$ such that $\forall p \in \mathbb{P}^3, p \in J(p)$. Now for every $p \in \mathbb{P}^3$, consider the line N_p given by all the lines in $J(p)$ through p. We get a $\mathbb{P}^1$-bundle which is the projective bundle $\mathbb{P}(N)$.

Chapter 8

Jumping lines, jumping conics and the Barth morphism

8.1 Generalities about jumping lines

Let E be a stable rank 2 vector bundle on $\mathbb{P}^r$ ($r \geq 2$) with Chern classes $c_1 = 0$ or $c_1 = -1$ and $c_2 = n$. We will speak about even bundle when $c_1 = 0$ and odd bundle in the other case. It is well known that, when the rank of the bundle E is equal to 2, it is stable (resp. semi-stable) if and only if $H^0(E) = 0$ (resp. $H^0(E(-1)) = 0$). The notions of semi-stability and stability coincide for odd bundle. The first important result is the following theorem due to Grauert and Mulich

8.1. Theorem. *Let E a semi-stable rank two vector bundle on $\mathbb{P}^r$ and l a general line. Then $E_l = O_l \oplus O_l$ if E is even, $E_l = O_l \oplus O_l(-1)$ if E is odd.*

Over some lines, called jumping lines, the bundle does not split in the above way. More precisely we can describe the set $S(E)$ of jumping lines of E in the following way

$$S(E) = \{l, H^0(E_l(-1)) \neq 0\}$$

Example 1 Assume that E is semi-stable but not stable. Then we have $h^0(E) = 1$ and the unique (modulo multiplication by scalar) non zero section of E gives the following exact sequence

$$0 \to O_{\mathbb{P}^r} \longrightarrow E \longrightarrow \mathcal{I}_Z(s) \to 0$$

where the zero-scheme $Z(s)$ of degree n is a scheme of codimension two in $\mathbb{P}^r$. Let l a line in $\mathbb{P}^r$, we restrict the above exact sequence to l

$$0 \to O_l \longrightarrow E_l \longrightarrow \mathcal{I}_Z(s) \otimes O_l \to 0$$

Since we have the following isomorphism of sheaves

$$\mathcal{I}_Z(s) \otimes O_l = \mathcal{I}_{Z(s)\cap l/l} \oplus \mathcal{R}$$

where $\mathcal{R}$ is a torsion sheaf supported by $Z(s) \cap l$ we obtain that
• $E_l = O_l \oplus O_l$ when l does not meet $Z(s)$
• $E_l = O_l(a) \oplus O_l(-a)$ when $l(O_{Z(s)\cap l}) = a$.

We would like now to show that $S(E)$ possesses a natural scheme structure. For this observe that the cohomological condition $H^0(E_l(-1)) \neq 0$ is equivalent to the condition
- $H^1(E_l(-1)) \neq 0$ when E is even
- $H^1(E_l) \neq 0$ when E is odd.

Now, consider the incidence variety point-line and the canonical projection morphism

$$\mathbb{P}^r \xleftarrow{p} \mathbb{F} \xrightarrow{p} G(1, \mathbb{P}^r)$$

The jumping lines of E define a closed subset in $G(1, \mathbb{P}^r)$ which is the support of the coherent sheaf
- $R^1 q_*(p^* E(-1))$ when E is even
- $R^1 q_* p^* E$ when E is odd

8.2. Theorem. *(Barth, [Ba1] thm****) When $c_1(E)$ is even $S(E)$ is a divisor of degree n.*
When $c_1(E)$ is odd (and E general in the moduli space) $S(E)$ is a codimension two subvariety of degree $\frac{n(n-1)}{2}$.

Proof We give the proof on $\mathbb{P}^2$. For the general case you can refer to the original Barth's paper ([Ba1]). In that case the incidence variety $\mathbb{F}$ is a divisor in $\mathbb{P}^2 \times \mathbb{P}^{2*}$ defined by the equation $\sum_{i=0}^{i=2} X_i X_i^* = 0$ where X_i (resp. X_i^*) are the homogeneous coordinates on $\mathbb{P}^2$ (resp. on $\mathbb{P}^{2*}$). Thus we have the canonical resolution

$$0 \to O_{\mathbb{P}^2 \times \mathbb{P}^{2*}}(-1, -1) \longrightarrow O_{\mathbb{P}^2 \times \mathbb{P}^{2*}} \longrightarrow O_{\mathbb{F}} \to 0$$

<u>First case : E even</u> Now tensorize this exact sequence by $p^* E(-1)$ and take the direct image by q on $\mathbb{P}^{2*}$. Since $H^0(E_l) = 0$ for the general line we find that $q_*(p^* E(-1)) = 0$. Then we obtain

$$0 \to H^1(E(-2)) \otimes O_{\mathbb{P}^{2*}}(-1) \xrightarrow{\phi} H^1(E(-1)) \otimes O_{\mathbb{P}^{2*}} \longrightarrow R^1 q_*(p^* E(-1)) \to 0$$

The Riemann-Roch-Grothendieck formula gives $\mathcal{X}(E(t)) = (t+1)(t+2) - n$. From the stability of E and the Serre duality we find $h^1(E(-1)) = h^1(E(-2)) = n$ and $H^1(E(-1)) = H^1(E(-2))^*$ which implies that ϕ could be represented by a square symmetric matrix with linear forms as coefficients. ***add theta characteristic It follows that the support of the scheme $R^1 q_*(p^* E(-1))$ is defined by the equation $\det(\phi) = 0$.
<u>Second case : E odd</u> Tensorize the above exact sequence by $p^* E$ and take the direct image by q on $\mathbb{P}^{2*}$. Since $H^0(E_l) = 1$ for the general line we find that $q_* p^* E$ is a rank 1 sheaf on $\mathbb{P}^{2*}$. Moreover we can verify that E is a line bundle. Then we obtain

$$0 \to q_* p^* E \longrightarrow H^1(E(-1)) \otimes O_{\mathbb{P}^{2*}}(-1) \xrightarrow{\phi} H^1(E) \otimes O_{\mathbb{P}^{2*}} \longrightarrow R^1 q_* p^* E \to 0$$

The Riemann-Roch-Grothendieck formula gives $\mathcal{X}(E(t)) = (t+1)^2 - n$. From the stability of E we find $h^1(E(-1)) = n$ and $h^1(E(-2)) = n - 1$ which implies that ϕ could be represented by a $n \times (n-1)$ matrix with linear forms as coeficients. It follows that the support of the scheme $R^1 q_* p^* E$ is a two codimension subscheme of degree $\frac{n(n-1)}{2}$ defined by the $V(\wedge^{n-1} \phi)$, except when the maximal minors possed a common factor.

8.3. Remark. *For any stable rank two vector bundle E such that $c_1(E)$ is odd the codimension of $S(E)$ is at most 2.*

8.2 Barth morphism on $\mathbb{P}^2$

Let $M(0,n)$ be the coarse moduli scheme of semi-stable coherent sheaves of rank 2 with Chern classes $(0,n)$ on $\mathbb{P}^2$. This variety is a projective, irreducible variety of dimension $4n-3$. We recall that a locally free sheaf representing a point of $M(0,n)$ is necessarily stable. Let us denote by $\mathbf{U}(0,n)$ the open set of points representing locally free sheaves. The closed set of classes of singular sheaves is an hypersurface in $M(0,n)$ that we denote by $\delta M(0,n)$. By the property of a coarse moduli scheme we obtain a morphism

$$M(0,n) \xrightarrow{\gamma} \mathbb{P}(H^0(\mathbb{P}^{2*}, O_{\mathbb{P}^{2*}}(n))),\ [E] \mapsto S(E)$$

This map was first considered by Barth. It is known that the restriction of the morphism γ to the open set $\mathbf{U}(0,n)$ is quasi-finite (**** number of theta-char on one curve is finite), that the the restriction to $\delta M(0,n)$ has fibers of dimension ≥ 1 (**** cite Maruyama), and the image of the boundary is contained in the closed set of reducible curves (**** cite Maruyama). This implies that the image of γ is also an irreducible variety of dimension $4n-3$. It is also known that there exists a smooth curve in the image (see Barth [Ba1], prop¿ ****).

Very recently LePotier and Tikhomirov have showed that the degree of the map $\gamma : M(0,n) \longrightarrow \mathrm{Im}\gamma$ is 1 for $n \geq 4$. The computation of the degree of the image is related to the computation of Donaldson numbers on $\mathbb{P}^2$. When $n < 4$ we have
• if $n = 2$ the map is an isomorphism
• if $n = 3$ this map is surjective and of degree 3.

The dimension of the linear system $\mathbb{P}(H^0(\mathbb{P}^{2*}, O_{\mathbb{P}^{2*}}(n)))$ is $\frac{n(n+3)}{2}$, and for $n \geq 4$ we have $\dim M(0,n) < \frac{n(n+3)}{2}$. When $n = 4$ the dimension of the moduli space is 13 and the dimension of the projective space of quartics is 14. The curves of the divisor $\gamma(M(0,4))$ are the so called Luroth quartics, i.e plane quartics circumscribed to a true pentagon. We will call the hypersurface $\mathcal{L} := \gamma(M(0,4))$ the Luroth hypersurface. It is not easy to find the equation of this hypersurface, but in 1918 Frank Morley already wrote that the Luroth's invariant is of degree 54.

8.4. Proposition. *Let E be a general bundle in $M(0,4)$. Then $S(E)$ is a Luroth quartic.*

Proof. By computing its Euler-Poincare polynomial we find $\mathcal{X}(E(1)) = 2$.

8.1. Exercise. *Assume that $h^1(E(1)) = 1$. Show that there exists a jumping line of order 3.*

Thus in general we have $h^1(E(1)) = 0$ and $h^0E(1) = 2$. Since all the jumping lines of E are of order 1 the determinant of the two independant sections of $E(1)$ is a smooth conic, i.e we have

$$0 \to 2O_{\mathbb{P}^2} \longrightarrow E(1) \longrightarrow \Theta \to 0$$

where Θ is supported by a smooth conic C. Dualize this exact sequence we get

$$0 \to E(-1) \longrightarrow 2O_{\mathbb{P}^2} \longrightarrow \mathcal{E}xt^1(\Theta, O_{\mathbb{P}^2}) \to 0$$

By computing the Chern classes we find that $\mathcal{E}xt^1(\Theta, O_{\mathbb{P}^2}) = O_C(\frac{5}{2})$. Now apply the functor p_*q^* to the last exact sequence. Since for the general line l we have $h^0E_l(-1) = 0$ we have $p_*q^*E(-1) = 0$ so a short exact sequence

$$0 \to 2O_{\mathbb{P}^{2*}} \longrightarrow p_*q^*O_C(\frac{5}{2}) \longrightarrow R^1p_*q^*E(-1) \to 0$$

Since a Poncelet related curve of degree 4 is the determinant of two linearly independant sections of the Schwarzenberger's bundle $E_5 = p_*q^*O_C(\frac{5}{2})$ this proves that the curve of jumping lines of E is a Luroth quartic. □

8.2. Exercise. *More generally show that the Poncelet's curves belong to the image of γ for any n. For this consider the family of bundles E such that $h^0(E(1)) = 2$ (called Hulsbergen's bundle).*

For $n \geq 5$ the dimension of the family of Poncelet's curves is strictly smaller than the dimension of $Im\gamma$. When $n > 4$ Matei Toma in [To] has shown that the Barth morphism restricted to the subscheme of Hulsbergen bundles is generically injective. It means that a general Poncelet's curve is associated to only one smooth conic.

8.3 Barth morphism on Schwarzenberger bundles

Let C a smooth conic with equation $f = 0$ on $\mathbb{P}^{2*}$ and E_{2n+1} the even Schwarzenberger's bundle associated to C. The second Chern classes of the normalized bundle $E_{2n+1}(-n)$ is $n(n+1)$. We denote by mC the divisor defined by the equation $f^m = 0$.

8.5. Proposition. $S(E_{2n+1}) = \frac{n(n+1)}{2}C$

*** preuve a developper avec Daniele *** **Proof.** Since E_{2n+1} is $SL(2,\mathbb{C}) = \mathrm{Aut}(C)$-invariant its divisor of jumping lines is supported by C. Its degree is $n(n+1)$, thus the proposition is proved. □

Let $\gamma : M(0, n(n+1)) \longrightarrow \mathbb{P}(H^0(\mathbb{P}^{2*}, O_{\mathbb{P}^{2*}}(n(n+1))))$. The above proposition showed that the image of E_{2n+1} is the curve $\frac{n(n+1)}{2}C$. Since this curve does not contain a linear component we know that the fiber $\gamma^{-1}(\frac{n(n+1)}{2}C)$ is finite. We show the following

8.6. Theorem. $\gamma^{-1}(\frac{n(n+1)}{2}C) = \{E_{2n+1}\}$ *(set-theoretically).*
Moreover if m is an integer which could not be written under the form $n(n+1)$ for any integer n then the curve mC does not belong to the image of γ.

Proof. Let $\mathcal{E}$ a vector bundle in the finite fiber $\gamma^{-1}(mC)$. Let $\sigma \in SL(2,\mathbb{C})$, since $S(\sigma^*\mathcal{E}) = \sigma.S(\mathcal{E}) = mC$ we deduce that $SL(2,\mathbb{C})$ acts on $\gamma^{-1}(mC)$. But $SL(2,\mathbb{C})$ is connected thus this action is trivial. It means that for any bundle $\mathcal{E} \in \gamma^{-1}(mC)$ and any $\sigma \in SL(2,\mathbb{C})$ we have $\sigma^*\mathcal{E} = \mathcal{E}$.

The following proposition proves the theorem

8.7. Proposition. *The only stable $SL(2,\mathbb{C})$-invariant bundles of rank 2 on $\mathbb{P}^2$ are the Schwarzenberger bundles.*

Proof of the proposition. We denote by C^* the image of $\mathbb{P}^1$ by the Veronese morphism $\mathbb{P}^1 \hookrightarrow \mathbb{P}^2 \simeq S^2\mathbb{P}^1$, π the morphism of degree 2 $\mathbb{P}^1 \times \mathbb{P}^1 \longrightarrow \mathbb{P}^2 \simeq S^2\mathbb{P}^1$ which send the couple (x,y) to the intersection point of the lines t_x and t_y tangent to $C^\vee$ in the points $\pi(x,x)$ and $\pi(y,y)$. We denote also p_1 and p_2 the canonical projections of $\mathbb{P}^1 \times \mathbb{P}^1$ on each factor. The action of $SL_2(\mathbb{C})$ on $\mathbb{P}^1$ induces an action on $\mathbb{P}^2$ which identifies $SL_2(\mathbb{C})$ to $\mathrm{Aut}(C^*)$ (resp. on $\mathbb{P}^{2*}$ which identifies $SL_2(\mathbb{C})$ to $\mathrm{Aut}(C)$ where C is the dual conic). More precisely, let $\sigma \in SL_2(\mathbb{C})$ and $z = \pi(x,y)$ a point of $\mathbb{P}^2$ the induced action on $\mathbb{P}^2$ is

$$\sigma.z = \pi(\sigma x, \sigma y).$$

Let F be a stable rank two vector bundle on $\mathbb{P}^2$ $SL_2(\mathbb{C})$-invariant, i.e. such that $\sigma^*F = F$ for all $\sigma \in SL_2(\mathbb{C})$. Then we have $(\sigma,\sigma)^*(\pi^*F) = \pi^*F$. We show thanks to a idea of Schwarzenberger ([Sch2], §3) that the bundle π^*F corresponds to an $SL_2(\mathbf{C})$-homorphism between two irreducible representations of $SL_2(\mathbf{C})$.

We recall that a line bundle on $\mathbb{P}^1 \times \mathbb{P}^1$ is of the following form

$$O_{\mathbb{P}^1\times\mathbb{P}^1}(a,b) \simeq p_1^*O_{\mathbb{P}^1}(a) \otimes p_2^*O_{\mathbb{P}^1}(b) \quad \text{and that} \quad \pi^*O_{\mathbb{P}^2}(1)) = O_{\mathbb{P}^1\times\mathbb{P}^1}(1,1)$$

We will assume that $c_1(F) = 0$ or $c_1(F) = -1$ and we will denote $c_1(F) = c_1$. Since the action of $SL_2(\mathbf{C})$ on $\mathbb{P}^2$ (resp. $\mathbb{P}^{2*}$) has two orbits say $C^\vee$ and $\mathbb{P}^2 \setminus C^\vee$ (resp. C and $\mathbb{P}^{2*} \setminus C$), it is clear that the support of $S(F)$ is C and that the order of jump is the same for every jumping line (i.e. it exists an integer $n > 0$ such that $h^0(F_l(-n)) = 1$ for any line $l \in C$).

Let $x \in \mathbb{P}^1$ and t_x the tangent to C coming from the point $\pi(x,x)$. By hypothesis on the jump order we have, $h^0(F_{t_x}(-n)) = 1$. We deduce that

$$p_{1*}\pi^*F(-n) = O_{\mathbb{P}^1}(-m) \quad \text{avec} \quad m > 0$$

Since the jump is uniform, the induced homomorphism $\pi^*F^\vee(n) \longrightarrow p_1^*O_{\mathbb{P}^1}(m)$ is surjective. Its kernel is a line bundle on $\mathbb{P}^1\times\mathbb{P}^1$. An easy computation of first Chern classes show that this kernel is $O_{\mathbb{P}^1\times\mathbb{P}^1}(2n-m-c_1, 2n-c_1)$. The vector bundle $\pi^*F^\vee(n)$ correspond to a non zero element (since it is not decomposed) of

$$\mathrm{Ext}^1(O_{\mathbb{P}^1\times\mathbb{P}^1}(m,0), O_{\mathbb{P}^1\times\mathbb{P}^1}(2n-m-c_1, 2n-c_1)) = H^1(O_{\mathbb{P}^1\times\mathbb{P}^1}(2n-2m-c_1, 2n-c_1))$$

The surjective homorphism $\pi^*F^\vee(n) \longrightarrow p_1^*O_{\mathbb{P}^1}(m)$ induces a non zero homorphism on $\mathbb{P}^2$

$$F^\vee(n) \longrightarrow E_{m,C}$$

Since $E_{1,C} = 2O_{\mathbb{P}^2}$ we have $m \geq 2$ if not $h^0(F(-n)) \neq 0$ which is a contradiction with the stability of F. Then the bundles $F^\vee(n)$ and $E_{m,C}$ are stable. Then the homomorphism is of maximal rank. Thus we find $c_1(F^\vee(n)) = 2n - c_1 \leq c_1(E_{m,C}) = m-1$. In particular $2n - 2m - c_1 - 1 < 0$ which implies by the Kunneth formula for sheaves [BS]

$$H^1(O_{\mathbb{P}^1\times\mathbb{P}^1}(2n-2m-c_1, 2n-c_1)) = H^1(O_{\mathbb{P}^1}(2n-2m-c_1)) \otimes H^0(O_{\mathbb{P}^1}(2n-c_1))$$

and by Serre duality

$$H^1(O_{\mathbb{P}^1\times\mathbb{P}^1}(2n-2m-c_1, 2n-c_1)) = \mathrm{Hom}(H^0(O_{\mathbb{P}^1}(2(m-n-1)+c_1), H^0(O_{\mathbb{P}^1}(2n-c_1)))$$

We denote by α_{π^*F} the homorphism (modulo a multiplicative scalar) corresponding to the bundle π^*F. Let $\Phi = (\sigma, \sigma)$ an automorphism of $\mathbb{P}^1 \times \mathbb{P}^1$ with $\sigma \in SL_2(\mathbf{C})$. The bundle $\Phi^*\pi^*F$ is represented by the homorphism

$$\alpha_{\Phi^*\pi^*F} = \sigma^*\alpha_{\pi^*F}(\sigma^*)^{-1}$$

In the other hand, $\Phi^*\pi^*F = \pi^*F$ which proves that the homomorphism α_{π^*F} is $SL_2(\mathbb{C})$-invariant. Since $H^0(O_{\mathbb{P}^1}(r)) \simeq S^r H^0(O_{\mathbb{P}^1}(1))$ is an irreducible representation of $SL_2(\mathbb{C})$ we deduce that $2(m-n-1)+c_1 = 2n-c_1$. It means $c_1(F^\vee(n)) = c_1(E_{m,C})$ and then $F^\vee(n) = E_{2n+1-c_1,C}$. $\square$

Unfortunately this result does not prove that the following Barth morphism

$$\gamma : M(0, n(n+1)) \longrightarrow \mathbb{P}(H^0(\mathbb{P}^{2*}, O_{\mathbb{P}^{2*}}(n(n+1))))$$

is generically injective. Indeed we will show that the Schwarzenberger's bundle belong to the ramification of this map. In other word the differential map

$$d\gamma_{[E_{2n+1}]} : T_{[E_{2n+1}]}M(0, n(n+1)) \longrightarrow T_{\frac{n(n+1)}{2}C}\mathbb{P}(H^0(\mathbb{P}^{2*}, O_{\mathbb{P}^{2*}}(n(n+1))))$$

is not injective. Indeed this map is certainly an equivariant map and (according to the $SL(2) = SL(A)$-action on $\mathbb{P}^2 = \mathbb{P}(S^2A)$)

$$T_{[E_{2n+1}]}M(0, n(n+1)) = H^1(\mathrm{End}E_{2n+1}) = \sum_{i=2}^{n+1} S^{2i}A$$

$$T_{\frac{n(n+1)}{2}C}\mathbb{P}(H^0(\mathbb{P}^{2*}, O_{\mathbb{P}^{2*}}(n(n+1)))) = H^0(O_{\frac{n(n+1)}{2}C}(n(n+1)) = \sum_{i=0}^{[\frac{n(n+1)}{4}]-1} S^{n(n+1)-4i}A$$

Let $n \geq 2$. We see that S^6A belongs to the kernel of this map when $n(n+1) = 0(mod4)$ and that S^4A belongs to the kernel of this map when $n(n+1) = 2(mod4)$.

8.3. Exercise. *Prove the above equalities.*

8.4 Hulek curve of jumping lines of second kind

What's about odd bundle. We have seen that in general the scheme of jumping lines of an odd bundle on $\mathbb{P}^2$ is a finite scheme. In some cases, for instance for Schwarzenberger's bundle, this scheme contain a divisor. For schwarzenberger's bundle it is easy to prove (by using $SL(2,\mathbb{C})$-invariance) that this scheme is a divisor supported by a smooth conic. We are able to caracterize those bundles which possed a line in their scheme of jumping lines (see for instance [Va4]). Hulek associates to any odd bundle of Chern classes $(-1, n)$ a plane curve in the dual plane $\mathbb{P}^{2*}$ of degree $2(n-1)$. For a bundle E this curve is denoted $C(E)$ and called curve of jumping lines of second kind. Set theoretically this curve is

$$C(E) = \{l \in \mathbb{P}^{2*} \mid H^0E_{l^2} \neq 0\}$$

First of all, like for jumping lines, we need the following important result due to Hulek.

8.8. Theorem. *(Hulek) For the generic line l we have $H^0 E_{l^2} = 0$.*

Now let $\mathbb{F}_2$ the divisor in $\mathbb{P}^2 \times \mathbb{P}^{2*}$ defined by the equation $(\sum_{i=0}^{i=2} X_i X_i^*)^2 = 0$ where X_i (resp. X_i^*) are the homogeneous coordinates on $\mathbb{P}^2$ (resp. on $\mathbb{P}^{2*}$). Thus we have the canonical resolution

$$0 \to O_{\mathbb{P}^2 \times \mathbb{P}^{2*}}(-2,-2) \longrightarrow O_{\mathbb{P}^2 \times \mathbb{P}^{2*}} \longrightarrow O_{\mathbb{F}_2} \to 0$$

We denote by p and q the projections on each factor $\mathbb{P}^2$ and $\mathbb{P}^{2*}$ and $\bar{p}$, $\bar{q}$ their restrictions to $\mathbb{F}_2$. Since $\bar{q}^{-1}(l) = l^2$, the above theorem assure that $\bar{q}_* \bar{p}^* E = 0$. Now we are able to give the proof of the following

8.9. Theorem. *(Hulek) $C(E)$ is a curve of degree $2(c_2 E - 1)$.*

Proof. We tensorize the resolution of $\mathbb{F}_2$ by $p^* E$ and we apply the functor q_*. Since $\bar{q}_* \bar{p}^* E = 0$ we find

$$0 \to H^1(E(-2)) \otimes O_{\mathbb{P}^{2*}}(-2) \xrightarrow{\phi} H^1(E) \otimes O_{\mathbb{P}^{2*}} \longrightarrow R^1 \bar{q}_* \bar{p}^* E \to 0$$

Since $h^1 E = h^1 E(-2) = c_2 - 1$ and $H^1 E = (H^1 E(-2))^*$ the map ϕ could be represented by a symmetric matrix of quadratic forms. Then $\mathrm{supp}(R^1 \bar{q}_* \bar{p}^* E) = \det(\phi)$ is a curve of degree $2(c_2 - 1)$. □

We would like to give an other result about the curve of jumping lines of second kind without proof. Then we will finish this part about Hulek morphism with examples.

8.10. Proposition. *(Hulek) For every odd bundle E with $c_2 E = n$ we have $S(E) \subset \mathrm{Sing}C(E)$ with equality for the general bundle.*

Proof. Remember that the incidence variety $\mathbb{F}$ is defined in $\mathbb{P}^2 \times \mathbb{P}^{2*}$ by the equation $\sum_{i=0}^{i=2} X_i X_i^* = 0$. We do not make a distinction between the projections p and q and their restrictions on $\mathbb{F}$. Then we have the following exact sequence

$$0 \to O_{\mathbb{F}}(-1,-1) \longrightarrow O_{\mathbb{F}_2} \longrightarrow O_{\mathbb{F}} \to 0$$

which is the relative version of

$$0 \to O_l(-1) \longrightarrow O_{l^2} \longrightarrow O_l \to 0$$

We tensorize this exact sequence by $p^* E$ and we apply q_*. Thus we obtain the long exact sequence

$$0 \to q_* p^* E \to (R^1 q_* p^* E(-1))(-1) \to R^1 \bar{q}_* \bar{p}^* E \to R^1 q_* p^* E \to 0$$

The surjective arrow $R^1 \bar{q}_* \bar{p}^* E \to R^1 q_* p^* E$ implies that every jumping lines is a jumping lines of the second kind. Moreover, since the rank 1 sheaf $R^1 q_* p^* E(-1)$ is not locally free over the jumping lines, this exact sequence shows that any jumping line is singular in $C(E)$. We omit to prove the equality for the general bundle. □

We want to study the image of the morphism

$$C : M(-1,n) \longrightarrow \mathbb{P}(H^0(O_{\mathbb{P}^{2*}}(2n-2)))$$

8.4. Exercise. *For $c_2 = 1$ the only bundle of $M(-1,1)$ does not have jumping lines. Show that it also does not have any jumping lines of the second kind.*

If $c_2 = 2$, let $s \in H^0E(1)$ a non zero section, we have the exact sequence induced by s

$$0 \to O_{\mathbb{P}^2} \longrightarrow E(1) \longrightarrow \mathcal{I}_{Z(s)}(1) \to 0$$

where $Z(s)$ consists of two distinct points x and y. The only jumping lines is the line passing through x and y. The curve $C(E)$ is a curve of degree 2 with one singular point. Since $h^0E(1) = 2$ and the determinant of this pencil is the line passing through x and y, we deduce by Hurwitz formula that there are two sections s and t such that their zero schemes are double points. So every lines D passing through $Z(s)$ or $Z(t)$ verify $H^0E_{D^2} \neq 0$.

If $c_2 = 4$ show that the curve $C(E)$ of the general odd bundle is a sextic with six singular points. When E is the Schwarzenberger's bundle E_4 associated to the conic C then $C(E_4) = 3C$.

8.5 Jumping conics

Let E be a stable rank two vector bundle on $\mathbb{P}^2$. We assume that E is normalized i.e. $c_1(E) = 0$ or -1. Let C be a smooth conic of $\mathbb{P}^2$. Since this conic is isomorphic to $\mathbb{P}^1$ the Grothendieck's theorem implies that $E_C = O_C(\frac{a}{2}) \oplus O_C(\frac{b}{2})$ where $O_C(\frac{a}{2})$ means the line bundle on C with degree a and $a + b = 2c_1(E)$. Moreover the Grauert-Mulich theorem implies that for the general conic of $\mathbb{P}^2$ we have $E_C = O_C \oplus O_C$ when E is even, $E_C = O_C(\frac{-1}{2}) \oplus O_C(\frac{-1}{2})$ when E is odd. Now we can define the jumping conics for smooth conics. They are the one such that the decomposition is not as above. Since the Grothendieck's theorem is not valid over singular conics, we need to define the jumping conics with a cohomological condition (equivalent for smooth conics).

8.11. Definition. *A conic C is a jumping conic for E if*

$$\begin{cases} E_C \neq 2O_C & \text{when } c_1 = 0 \\ h^0(E_C) \neq 0 & \text{when } c_1 = -1 \end{cases}$$

We denote by $J(E)$ the set of jumping conics. The Grauert-Mulich theorem applied to conics leads to the following result ([Man] thm 1.8)

8.12. Theorem. *$J(E)$ is a divisor of* $\mathbb{P}(H^0(O_{\mathbb{P}^2}(2)))$ *and* $\deg J(E) = c_2 + c_1$.

Proof. Consider the incidence variety point-conic ie the divisor $\mathbb{F} \subset \mathbb{P}^2 \times \mathbb{P}^5$ defined set theoretically by

$$\mathbb{F} = \{(x, C) \mid x \in C\}$$

and defined by the equation

$$X_0^2Y_0 + X_0X_1Y_1 + X_0X_2Y_2 + X_1^2Y_3 + X_1X_2Y_4 + X_2^2Y_5 = 0$$

We denote by p and q the projections on $\mathbb{P}^2$ and $\mathbb{P}^5$ respectively. Thus we have the following resolution of $\mathbb{F}$ in $\mathbb{P}^2 \times \mathbb{P}^5$

$$0 \to 0_{\mathbb{P}^2\times\mathbb{P}^5}(-2,-1) \longrightarrow 0_{\mathbb{P}^2\times\mathbb{P}^5} \longrightarrow O_{\mathbb{F}} \to 0$$

— Assume first that the bundle E is odd. After tensoring this exact sequence by p^*E and taking the direct image on $\mathbb{P}^5$ we obtain

$$0 \to H^1E(-2) \otimes O_{\mathbb{P}^5}(-1) \longrightarrow H^1E \otimes O_{\mathbb{P}^5} \longrightarrow R^1q_*p^*E \to 0$$

Indeed, the Grauert-Mulich 's theorem implies that $q_*p^*E = 0$. Since $h^1E(-2) = h^1E = c_2 - 1$ the support of $R^1q_*p^*E$ is a divisor of degree $c_2 - 1$.

Remark. Consider the Veronese morphism $\mathbb{P}^{2*} \hookrightarrow \mathbb{P}^5$ which sends a line l to the conic l^2. We denote by V the image of $\mathbb{P}^{2*}$. The support of $R^1q_*p^*E \otimes O_V$ is a curve of degree $4(n-1)$. Since the Veronese morphism is of degree 2 this proves the Hulek's theorem.

— Assume now that the bundle E is even. The sheaf q_*p^*E is a rank 2 reflexive sheaf on $\mathbb{P}^5$. Its first Chern class is $-c_2$. Indeed, consider the following exact sequence

$$0 \to q_*p^*E \to H^1E(-2) \otimes O_{\mathbb{P}^5}(-1) \to H^1E \otimes O_{\mathbb{P}^5} \longrightarrow R^1q_*p^*E \to 0$$

Since $h^1E(-2) = c_2$ and $h^1E = c_2 - 2$ the codimension of the support of $R^1q_*p^*E$ is generically 3. Let l a general line in $\mathbb{P}^5$, here general means : l do not meet neither the support of $R^1q_*p^*E$ neither the singular locus of q_*p^*E (in fact they certainly coincide). Then the restriction of the above exact sequence is

$$0 \to q_*p^*E \otimes O_l \to H^1E(-2) \otimes O_l(-1) \to H^1E \otimes O_l \to 0$$

This proves that $c_1(q_*p^*E) = -c_2$.

The canonical map (evaluation) $ev : q^*q_*p^*E \to p^*E$ becomes over a conic C

$$O_C \oplus O_C \longrightarrow E_C$$

So ev is lying over the jumping conics. But the zero locus of ev is defined by its determinant which is an hypersurface of degree $c_1(p^*E) - c_1(q^*q_*p^*E) = c_2$. □

Singular conics

Let $l_1 \cup l_2$ a couple of lines meeting transversally and l^2 the singular conic supported by l.

8.13. Proposition. *If E is even, then*

— *$l_1 \cup l_2$ is a jumping conic if and only if at least one of the two lines l_i is a jumping line for E.*

— *l^2 is a jumping conic if and only if l is a jumping line.*

Proof. Assume first that $l_1 \cup l_2 \notin J(E)$. The following exact sequence

$$O \to O_{l_1}(-1) \longrightarrow O_{l_1 \cup l_2} \longrightarrow O_{l_2} \to 0$$

tensorized by E gives

$$O \to E_{l_1}(-1) \longrightarrow O_{l_1 \cup l_2} \oplus O_{l_1 \cup l_2} \longrightarrow E_{l_2} \to 0$$

Since the last arrow is surjective we have $E_{l_2} = O_{l_2} \oplus O_{l_2}$. By changing the role of l_1 and l_2, we prove also that $l_1 \notin S(E)$. In the same way if l_1 and l_2 are not in $S(E)$ then we obtain $H^0E_{l_1 \cup l_2} = H^0(O_{l_2} \oplus O_{l_2})$. Then any section is non zero everywhere, so $E_{l_1 \cup l_2} = O_{l_1 \cup l_2} \oplus O_{l_1 \cup l_2}$.

The same proofs work for l^2 instead of $l_1 \cup l_2$. □

8.14. Proposition. *Assume that E is odd. If at least one of the two lines l_i is a jumping line for E then $l_1 \cup l_2$ is a jumping conic for E.*

Proof. Assume first that $l_1 \in S(E)$. The following exact sequence

$$O \to O_{l_1}(-1) \longrightarrow O_{l_1 \cup l_2} \longrightarrow O_{l_2} \to 0$$

tensorized by E shows that $h^0 E_{l_1 \cup l_2} \neq 0$.
The converse is in general false. Indeed we have $\dim(J(E) \cap S) = 3$ but $\dim(\mathbb{P}^{2*} \times S(E)) = 2$. $\square$
We have already proven the following proposition in the text concerned with Hulek curve.

8.15. Proposition. *If E is odd and l is a jumping line then l^2 is a jumping conic.*

In fact we know that $J(E) \cap V$ is exactly the image of the Hulek curve $C(E)$ by the Veronese morphism.

8.5. Exercise. *Assume that E is odd and that $l^2 \in J(E)$ but $l \notin S(E)$. Then prove that $E_{l^2} = O_{l^2} \oplus O_{l^2}(-1)$.*
Find by yourself the following exact sequences

$$O \to O_{l_1 \cup l_2} \longrightarrow O_{l_1} \oplus O_{l_2} \longrightarrow O_{l_1 \cap l_2} \to 0$$

$$O \to O_{l_1}(-1) \longrightarrow O_{l_1 \cup l_2} \longrightarrow O_{l_2} \to 0$$

$$O \to O_l(-1) \longrightarrow O_{l^2} \longrightarrow O_l \to 0$$

8.6 Jumping lines for Logarithmic bundles on $\mathbb{P}^2$.

Let $\mathcal{H} = \{H_1, \cdots, H_{n+k+1}\}$ be an arrangement of $n+k+1$ hyperplanes in general linear position in $\mathbb{P}^n$. We denote by $E(\mathcal{H})$ the bundle of meromorphic forms having at most logarithmic poles over $\mathcal{H}$. It is a basic fact, proved first by Deligne, that $E(\mathcal{H})$ appears in a exact sequence

$$0 \to \Omega_{\mathbb{P}^n} \longrightarrow E(\mathcal{H}) \longrightarrow \oplus_{i=1}^{n+k+1} O_{H_i} \to 0$$

The aim of this section is to prove that a Logarithmic bundle is a Steiner bundle.

8.16. Proposition. $E(\mathcal{H})$ *is a Steiner bundle.*

Proof. First of all we will prove that one extension

$$0 \to \Omega_{\mathbb{P}^n} \longrightarrow . \longrightarrow \oplus_{i=1}^{n+k+1} O_{H_i} \to 0$$

is a Steiner's bundle. Then we will show that all such extensions (which are bundle of course) are isomorphic.

Let (X_i) be the coordinates on $\mathbb{P}^n$ and (α_{ij}) the complex numbers such that $H_j = \sum_{i=0,\cdots,n} \alpha_{ij} X_i$. Since the hyperplanes of $\mathcal{H}$ are in general linear position there are exactly k relations between them, say $H_{n+k+1} = (\sum_{j=1,\cdots,n+k} a_{lj} H_j)_{l=1,\cdots,k}$. Consider the following commutative diagram.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & O_{\mathbb{P}^n}(-1) & \xrightarrow{A} & O_{\mathbb{P}^n}^{n+k+1}(-1) & \xrightarrow{B} & O_{\mathbb{P}^n}^{n+k}(-1) & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle (X_i)} & & \Big\downarrow{\scriptstyle M} & & \Big\downarrow{\scriptstyle N} & & \\
0 & \longrightarrow & O_{\mathbb{P}^n}^{n+1} & \xrightarrow{(\alpha_{ij})} & O_{\mathbb{P}^n}^{n+k+1} & \xrightarrow{D} & O_{\mathbb{P}^n}^{k} & \longrightarrow & 0
\end{array}$$

Where $A = {}^t(1, \cdots, 1)$, $M = \begin{pmatrix} H_{n+k+1} & & & \\ & H_1 & & \\ & & H_2 \ \ldots & \\ & & \ldots & H_{n+k} \end{pmatrix}$,

$$B = \begin{pmatrix} 1 & -1 & & & \\ 1 & 0 & -1 & & \\ 1 & 0 & 0 & -1\ldots & \\ 1 & 0 & & \ldots 0 & -1 \end{pmatrix}, \ N = \begin{pmatrix} a_{1,1}H_1 & a_{1,2}H_2 & & a_{1,n+k}H_{n+k} \\ a_{2,1}H_1 & a_{22}H_2 & & a_{2,n+k}H_{n+k} \\ \ldots & & \ldots & \\ a_{k,1}H_1 & a_{k,2}H_2 & \ldots & a_{k,n+k}H_{n+k} \end{pmatrix}, \text{ and}$$

where D is the matrix with only 1 on the first column and $(-a_{lj})$ on the others.

Now you can verify as an exercise that N is surjective. Its kernel is a bundle $S^*(-1)$ such that S is Steiner. Apply the snake lemma, then you find

$$0 \to S^*(-1) \longrightarrow \Omega^*_{\mathbb{P}^n}(-1) \longrightarrow \oplus_{i=1}^{n+k+1} O_{H_i} \to 0$$

Dualize this exact sequence, then you get

$$0 \to \Omega_{\mathbb{P}^n} \longrightarrow S \longrightarrow \oplus_{i=1}^{n+k+1} O_{H_i} \to 0$$

The following lemma will prove that two bundles which appear as extension

$$0 \to \Omega_{\mathbb{P}^n} \longrightarrow . \longrightarrow \oplus_{i=1}^{m} O_{H_i} \to 0$$

are isomorphic.

8.17. Lemma. *Let $(H_1, \cdots, H_m)$, m hyperplanes in $\mathbb{P}^n$, ϕ and ψ two surjective homomorphisms $\phi, \psi : \Omega^*_{\mathbb{P}^n}(-1) \longrightarrow \oplus_{i=1}^m O_{H_i}$. Then the bundles* $\ker(\phi)$ *and* $\ker(\psi)$ *are isomorphic.*

Proof. Since $\Omega^*_{\mathbb{P}^n}(-1)_{|H_i} = O_{H_i} \oplus \Omega^*_{H_i}(-1)$ and $H^0(\Omega_{H_i}(1)) = 0$ we have

$$\mathrm{Hom}(\Omega^*_{\mathbb{P}^n}(-1), \oplus_{i=1}^m O_{H_i}) \simeq \mathrm{Hom}(\oplus_{i=1}^m O_{H_i}, \oplus_{i=1}^m O_{H_i})$$

Since the homomorphisms ϕ and ψ are surjectives the induced matrices M_ϕ, M_ψ of complex numbers are of maximal rank. Then we have the following relation $M_\psi M_\phi^{-1} \phi = \psi$, in other words the following commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \ker(\phi) & \longrightarrow & \Omega^*_{\mathbb{P}^n}(-1) & \xrightarrow{\phi} & \oplus_{i=1}^m O_{H_i} & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle \simeq} & & \Big\downarrow{\scriptstyle \simeq} & & \Big\downarrow{\scriptstyle M_\psi M_\phi^{-1}} & & \\
0 & \longrightarrow & \ker(\psi) & \longrightarrow & \Omega^*_{\mathbb{P}^n}(-1) & \xrightarrow{\psi} & \oplus_{i=1}^m O_{H_i} & \longrightarrow & 0
\end{array}$$

$\square$

Add the proof that singular plane cubics form a hypersurface of degree 12 , S. Gimignano
Include Torelli with simplified proof (work on P(W) in fact...) *

8.7 Jumping conics of Schwarzenberger's bundle.

With the same notations than before we define $E_n = p_* q^* O_{D^*}(\frac{n}{2})$. We recall that $S(E_n)$ is supported by D^*. Moreover if l is a jumping line for E_n we have $h^0(E_{n|l}(1-n)) = 1$. By definition, or by construction E_n is invariant under the action of $\mathrm{SL}_2(\mathbb{C}) \simeq \mathrm{Aut}(D)$.

Some notations

Let $\psi : \mathbb{P}^{2*} \times \mathbb{P}^{2*} \to \mathbb{P}^5$ the morphism which sends a couple of lines on the conic union of these lines. The threefold $\mathbb{T} = \psi(\mathbb{P}^{2*} \times D^*)$ consist in degenerated conic such that one line of its support is tangent to D. The morphism ψ restricted to the diagonal is the Veronese morphism $v : \mathbb{P}^{2*} \hookrightarrow \mathbb{P}^5$. The variety $\mathrm{Sec}(v(D^*))$ of lines bisecants to $v(D^*)$ is a threefold of degree 3. You can obtain it by intersection of the hypersurface $S = \psi(\mathbb{P}^{2*} \times \mathbb{P}^{2*})$ with the hyperplane $< v(D^*) >$.
From now we assume that $n \geq 3$.

8.18. Theorem. $J(E_n) = \mathcal{C}_n$

Proof : First we remark that the two hypersurfaces have the same degree and that $\mathcal{C}_n$ is reduced. Then it is enough to show that we have $\mathcal{C}_n \subset J(E_n)$ on the open set of smooth conics.

Let C a smooth conic n-circumscribed to D. The $\binom{n}{2}$ vertices defined by the data of n-tangent lines are the zeroes of one section $s \in H^0(E_n)$ ([?], proposition 1.4). The following exact sequence

$$0 \to O_{\mathbb{P}^2} \xrightarrow{s} E_n \longrightarrow I_{Z(s)}(n-1) \to 0$$

restricted to C shows that $E_{n|C} = O_C(\frac{n}{2}) \oplus O_C(\frac{n-2}{2})$. □

8.19. Proposition. *Let C be a smooth osculating conic (resp.surosculating) to D. Then $C \notin M_n(D)$.*

Proof. A point in $v(D^*)$ corresponds to a degenerated conic supported by a tangent line to D. A point in the developable surface of $v(D^*)$ corresponds to a degenerated conic supported by a tangent line l to D and by an other line meeting l at the tangent point $l \cap D$. The variety of osculating conic (resp. surosculating) is the cone with vertex D and basis the developable surface of $v(D^*)$ (resp. the cone with vertex D and basis $v(D^*)$). But $\mathrm{SL}_2(\mathbb{C})$ acts transitively on the smooth conics of these two cones. It follows that if one of them belongs to $M_n(D)$ then all of them belong to $M_n(D)$. In particular the vertex of the cone belong to M_n. But the following exact sequence

$$0 \to E_n(-1) \longrightarrow E_n \longrightarrow O_D(\frac{n-1}{2}) \to 0$$

proves that $D \notin J(E_n)$. We deduce that M_n meets the osculating cone along the developable and the surosculating cone along the rational quartic. □

8.6. Exercise. *Prove the existence of this above exact sequence.*

Of course the divisor $J(E_n)$ meets the hypersurface S. When n is odd $\mathrm{supp}(J(E_n) \cap S) \subset \mathbb{T}$. It is a consequence of $S(E_n)$ supported by D^* ([Man], remarque 1.2 et lemme 1.3). Moreover, since $J(E_n) \cap S$ and $\mathbb{T}$ are both $\mathrm{SL}_2(\mathbb{C})$-invariant threefolds, we have $\mathrm{supp}(J(E_n) \cap S) = \mathbb{T}$. When n is even we still have $\mathbb{T} \subset \mathrm{supp}(J(E_n) \cap S)$ but there are others jumping conics in S. To determine which one are jumpinc conics we will study $M_n \cap S$.

8.20. Theorem. *(i)* $[M_{2n+1} \cap \mathrm{Sec}(v(\mathbb{P}^{2*}))]_{\mathrm{red}} = \mathbb{T}$
(ii) $[M_{2n} \cap \mathrm{Sec}(v(\mathbb{P}^{2*}))]_{\mathrm{red}} = \mathbb{T} \cup \bigcup_{z \in \mathfrak{P}_{2n}} \bar{\Omega}_{(\frac{1+z^2}{2z})^2}$

8.7. Exercise. *Prove this theorem.*

8.21. Corollary. *The degenerated jumping conics are*
(i) $[J(E_{2n+1}) \cap \mathrm{Sec}(v(\mathbb{P}^{2*}))]_{\mathrm{red}} = \mathbb{T}$
(ii) $[J(E_{2n}) \cap \mathrm{Sec}(v(\mathbb{P}^{2*}))]_{\mathrm{red}} = \mathbb{T} \cup \bigcup_{k \geq 2, k|n} \bigcup_{z \in \mathfrak{P}_{2k}} \bar{\Omega}_{(\frac{1+z^2}{2z})^2}$

Proof. It is an immediate consequence of the above theorem and the fact that $J(E_n) = \bigcup_{r \geq 3, r|n} M_r$. $\square$

8.8 Singular locus of $J(E_n)$

Quite nothing is known about this problem. From Barth and Bauer's paper it follows that a smooth conic n-circumscribed to D (and meeting D in four distinct points) is a smooth point of M_n. Then this conic is a smooth point of $J(E_n)$ too. This is not suprising since the jump order is as small as possible. But the link between jump order and singular point is not established yet, and we could expect, like in the case of jumping lines, that there is no coincidence. We could observe that the conic of $\mathbb{T}$ are singular in $J(E_n)$ when $n \geq 5$ is not a prime number. Moreover the jump order is the greatest one.

8.22. Proposition. *Any conic $C \notin \mathbb{T}$ verify $h^0(E_{n|C}(-[\frac{n}{2}] - 1)) = 0$.*
If $C \in \mathbb{T}$ we have $h^0(E_{n|C}(2-n)) \neq 0$ and $h^0(E_{n|C}(1-n)) = 0$.

Proof. Assume first that $C \notin \mathbb{T}$. When C is smooth you saw (in the proof of the first thm) that $h^0(E_{n|C}(-[\frac{n}{2}] - 1)) = 0$. It proves the proposition when n is odd. When n is even, the exact sequence ([Va1] page 435, *suites de liaison*)

$$0 \to E_n \longrightarrow E_{n+1} \longrightarrow O_l \to 0$$

where l is a tangent line to D proves that

$$h^0(E_{n|C}(-[\frac{n}{2}] - 1)) \neq 0 \Rightarrow h^0(E_{n+1|C}(-[\frac{n}{2}] - 1)) \neq 0.$$

Since $[\frac{n}{2}] = [\frac{n+1}{2}]$ this implies $C \in J(E_{n+1})$ which is a contradiction.

Assume now that $C = l \cup d$ with one of the two lines l and d is tangent to D (if $l = d$, the conic $l \cup d$ corresponds to the double line). When l eis tangent to D, the exact sequence

$$0 \to O_l(-1) \longrightarrow O_{l \cup d} \longrightarrow O_d \to 0$$

proves, after tensorisation by E_n, that $h^0(E_{n|l\cup d}(2-n)) \neq 0$.
Next we shows $h^0(E_n(1-n)) = h^1(E_n(-1-n)) = 0$ with the resolution

$$0 \to (n-1)O_{\mathbb{P}^2}(-1) \to (n+1)O_{\mathbb{P}^2} \to E_n \to 0$$

This implies $h^0(E_{n|l\cup d}(1-n)) = 0$.

Chapter 9

Hyperdeterminants

9.1 Multidimensional boundary format matrices

Hyperdeterminants were introduced by Cayley in [Cay], then this notion was forgotten until Gelfand, Kapranov and Zelevinsky recently rediscovered Cayley's results and gave a modern account of the subject. It is well known that a square matrix A is nondegenerate if and only if the homogeneous linear system $A \cdot x = 0$ has only the zero solution. This notion of nondegeneracy can be generalized to the multidimensional matrices, in this case the above linear system will be replaced by a suitable multilinear system. In the case of plane (bidimensional) matrices, the determinant is defined only for square matrices. So we have to expect for analogous restrictions on the format of multidimensional matrices in order to define their hyperdeterminant. We consider first the case of boundary format, that comes in a natural way, then we generalize the hyperdeterminant to a large class of matrices.
Let V_i for $i = 0, \ldots, p$ be a complex vector space of dimension $k_i + 1$. We assume $k_0 = \max_i k_i$. It is not necessary to assume $k_0 \geq k_1 \geq \ldots \geq k_p$ (see remark 9.10).
We remark that a multidimensional matrix $A \in V_0 \otimes \ldots \otimes V_p$ can be regarded as a map $V_0^\vee \to V_1 \otimes \cdots \otimes V_p$, hence taken the dual map $V_1^\vee \otimes \ldots \otimes V_p^\vee \to V_0$ (that we call also A), we can give the following definition:

9.1. Definition. *A multidimensional matrix A is called* degenerate *if there are $v_i \in V_i^*$, $v_i \neq 0$ for $i = 1, \ldots, p$ such that $A(v_1 \otimes \ldots \otimes v_p) = 0$.*

Such a solution $v_1 \otimes \ldots \otimes v_p$ is called a nontrivial solution.
If $p = 1$ nondegenerate matrices are exactly the matrices of maximal rank.
The following theorem, even if in a special case, gives the flavour of the utility of hyperdeterminants.

9.2. Theorem. *(Cayley) Let A be a $3 \times 2 \times 2$ matrix and let A_{00}, A_{01}, A_{10}, A_{11} be the 3×3 submatrices obtained by*

$$\begin{bmatrix} a_{000} & a_{001} & a_{010} & a_{011} \\ a_{100} & a_{101} & a_{110} & a_{111} \\ a_{200} & a_{201} & a_{210} & a_{211} \end{bmatrix}$$

deleting respectively the first column (00)*, the second* (01)*, the third* (10) *and the fourth one* (11)*. The multilinear system* $A(x \otimes y) = 0$ *has a nontrivial solution if and only if* $\det A_{01} \det A_{10} - \det A_{00} \det A_{11} = 0$

Proof. We may assume that the 3×4 matrix in the statement has maximal rank, otherwise one equation in the system is a combination of the other two and it is easy to check that a $2 \times 2 \times 2$ system has always nontrivial solutions.
Any solution $(x_0y_0, x_0y_1, x_1y_0, x_1y_1)$ of the system is proportional to

$$(\det A_{00}, -\det A_{01}, \det A_{10}, -\det A_{11})$$

Also a solution $(x_0y_0, x_0y_1, x_1y_0, x_1y_1) = (z_0, z_1, z_2, z_3)$ can be thought as a point in the quadric $\mathbb{P}^1 \times \mathbb{P}^1$ with equation $z_0z_3 - z_1z_2 = 0$. Hence the result follows. $\square$

9.3. Definition. *The expression* $\det A_{01} \det A_{10} - \det A_{00} \det A_{11}$ *is called the hyperdeterminant of the* $3 \times 2 \times 2$ *multidimensional matrix* A *and we denote it as* $Det(A)$ *(note the capital letter!).*

9.4. Lemma. *If* $k_0 < \sum_{i=1}^p k_i$ *then all matrices in* $V_0 \otimes \ldots \otimes V_p$ *are degenerate.*

Proof. The kernel of the map induced by A $V_1^\vee \otimes \ldots \otimes V_p^\vee \to V_0$ has codimension $\leq k_0 + 1 < \sum_{i=1}^p k_i + 1$. Hence $P(KerA) \subset P(V_1 \otimes \ldots \otimes V_p)$ meets the Segre variety. $\square$

9.1. Exercise. *Let* $v_i \in V_i^\vee$ *nonzero elements for* $i = 1, \ldots, p$*. Then* $\{A \in V_0 \otimes \ldots \otimes V_p | A(v_1 \otimes \ldots \otimes v_p) = 0\}$ *is a linear space of codimension* $k_0 + 1$*.*

9.5. Theorem. *If* $k_0 \geq \sum_{i=1}^p k_i$ *the degenerate matrices fill an irreducible variety of codimension* $k_0 - \sum_{i=1}^p k_i + 1$*.*

Proof. Consider the incidence variety

$$Z = \{(A, ([v_1], \ldots, [v_p]) \in (V_0 \otimes \ldots V_p) \times [\mathbb{P}(V_1) \times \ldots \times \mathbb{P}(V_p)] \,|A(v_1 \otimes \ldots \otimes v_p) = 0\}$$

By the previous exercise Z is a vector bundle over $\mathbb{P}(V_1) \times \ldots \times \mathbb{P}(V_p)$, and $dimZ = dimV_0 \otimes \ldots \otimes V_p - (k_0 - \sum_{i=1}^p k_i + 1)$. Hence Z is irreducible and its projection over $V_0 \otimes \ldots V_p$ is D which is still irreducible. Moreover the generic fiber of the projection $Z \to D$ is 0-dimensional (consider a linear space cutting in one point the Segre variety). It follows the result.

$\square$

9.6. Definition. *If* $k_0 = \sum_{i=1}^p k_i$ *the matrices* $A \in V_0 \otimes \ldots \otimes V_p$ *are called* of boundary format. *It follows from the previous theorem that boundary format degenerate matrices fill a irreducible hypersurface.*

9.2 Hyperdeterminants in the boundary format case

Let $A \in V_0 \otimes \ldots \otimes V_p$ be of boundary format and let $m_j = \sum_{i=1}^{j-1} k_i$ with the convention $m_1 = 0$.
We remark that the definition of m_i depends on the order we have chosen among the k_j's (see remark 9.10).

9.2. Exercise. *With the above notations the vector spaces* $V_0^\vee \otimes S^{m_1}V_1 \otimes \ldots \otimes S^{m_p}V_p$ *and* $S^{m_1+1}V_1 \otimes \ldots \otimes S^{m_p+1}V_p$ *have the same dimension* $N = \frac{(k_0+1)!}{k_1!\ldots k_r!}$.

9.7. Theorem. (and definition of ∂_A). *Let* $k_0 = \sum_{i=1}^p k_i$. *Then the hypersurface of degenerate matrices has degree* $N = \frac{(k_0+1)!}{k_1!\ldots k_r!}$ *and its equation is given by the determinant of the natural morphism*

$$\partial_A : V_0^\vee \otimes S^{m_1}V_1 \otimes \ldots \otimes S^{m_p}V_p \longrightarrow S^{m_1+1}V_1 \otimes \ldots \otimes S^{m_p+1}V_p$$

Proof. If A is degenerate then we get $A(v_1 \otimes \ldots \otimes v_p) = 0$ for some $v_i \in V_i^*$, $v_i \neq 0$ for $i = 1, \ldots, p$. Then $(\partial_A)^t \left(v_1^{\otimes m_1+1} \otimes \ldots \otimes v_p^{\otimes m_p+1}\right) = 0$.
Conversely if A is nondegenerate we get a surjective natural map of vector bundles over $X = \mathbb{P}(V_2) \times \ldots \times \mathbb{P}(V_p)$

$$V_0^\vee \otimes \mathcal{O}_X \xrightarrow{\phi_A} V_1 \otimes \mathcal{O}_X(1, \ldots, 1).$$

Indeed, by our definition, ϕ_A is surjective if and only if A is nondegenerate.
We construct a vector bundle S over $\mathbb{P}(V_2) \times \ldots \times \mathbb{P}(V_p)$ whose dual S^* is the kernel of ϕ_A so that we have the exact sequence

$$0 \longrightarrow S^* \longrightarrow V_0^\vee \otimes \mathcal{O} \longrightarrow V_1 \otimes \mathcal{O}(1, \ldots, 1) \longrightarrow 0. \tag{9.1}$$

After tensoring by $\mathcal{O}(m_2, \ldots, m_p)$ and taking cohomology we get

$$H^0(S^*(m_2, m_3, \ldots, m_p)) \longrightarrow V_0^\vee \otimes S^{m_1}V_1 \otimes \ldots \otimes S^{m_p}V_p \xrightarrow{\partial_A} S^{m_1+1}V_1 \otimes \ldots \otimes S^{m_p+1}V_p$$

and we need to prove

$$H^0(S^*(m_2, m_3, \ldots, m_p)) = 0. \tag{9.2}$$

Let $d = \dim\left(\mathbb{P}(V_2) \times \ldots \times \mathbb{P}(V_p)\right) = \sum_{i=2}^p k_i = m_{p+1} - k_1$.
Since $det(S^*) = \mathcal{O}(-k_1 - 1, \ldots, -k_1 - 1)$ and $rank\ S^* = d$ from remark 5.11 it follows that

$$S^*(m_2, m_3, \ldots, m_p) \simeq \wedge^{d-1} S(-1, -k_1 - 1 + m_3, \ldots, -k_1 - 1 + m_p) \tag{9.3}$$

Hence, by taking the $(d-1)$-th wedge power of the dual of the sequence (9.1), and using Künneth formula to calculate the cohomology as in [GKZ1], the result follows. □

9.8. Definition. The hyperdeterminant of $A \in V_0 \otimes \ldots \otimes V_p$ is the usual determinant of ∂_A, that is

$$Det(A) := det \partial_A \tag{9.4}$$

where $\partial_A = H^0(\phi_A)$ and $\phi_A : V_0^\vee \otimes \mathcal{O}_X \xrightarrow{\phi_A} V_1 \otimes \mathcal{O}_X(1, \ldots, 1)$ is the sheaf morphism associated to A. In particular

$$deg\, Det = \frac{(k_0+1)!}{k_1! \ldots k_r!}$$

This is theorem 3.3 of chapter 14 of [GKZ]. Now, applying remark 5.11, we have a $GL(V_0) \times \ldots \times GL(V_p)$-equivariant function

$$Det \colon V_0 \otimes \ldots \otimes V_p \to \bigotimes_{i=0}^{p} (detV_i)^{\frac{N}{k_i+1}}$$

$$A \mapsto det(\partial_A)$$

9.9. Corollary. *Let $A \in V_0 \otimes \ldots \otimes V_p$ of boundary format. A is nondegenerate if and only if $Det(A) \neq 0$*

9.10. Remark. Any permutation of the p numbers $k_1, \ldots, k_p$ gives different m_i's and hence a different map ∂_A. As noticed by Gelfand, Kapranov and Zelevinsky, in all cases the determinant of ∂_A is the same by theorem 9.7.

9.11. Example. The $3 \times 2 \times 2$ case. *In this case the morphism $V_0^* \otimes V_1 \to S^2 V_1 \otimes V_2$ is represented by a 6×6 matrix, which, with the obvious notations, is the following*

$$\begin{bmatrix} a_{000} & a_{010} & a_{001} & a_{011} & & \\ & a_{000} & & a_{001} & a_{010} & a_{011} \\ a_{100} & a_{110} & a_{101} & a_{111} & & \\ & a_{100} & & a_{101} & a_{110} & a_{111} \\ a_{200} & a_{210} & a_{201} & a_{211} & & \\ & a_{200} & & a_{201} & a_{210} & a_{211} \end{bmatrix}$$

The hyperdeterminant is the determinant of this matrix. This determinant is symmetric exchanging the second and the third index (this is not trivial from the above matrix!).

9.12. Example. *In the case $4 \times 3 \times 2$ the hyperderminant can be obtained as the usual determinant of one of the following two maps*

$$V_0^* \otimes V_1 \to S^2 V_1 \otimes V_2$$

$$V_0^* \otimes S^2 V_2 \to V_1 \otimes S^3 V_2$$

Alternative proof that the degree of the hypersurface of degenerate matrices is $N = \frac{(k_0+1)!}{k_1! \ldots k_p!}$. We know that A is degenerate iff the corresponding $\mathbb{P}(V_0)^\vee$ meets the Segre variety. Hence the condition is given by a polynomial $P(x_1, \ldots, x_m)$ in the variables $x_i \in \mathbb{P}(\wedge^{k_0+1} V_1 \otimes \ldots \otimes V_p)$ of degree equal to the degree of the Segre variety which is $\frac{k_0!}{k_1! \ldots k_p!}$. Since x_i have degree $k_0 + 1$
in terms of the coefficients of A, the result follows.

A case where this technique simplifies a lot the formulas is the $3 \times 2 \times 2$ case. In this case $\mathbb{P}(V_1) \times \mathbb{P}(V_2)$ is a quadric surface and $\mathbb{P}(V_0)^*$ is a point in the $\mathbb{P}(V_1 \otimes V_2)$ where the quadric is embedded. Hence consider the following 3×4 matrix

$$\begin{bmatrix} a_{000} & a_{010} & a_{001} & a_{011} \\ a_{100} & a_{110} & a_{101} & a_{111} \\ a_{200} & a_{210} & a_{201} & a_{211} \end{bmatrix}$$

Call p_{00} the determinant obtained by deleting the first column, p_{10} the determinant obtained by deleting the second column, and so on. The coordinates of $\mathbb{P}(V_0)^*$ are $(p_{00}, -p_{10}, p_{01}, -p_{11})$. Then the hyperdeterminant is obtained by the formula

$$DetA = p_{00}p_{11} - p_{01}p_{10}$$

Let $A = (a_{i_0,\dots,i_p})$ a matrix of format $(k_0+1)\times\cdots\times(k_p+1)$ and $B = (b_{j_0,\dots,j_q})$ of format $(l_0+1)\times\cdots\times(l_q+1)$, if $k_p = l_0$ it is defined (see [GKZ]) the convolution (or product) $A*B$ of A and B as the $(p+q)$-dimensional matrix C of format $(k_0+1)\times\cdots\times(k_{p-1}+1)\times(l_1+1)\times\cdots\times(l_q+1)$ with entries

$$c_{i_0,\dots,i_{p-1},j_1,\dots,j_q} = \sum_{h=0}^{k_p} a_{i_0,\dots,i_{p-1},h}b_{h,j_1,\dots,j_q}.$$

Similarly, we can define the convolution $A*_{r,s}B$ with respect to a pair of indices r, s such that $k_r = l_s$.

9.13. Theorem. *If $A \in V_0\otimes\cdots\otimes V_p$ and $B \in W_0\otimes\cdots\otimes W_q$ are nondegenerate boundary format matrices with $dimV_i = k_i+1$, $dimW_j = l_j+1$ and $W_0^\vee \simeq V_p$ then $A*B$ is also nondegenerate and*

$$Det(A*B) = (DetA)^{\binom{l_0}{l_1,\dots,l_q}}(DetB)^{\binom{k_0+1}{k_1,\dots,k_{p-1},k_p+1}} \tag{9.5}$$

We remark that equation (9.5) generalizes the Binet-Cauchy theorem for determinant of usual square matrices.

Proof. [DO] □

9.3. Exercise. *From (Definition 9.8) the degree of the hyperdeterminant of a boundary format $(k_0+1)\times\cdots\times(k_p+1)$ matrix A is given by the multinomial coefficient:*

$$N_A = \binom{k_0+1}{k_1,\dots,k_p}$$

Thus, (9.5) can be rewritten as

$$Det(A*B) = [(DetA)^{N_B}(DetB)^{N_A}]^{\frac{1}{l_0+1}}$$

9.14. Remark. *The same result of the above theorem works for the convolution with respect to the pair of indices $(j,0)$ with j varying in $\{1,\dots p\}$. Indeed the condition $W_0^\vee \simeq V_j$ ensure that $A*_{j,0}$ is again of boundary format and we can arrange the indices as in the proof because for any permutation σ we have $Det(A) = Det(\sigma A)$.*

9.15. Corollary. *If A and B are boundary format matrices then*

$$A \text{ and } B \text{ are nondegenerate} \iff A *_{0,j} B \text{ are nondegenerate}$$

The implication $\Longleftarrow$ of the previous corollary is true without the assumption of boundary format, see proposition 1.9 of [GKZ].

9.16. Remark. *Theorem 9.13 and the implication $\Longrightarrow$ of the corollary 9.15 work only for boundary format matrices. Indeed, if, for instance, A and B are $2 \times 2 \times 2$ matrices with*

$$\begin{aligned} a_{ijk} = 0 &\quad \textit{for all} \quad (i,j,k) \notin \{(0,0,0),(1,1,1)\} \quad \textit{and} \\ b_{krs} = 0 &\quad \textit{for all} \quad (k,r,s) \notin \{(0,0,1),(1,1,0)\} \end{aligned}$$

then A and B are nondegenerate since, applying Cayley formula (see [Cay] pag.89 or [GKZ] pag.448), their hyperdeterminants are respectively:

$$Det(A) = a_{000}^2 a_{111}^2 \quad \textit{and} \quad Det(B) = b_{001}^2 b_{110}^2$$

but the convolution $A * B$ is degenerate. In this case, by using Schläfli's method of computing hyperdeterminant ([GKZ]), it easy to find that $Det(A*B)$ corresponds to the discriminant of the polynomial $F(x_0, x_1) = a_{000}^2 a_{111}^2 b_{001}^2 b_{110}^2 x_0^2 x_1^2$ which obviously vanishes.

9.3 The dual variety and hyperdeterminants in the general case

We want to mention how the hyperdeterminant can be generalized to cases not of boundary format (see [GKZ]. The basic notion is that of dual variety. Let $X \subset \mathbb{P}^n$ be a variety. A hyperplane $H \in \mathbb{P}^{n*}$ is said to be tangent to X if there is a smooth point $x \in X$ such that $T_x X \subset H$. The **dual variety** $X^\vee$ is defined to be the Zariski closure of the set of tangent hyperplanes.

9.4. Exercise. *The dual variety of the rational normal curve $C \subset \mathbb{P}(S^d U)$ is a hypersurface in $\mathbb{P}(S^d U)$ which is the equation of the discriminant, that is $f \in C^\vee$ iff f has a multiple root. The degree of $C^\vee$ is $2(d-1)$.*

9.17. Example. *The previous exercise can be generalized to the higher dimensional case. The dual of the Veronese variety $V \subset \mathbb{P}(S^d\mathbb{C}^{n+1})$ is the discriminant hypersurface in $\mathbb{P}(S^d\mathbb{C}^{n+1})$ which contains the singular hypersurfaces of degree d in $\mathbb{P}(\mathbb{C}^{n+1})$. Its degree is $(n+1)(d-1)^n$.*

9.18. Example. *Let $k_0 = \max_{i=0,\dots,p} k_i$. Then the dual variety of the Segre variety $\mathbb{P}(V_0) \times \ldots \times \mathbb{P}(V_p)$ is a hypersurface if and only if $k_0 \le \sum_{i=1}^p k_i$ ([GKZ]). We underline that the boundary format corresponds to put an equality in the above inequality, which justifies its name.*

9.19. Definition. *Let $k_0 \le \sum_{i=1}^p k_i$. Then the equation of the hypersurface*

$$(\mathbb{P}(V_0) \times \ldots \times \mathbb{P}(V_p))^\vee$$

defines (up to a constant) the hyperdeterminant of $A \in \mathbb{C}^{k_0} \times \ldots \times \mathbb{C}^{k_p}$.

To show that the previous definition fits with the Definition 9.8 we have to check, according to the Corollary 9.9 that in the boundary format case the dual variety coincides with the variety of degenerate matrices. This follows from the following

9.20. Theorem. *Let $k_0 \geq \sum_{i=1}^{p} k_i$.*
(i) The dual variety $(\mathbb{P}(V_0) \times \ldots \times \mathbb{P}(V_p))^\vee$ coincides with the variety of degenerate matrices in $\mathbb{P}(V_0^* \otimes \ldots \otimes V_p^*)$.
(ii) In particular the dual variety has codimension $k_0 - \sum_{i=1}^{p} k_i + 1$.
(iii) deg $(\mathbb{P}(V_0) \times \ldots \times \mathbb{P}(V_p))^\vee = \binom{k_0+1}{k_1 \ldots k_p}$

Proof. We can identify (up to scalar multiples) $\phi \in \mathbb{P}(V_0^* \otimes \ldots \otimes V_p^*)$ with a hyperplane in $\mathbb{P}(V_0 \otimes \ldots \otimes V_p)$ and with $\phi\colon V_1^* \otimes \ldots \otimes V_p^* \to V_0$. So $\phi \in \left(\mathbb{P}^{k_0} \times \ldots \times \mathbb{P}^{k_p}\right)^\vee$ if and only if there exist nonzero $\tilde{x}_i \in V_i^*$ for $i = 0, \ldots, p$ such that ϕ contains the projective space generated by $\mathbb{P}(V_0^*) \times \{\tilde{x}_1\} \times \ldots \times \{\tilde{x}_p\}$ and $\{\tilde{x}_1\} \times \ldots \times \{\hat{x_i}\} \times \ldots \times \{\tilde{x}_p\} \times \mathbb{P}(V_i^*)$ for $i = 1, \ldots, p$. This is equivalent to the existence of nonzero $\tilde{x}_i \in V_i^*$ for $i = 0, \ldots, p$ such that
(a) $\phi(\tilde{x}_1 \otimes \ldots \otimes \tilde{x}_p)(x_0) = 0 \qquad \forall x_0 \in V_0^*$
(b) $\phi(\tilde{x}_1 \otimes \ldots \otimes x_i \otimes \ldots \otimes \tilde{x}_p)(\tilde{x}_0) = 0 \qquad \forall x_i \in V_i^*$
Condition (a) is equivalent to nondegeneracy of ϕ. Now with our assumption condition (a) implies condition (b). In fact denote, for a fixed ϕ

$$H_{\tilde{x}_1, \ldots, \tilde{x}_p} = \{x_0 \in V_0^* | \phi(\tilde{x}_1 \otimes \ldots \otimes \tilde{x}_p)(x_0) = 0\}$$

$$H_{\tilde{x}_1, \ldots, \hat{x_i}, \ldots, \tilde{x}_p} = \{x_0 \in V_0 | \phi(\tilde{x}_1 \otimes \ldots \otimes x_i \otimes \ldots \tilde{x}_p)(x_0) = 0 \quad \forall x_i \in V_i^*\}$$

$H_{\tilde{x}_1, \ldots, \tilde{x}_p}$ has codimension 1 for a general ϕ and coincides with V_0^* if ϕ satisfies condition (a).
$H_{\tilde{x}_1, \ldots, \hat{x_i}, \ldots, \tilde{x}_p} = \cap_{x_i \in V_i} H_{\tilde{x}_1, \ldots, x_i, \ldots, \tilde{x}_p}$ has codimension $\leq k_i + 1$ for a general ϕ and codimension $\leq k_i$ if ϕ satisfies condition (a). Hence if ϕ satisfies condition (a) the intersection of $H_{\tilde{x}_1, \ldots, \hat{x_i}, \ldots, \tilde{x}_p}$ for $i = 1, \ldots, p$ contains a nonzero element and this means that ϕ satisfies condition (b).
We have proved (i) and (ii).
(iii) (and (ii)) follow also from Proposition 6.2.

□

For the convenience of the reader we state the following theorem which is a straightforward generalization of Theorem 6.3 and contains Theorem 9.20 (i) and Corollary 9.9.

9.21. Theorem. *Let $A \in Hom(V_1 \otimes \ldots \otimes V_p, V_0)$ and let $k_0 \geq \sum_{i=1}^{p} k_i$. The following conditions are equivalent*
(i) A is degenerate
(ii) $A \in (\mathbb{P}(V_0) \times \ldots \times \mathbb{P}(V_p))^\vee$
(iii) $\mathbb{P}(V_0)^\vee \cap \mathbb{P}(V_1) \times \ldots \times \mathbb{P}(V_p) \neq \emptyset$ (where the embedding of $PP(V_0)$ is induced by A)
(iv) $V_0^\vee \otimes \mathcal{O}_X \xrightarrow{\phi_A} V_1 \otimes \mathcal{O}_X(1, \ldots, 1)$ is surjective.

In [GKZ] and in [WZ] a wellposed definition of hyperdeterminant was found, not depending to any constant.
The hyperdeterminant of a $2 \times 2 \times 2$ matrix (NOT of boundary format!) has degree 4. Its geometrical interpretation is the following. Let A be a nonzero $2 \times 2 \times 2$ matrix. You have an exact sequence on $\mathbb{P}^1$

$$0 \to \mathcal{O}(-1)^2 \xrightarrow{A} \mathcal{O}^2 \to \mathcal{I}_Z \to 0$$

where Z is a scheme . Then $Det A \neq 0$ if and only if Z is a reduced scheme of length 2 (that is consists of two distint points, Schläfli).

9.22. Remark. *The given definition of hyperdeterminant can be generalized to other cases where the codimension of the degenerate matrices is bigger than one, these cases are not covered in [GKZ]. If $k_0, \ldots, k_p$ are nonnegative integers satisfying $k_0 = \sum_{i=1}^p k_i$ then we denote again $m_j = \sum_{i=1}^{j-1} k_i$ with the convention $m_1 = 0$.*
Assume we have vector spaces $V_0, \ldots, V_p$ and a positive integer q such that $\dim V_0 = q(k_0+1), \dim V_1 = q(k_1+1)$ and $\dim V_i = (k_i+1)$ for $i = 2, \ldots, p$. Then the vector spaces $V_0 \otimes S^{m_1}V_1 \otimes \ldots \otimes S^{m_p}V_p$ and $S^{m_1+1}V_1 \otimes \ldots \otimes S^{m_p+1}V_p$ still have the same dimension.
In this case degenerate matrices form a subvariety of codimension bigger than 1.
The case $q = p = 2$ has been explored in [CO] leading to the proof that the moduli space of instanton bundles on $\mathbb{P}^3$ is affine.

9.5. Exercise. *Let A a $2 \times 2 \times 4$ matrix. In this format there is no good notion of hyperdeterminant.*
(i) Prove that if it is degenerate then the $2 \times 2 \times 3$ minors vanish.
(ii) Find a nondegenerate A such that all the $2 \times 2 \times 3$ minors vanish.
(iii) Define $I(A)$ to be the det of the 4×4 matrix obtained by A by stacking the two faces (do not worry about the way in doing it!). Prove that A is degenerate if and only if all the $2 \times 2 \times 3$ minors vanish and $I(A) = 0$.

9.4 Multidimensional matrices and bundles

A multidimensional matrix is an element $A \in V_0 \otimes \ldots \otimes V_p$ where V_i is a complex vector space of dimension $k_i + 1$ for $i = 0, \ldots, p$. We will say that the type of A is $k_0 \times \ldots \times k_p$. We want to consider the action of $SL(V_0) \times \ldots \times SL(V_p)$ on $\mathbb{P}(V_0 \otimes \ldots \otimes V_p)$. If $p = 1$ there are finitely many orbits determined by the rank. In particular all bidimensional matrices of maximal rank are equivalent under the action of $SL(V_0) \times SL(V_1)$. It is immediate to check that by dimensional reasons this property cannot hold in general for $p \geq 2$. We will restrict to the boundary format case, that is

$$k_0 = \sum_{i=1}^{p} k_i$$

We denote by $Det\ A$ the hyperdeterminant of A. Let $e_0^{(j)}, \ldots, e_{k_j}^{(j)}$ be a basis in V_j so that every $A \in V_0 \otimes \ldots \otimes V_p$ has a coordinate form

$$A = \sum a_{i_0, \ldots, i_p} e_{i_0}^{(0)} \otimes \ldots \otimes e_{i_p}^{(p)}$$

Let $x_0^{(j)}, \ldots, x_{k_j}^{(j)}$ be the coordinates in V_j. Then A has the following different descriptions:
1) A multilinear form

$$\sum_{(i_0,\ldots,i_p)} a_{i_0,\ldots,i_p} x_{i_0}^{(0)} \otimes \ldots \otimes x_{i_p}^{(p)}$$

2) An ordinary matrix $M_A = (m_{i_1 i_0})$ of size $(k_1+1)\times(k_0+1)$ whose entries are multilinear forms

$$m_{i_1 i_0} = \sum_{(i_2,\ldots,i_p)} a_{i_0,\ldots,i_p} x_{i_2}^{(0)} \otimes \ldots \otimes x_{i_p}^{(p)}$$

3) A sheaf morphism f_A on the product $X = \mathbb{P}^{k_2} \times \ldots \times \mathbb{P}^{k_p}$

$$\mathcal{O}_X^{k_0+1} \xrightarrow{f_A} \mathcal{O}_X(1,\ldots,1)^{k_1+1}$$

We have seen in the case $p=2$ the following

9.23. Theorem. *The following properties are equivalent*
i) Det $A \neq 0$.
ii) the matrix M_A *has constant rank* k_1+1 *on* $X = \mathbb{P}^{k_2} \times \ldots \times \mathbb{P}^{k_p}$.
iii) the morphism f_A *is surjective so that* $S_A^* = Ker f_A$ *is a vector bundle of rank* $k_0 - k_1$.

9.24. Theorem. *All nondegenerate matrices of type* $2 \times k \times (k+1)$ *are* $GL(2) \times GL(k) \times GL(k+1)$ *equivalent.*

Proof. Let A, A' two such matrices. They define two exact sequences on $\mathbb{P}^1$

$$0 \to \mathcal{O}(-k) \longrightarrow \mathcal{O}^{k+1} \xrightarrow{A} \mathcal{O}(1)^k \to 0$$

$$0 \to \mathcal{O}(-k) \longrightarrow \mathcal{O}^{k+1} \xrightarrow{A'} \mathcal{O}(1)^k \to 0$$

We want to show that there is a commutative diagram

$$\begin{array}{ccccccccc}
0 & \to & \mathcal{O}(-k) & \longrightarrow & \mathcal{O}^{k+1} & \xrightarrow{A} & \mathcal{O}(1)^k & \to & 0 \\
 & & \downarrow 1 & \searrow & \downarrow f & & & & \\
0 & \to & \mathcal{O}(-k) & \longrightarrow & \mathcal{O}^{k+1} & \xrightarrow{A'} & \mathcal{O}(1)^k & \to & 0
\end{array}$$

In order to show the existence of f we apply the functor $Hom(-, \mathcal{O}^{k+1})$ to the first row. We get $Hom(\mathcal{O}^{k+1}, \mathcal{O}^{k+1}) \xrightarrow{g} Hom(\mathcal{O}(-k), \mathcal{O}^{k+1}) \to Ext^1(\mathcal{O}(1)^k, \mathcal{O}^{k+1}) \simeq H^1(\mathcal{O}(-1)^{k(k+1)}) = 0$. Hence g is surjective and f exists. Now it is straightforward to complete the diagram with a morphism $\phi \colon \mathcal{O}(1)^k \to \mathcal{O}(1)^k$, which is a isomorphism by the snake lemma. $\square$

Let $(x_0, \ldots, x_1)$ be homogeneous coordinates on $\mathbb{P}(V)$. We set

$$I_k(x_0,x_1) := \begin{pmatrix} x_0 & x_1 & & \\ & \ddots & \ddots & \\ & & x_0 & x_1 \end{pmatrix} \quad \text{and} \quad \tilde{I}_k(x_0,x_1) := \begin{pmatrix} x_1 & & & \\ x_0 & x_1 & & \\ & \ddots & \ddots & \\ & & x_0 & x_1 \\ & & & x_0 \end{pmatrix}.$$

A reformulation of the previous theorem is the following

9.25. Proposition. *Every surjective morphism of vector bundles on* $\mathbb{P}^1$

$$\mathcal{O}_{\mathbb{P}^1}^{k+1} \to \mathcal{O}_{\mathbb{P}^1}(1)^k$$

is represented, in a suitable system of coordinates (x_0, x_1)*, by the matrix* I_k*.*

9.26. Corollary. *For* $k = 2$ *any Steiner bundle is Schwarzenberger.*

9.5 Multidimensional Matrices of boundary Format and Geometric Invariant Theory

It is well known that all one dimensional subgroups of the complex Lie group $SL(2)$ either are conjugated to the maximal torus consisting of diagonal matrices (which is isomorphic to $\mathbb{C}^*$) or are conjugated to the subgroup $\mathbb{C} \simeq \left\{ \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} | b \in \mathbb{C} \right\}$.

9.27. Definition. *A* $p+1$*-dimensional matrix of boundary format* $A \in V_0 \otimes \ldots \otimes V_p$ *is called triangulable if one of the following equivalent conditions holds:*
i) there exist bases in V_j *such that* $a_{i_0,\ldots,i_p} = 0$ *for* $i_0 > \sum_{t=1}^{p} i_t$
ii) there exist a vector space U *of dimension 2, a subgroup* $\mathbb{C}^* \subset SL(U)$ *and isomorphisms* $V_j \simeq S^{k_j}U$ *such that if* $V_0 \otimes \ldots \otimes V_p = \oplus_{n\in\mathbb{Z}} W_n$ *is the decomposition into direct sum of eigenspaces of the induced representation, we have* $A \in \oplus_{n\geq 0} W_n$

Proof of the equivalence between i) and ii)
Let x, y be a basis of U such that $t \in \mathbb{C}^*$ acts on x and y as tx and $t^{-1}y$. Set $e_k^{(j)} := x^k y^{k_j-k}\binom{k_j}{k} \in S^{k_j}U$ for $j > 0$ and $e_k^{(0)} := x^{k_0-k}y^k\binom{k_0}{k} \in S^{k_0}U$ so that $e_{i_0}^{(0)} \otimes \ldots \otimes e_{i_p}^{(p)}$ is a basis of $S^{k_0}U \otimes \ldots \otimes S^{k_p}U$ which diagonalizes the action of $\mathbb{C}^*$. The weight of $e_{i_0}^{(0)} \otimes \ldots \otimes e_{i_p}^{(p)}$ is $2\left(\sum_{t=1}^{p} i_t - i_0\right)$, hence ii) implies i). The converse is trivial.

9.28. Definition. *A* $p+1$*-dimensional matrix of boundary format* $A \in V_0 \otimes \ldots \otimes V_p$ *is called diagonalizable if one of the following equivalent conditions holds*
i) there exist bases in V_j *such that* $a_{i_0,\ldots,i_p} = 0$ *for* $i_0 \neq \sum_{t=1}^{p} i_t$
ii) there exist a vector space U *of dimension 2, a subgroup* $\mathbb{C}^* \subset SL(U)$ *and isomorphisms* $V_j \simeq S^{k_j}U$ *such that* A *is a fixed point of the induced action of* $\mathbb{C}^*$*.*

9.29. Definition. *A* $p+1$*-dimensional matrix of boundary format* $A \in V_0 \otimes \ldots \otimes V_p$ *is an identity if one of the following equivalent conditions holds*
i) there exist bases in V_j *such that*

$$a_{i_0,\ldots,i_p} = \begin{cases} 0 & \text{for} \quad i_0 \neq \sum_{t=1}^{p} i_t \\ 1 & \text{for} \quad i_0 = \sum_{t=1}^{p} i_t \end{cases}$$

ii) there exist a vector space U *of dimension* 2 *and isomorphisms* $V_j \simeq S^{k_j}U$ *such that* A *belongs to the unique one dimensional* $SL(U)$*-invariant subspace of* $S^{k_0}U \otimes S^{k_1}U \otimes \ldots \otimes S^{k_p}U$

The equivalence between i) and ii) follows easily from the following remark: the matrix A satisfies the condition ii) if and only if it corresponds to the natural multiplication map $S^{k_1}U \otimes \ldots \otimes S^{k_p}U \to S^{k_0}U$ (after a suitable isomorphism $U \simeq U^*$ has been fixed).
In the case $p = 2$ the identity matrices correspond exactly to the Schwarzenberger bundles. The definitions of triangulable, diagonalizable and identity apply to elements of $\mathbb{P}(V_0 \otimes \ldots \otimes V_p)$ as well. In particular all identity matrices fill a distinguished orbit in $\mathbb{P}(V_0 \otimes \ldots \otimes V_p)$. The function Det is $SL(V_0) \times \ldots \times SL(V_p)$-invariant, in particular if $Det\ A \neq 0$ then A is semistable for the action of $SL(V_0) \times \ldots \times SL(V_p)$. We denote by $Stab\ (A) \subset SL(V_0) \times \ldots \times SL(V_p)$ the stabilizer subgroup of A and by $Stab\ (A)^0$ its connected component containing the identity. The main results are the following.

9.30. Theorem. *([AO]) Let $A \in \mathbb{P}(V_0 \otimes \ldots \otimes V_p)$ of boundary format such that $Det\ A \neq 0$. Then*

$$A \text{ is triangulable} \iff A \text{ is not stable for the action of } SL(V_0) \times \ldots \times SL(V_p)$$

9.31. Theorem. *([AO]) Let $A \in \mathbb{P}(V_0 \otimes \ldots \otimes V_p)$ be of boundary format such that $Det\ A \neq 0$. Then*

$$A \text{ is diagonalizable} \iff Stab(A) \text{ contains a subgroup isomorphic to } \mathbb{C}^*$$

***add picture
The proof of the two above theorems relies on the Hilbert-Mumford criterion. The proof of the following theorem needs more geometry.

9.32. Theorem. *([AO] for $p = 2$, [D] for $p \geq 3$) Let $A \in \mathbb{P}(V_0 \otimes V_1 \otimes \ldots \otimes V_p)$ of boundary format such that $Det\ A \neq 0$. Then there exists a 2-dimensional vector space U such that $SL(U)$ acts over $V_i \simeq S^{k_i}U$ and according to this action on $V_0 \otimes \ldots \otimes V_p$ we have $Stab\ (A)^0 \subset SL(U)$. Moreover the following cases are possible*

$$Stab\ (A)^0 \simeq \begin{cases} 0 & \textit{(trivial subgroup)} \\ \mathbb{C} & \\ \mathbb{C}^* & \\ SL(2) & \textit{(this case occurs if and only if } A \textit{ is an identity)} \end{cases}$$

9.33. Remark. *When A is an identity then $Stab\ (A) \simeq SL(2)$.*

9.34. Theorem. *([AO99] theorem 6.14) Let E be a Steiner bundle of rank n on $\mathbb{P}^n$. Let $Sym(E) = \{g \in SL(n+1) | g^*E \simeq E\}$ and let $Sym^0(E)$ be its connected component containing the identity. Then $Sym^0(E)$ is always contained in $SL(2)$ and the equality holds if and only if E is a Schwarzenberger bundle.*

Chapter 10

Appendix

10.1 Grassmannians and Segre varieties

Let V be a vector space of dimension $n+1$ and consider $v \in V$, $v \neq 0$. Define

$$\phi_i \colon \wedge^i V \to \wedge^{i+1} V$$

by

$$\phi_i(\omega) := \omega \wedge v$$

10.1. Lemma. *(Koszul complex of a vector). The following sequence is exact*

$$0 \longrightarrow \wedge^0 V = \mathbb{C} \xrightarrow{\phi_0} \wedge^1 V \xrightarrow{\phi_1} \wedge^2 V \xrightarrow{\phi_2} \ldots \xrightarrow{\phi_n} \wedge^{n+1} V \longrightarrow 0$$

Proof. It is evident that the above sequence is a complex. Choose a basis of V given by $e_1, \ldots, e_n, e_{n+1} = v$. Choose $\omega \in \wedge^k V$ such that $\phi_k(\omega) = \omega \wedge v = 0$. If $\omega = \sum_{i_1 < \ldots < i_k} a_{i_1 \ldots i_k} e_{i_1} \wedge \ldots \wedge e_{i_k}$ then each nonzero coefficient $a_{i_1 \ldots i_k}$ has $i_k = n+1$. Hence $\psi = \sum_{i_1 < \ldots < i_{k-1}} a_{i_1 \ldots i_k} e_{i_1} \wedge \ldots \wedge e_{i_{k-1}}$ satisfies $\phi_{k-1}(\psi) = \psi \wedge v = \omega$. □

10.2. Remark. *The theorem 10.1 admits the following generalization [Serre, Algèbre locale, multiplicités, LNM 11, Springer]. Let E be a vector bundle of rank n over X and consider $s \in H^0(X, E)$ such that $Z = \{x | s(x) = 0\}$ has pure codimension n. Define $\phi_i \colon \wedge^i E \to \wedge^{i+1} E$ by $\phi_i(\omega) = \omega \wedge s$ and the dual $\phi_i^t \colon \wedge^{i+1} E^* \to \wedge^i E^*$. Then the following sequence is exact*

$$0 \longrightarrow \wedge^n E^* \xrightarrow{\phi_{n-1}^t} \wedge^{n-1} E^* \xrightarrow{\phi_{n-2}^t} \ldots \xrightarrow{\phi_1^t} E^* \xrightarrow{\phi_0^t} \mathcal{O}_X \longrightarrow \mathcal{O}_Z \longrightarrow 0$$

and it is called the Koszul sequence associated to s.

The Grassmannian

Let $\mathbb{P}^n = \mathbb{P}(V)$. Grassmannians parametrize the set of linear subspaces of dimension k in $\mathbb{P}^n$. The best way to give to this set the structure of an algebraic variety is the following definition.

10.3. Definition. . *$Gr(k, n) = Gr(\mathbb{P}^k, \mathbb{P}^n)$ is defined as the subset of $\mathbb{P}(\wedge^{k+1} V)$ consisting of decomposable tensors.*

10.4. Theorem. *Gr(k,n) is a projective variety of dimension* $(k+1)(n-k)$.

In order to prove the theorem we have the following

10.5. Lemma.

i) If $\omega \in \wedge^{k+1}V$ *then* $\dim\{v \in V | \omega \wedge v = 0\} \leq k+1$.

ii) $\omega \in \wedge^{k+1}V$ *is decomposable if and only if* $\dim\{v \in V | \omega \wedge v = 0\} = k+1$.

Proof of lemma 10.5. By the theorem 10.1

$$\omega \wedge v = 0 \quad \Leftrightarrow \quad \exists \psi \text{ such that } \omega = \psi \wedge v$$

Hence if $v_1, \ldots, v_j$ are independent elements in $\{v \in V | \omega \wedge v = 0\}$ it follows that

$$\omega = \psi' \wedge v_1 \wedge \ldots \wedge v_j$$

(choose a basis containing $v_1, \ldots, v_j$!) and the result is obvious.

Proof of the theorem 10.4 Consider the morphism

$$\begin{array}{rcl} \phi(\omega)\colon V & \to & \wedge^{k+2}V \\ v & \mapsto & \omega \wedge v \end{array}$$

By the lemma $\omega \in Gr(k,n)$ if and only if $rk\ \phi(\omega) = n-k$. $rk\ \phi(\omega)$ is always $\geq n-k$ by the lemma 10.5 i), so the last condition is satisfied if and only if $rk\ \phi(\omega) \leq n-k$. The map

$$\begin{array}{rcl} \wedge^{k+1}V & \to & Hom(V, \wedge^{k+2}V) \\ \omega & \mapsto & \phi(\omega) \end{array}$$

is linear, hence the entries of the matrix $\phi(\omega)$ are homogeneous coordinates on $\mathbb{P}(\wedge^{k+1}V)$ and $Gr(k,n)$ is defined by the vanishing of the $(n-k+1) \times (n-k+1)$ minors of this matrix.

The map $i\colon Gr(k,n) \to \mathbb{P}(\wedge^{k+1}V)$ is called the Plücker embedding. The equations that we have found define the Grassmannian as scheme but they do not generate the homogeneous ideal of $G = Gr(k,n)$. The ideal $I_{G,\mathbb{P}}$ is generated by quadrics that are called Plücker quadrics (see [Harris]).

10.1. Exercise. *Prove that i is a closed immersion*

Hint: writing down coordinates you can show injectivity.
In conclusion we have a biunivoc correspondence between points in $Gr(k,n)$ and linear subspaces $\mathbb{P}^k \subset \mathbb{P}^n$. The following construction shows that this correspondence is much more rich than a set correspondence.
Define the incidence variety $\mathcal{U} \subset Gr(k,n) \times \mathbb{P}^n$ given by $\{(g,x) | x \in g\}$ (really $\mathcal{U}$ is the projective bundle $\mathbb{P}(U)$ where U is the universal bundle on the Grassmannian). $\mathcal{U} \to Gr(k,n)$ satisfies the following universal property: for every subscheme $\mathcal{F} \subset S \times \mathbb{P}^n$ such that the projection $\mathcal{F} \to S$ is flat ($\mathcal{F}$ with this property is called a flat family) and $\mathcal{F}_s$ is a linear $\mathbb{P}^k$ for every $s \in S$ then there exists a unique morphism $\phi\colon S \to Gr(k,n)$ such that $\phi^*\mathcal{U} = \mathcal{F}$. This property says that the Grassmannians are *Hilbert schemes* (in fact they are the simplest Hilbert schemes). For an introduction to Hilbert schemes see ([Eis-Har]). It

is interesting to remark that in order to construct the Hilbert schemes, the Grassmannians are needed as first step. We will see in connections with vector bundles other examples of the ubiquity of Grassmannians in modern geometry.
When $k = 0$ or $n-1$, $Gr(k,n)$ is isomorphic to the projective space $\mathbb{P}^n$. The simplest Grassmannian which is not a projective space is $Gr(1,3)$.

10.2. Exercise. *Let* $p_{ij} = \begin{vmatrix} x_i & x_j \\ y_i & y_j \end{vmatrix}$ *for* $0 \le i < j \le 3$ *be Plücker coordinates in the embedding* $Gr(1,3) \to \mathbb{P}^5$. *Prove that* $Gr(1,3)$ *is given by the smooth quadric with equation*

$$p_{01}p_{23} - p_{02}p_{13} + p_{03}p_{12} = 0$$

10.3. Exercise. $Gr(k,n)$ *is a rational variety of dimension* $(k+1)(n-k)$

10.4. Exercise. $Gr(k,n)$ *is a homogeneous variety, in particular it is smooth.*

A more advanced property is that $Pic(Gr) = \mathbb{Z}$. This is equivalent to say that in the Plücker embedding every effective divisor on Gr is cutted as a scheme by a hypersurface in $\mathbb{P}(\wedge^{k+1}V)$. This is a famous result proved by Severi in 1915, which is the core of the definition of the Chow variety, which is one of the first examples of moduli spaces. The Plücker embedding corresponds to the embedding given by the complete linear system $H^0(\mathcal{O}(1))$.
In the case $k = 1$ we have the *Grassmannian of lines.*

10.5. Exercise. *Show that a Grassmannian of lines in its Plücker embedding can be seen as the (projective) variety of skew-symmetric matrices of rank* 2. *Its equation are given by* 4×4 *pfaffians, that are quadrics in the Plücker embedding.*

10.6. Exercise. *Show that the secant variety of* $Gr(\mathbb{P}^1, \mathbb{P}^n)$ *can be interpreted as the variety of skew-symmetric matrices of rank* ≤ 4 *and its dimension is* $4n-7$. *Compare this with the general variety of dimension* $2(n-1)$ *that has secant variety of dimension* $4n-3$. *Deduce that* $Gr(\mathbb{P}^1, \mathbb{P}^n)$ *projects smoothly in* $\mathbb{P}^{4n-7}$.

Kunneth formula for sheaves Let X, Y be projective varieties and let p and q be the two projections of the product $X \times Y$ onto the two factors. The formula says that for every coherent sheaves F over X and G over Y we have

$$H^i(X \times Y, p^*F \otimes q^*G) = \oplus_{j=1}^{i} H^j(X,F) \otimes H^{i-j}(Y,G)$$

Hirzebruch ([Hir]) attributes this formula to Grothendieck.

The Segre variety

Let V_i be complex vector spaces of dimension $k_i + 1$ for $i = 0, \ldots, p$ The Segre variety is the product $\mathbb{P}(V_0) \times \ldots \times \mathbb{P}(V_p)$. The Segre embedding describes this variety in the space of tensors in a manner similar to Grassmannians. In fact we have a natural closed immersion

$$\mathbb{P}(V_0) \times \ldots \times \mathbb{P}(V_p) \xrightarrow{i} \mathbb{P}(V_0 \otimes \ldots \otimes V_p)$$

given by $(v_0, \ldots, v_p) \mapsto v_0 \otimes \ldots \otimes v_p$ which describes the image of i as the set of decomposable tensors.

When $p = 1$ the tensor space $\mathbb{P}(V_0 \otimes V_1)$ can be interpreted as $\mathbb{P}(Hom(V_0^*, V_1))$ and in this case the Segre variety corresponds to morphisms of rank ≤ 1. Its equations in this case are given by 2×2 minors. It is known that in general the ideal of the Segre variety is generated by quadrics.

The Hilbert polynomial of the Segre variety in the Segre embedding is $\prod_{i=0}^{p} \binom{t+k_i}{k_i}$. It follows that the degree of the Segre embedding is the multinomial coefficient $\left(\frac{\sum k_i)!}{\prod(k_i!)}\right)$.

The Picard group of $\mathbb{P}(V_0) \times \ldots \times \mathbb{P}(V_p)$ is isomorphic to $\mathbb{Z}^{p+1}$ and its elements will be denoted by $\mathcal{O}(a_0, \ldots, a_p)$ where a_i are integers. The previous embedding corresponds to the embedding given by the complete linear system $H^0(\mathcal{O}(1, \ldots, 1))$.

10.2 Vector bundles

A vector bundle over X is unformally a family of vector spaces parametrized by X. $X \times \mathbb{C}^r \to X$ is called the trivial bundle of rank r. In general we require that a vector bundle is locally trivial. More precisely

10.6. Definition. *A vector bundle E of rank r over an algebraic variety X is an algebraic variety E with a surjective morphism*

$$\pi \colon E \to X$$

such that there exists an open covering $\{U_\alpha\}_{\alpha \in I}$ of X satisfying the two properties i) there exist isomorphisms $\phi_\alpha \colon \pi^{-1}(U_\alpha) \longrightarrow U_\alpha \times \mathbb{C}^r$ making commutative the diagram

$$\begin{array}{ccc} \pi^{-1}(U_\alpha) & \xrightarrow{\phi_\alpha} & U_\alpha \times \mathbb{C}^r \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle p_1} \\ U_\alpha & \xrightarrow{id} & U_\alpha \end{array}$$

ii) $\forall \alpha, \beta \in I$ the composition (restricted)

$$(U_\alpha \cap U_\beta) \times \mathbb{C}^r \xrightarrow{\phi_\beta^{-1}} \pi^{-1}(U_\alpha \cap U_\beta) \xrightarrow{\phi_\alpha} (U_\alpha \cap U_\beta) \times \mathbb{C}^r$$

has the form

$$\phi_\alpha \circ \phi_\beta^{-1}(x, v) = (x, g_{\alpha\beta}(x)v)$$

where

$$g_{\alpha\beta} \colon (U_\alpha \cap U_\beta) \to GL(r)$$

are algebraic.

i) means that the fibration is locally trivial and that each fiber $\pi^{-1}(x)$ is isomorphic to $\mathbb{C}^r$.
ii) means that the structure group of the bundle is linear.
$g_{\alpha\beta}$ are called the transition functions and satisfy the properties

$$g_{\alpha\beta}^{-1} = g_{\beta\alpha} \tag{10.1}$$

$$g_{\alpha\beta} \cdot g_{\beta\gamma} = g_{\alpha\gamma} \tag{10.2}$$

In equivalent way, given a covering $\{U_\alpha\}_{\alpha\in I}$ with a set of transition functions $g_{\alpha\beta}(x)$ satisfying (10.1) and (10.2) we can construct a vector bundle E as the quotient of the disjoint union

$$\coprod_\alpha (U_\alpha \times \mathbb{C}^r)$$

by the relation $\sim$ defined in the following way:

$$\forall (x,v) \in U_\alpha \times \mathbb{C}^r \quad (x',v') \in U_\beta \times \mathbb{C}^r$$

we have

$$(x,v) \sim (x',v') \text{ iff } x = x' \quad v = g_{\alpha\beta}(x)v'$$

10.7. Remark. *We can say synthetically that "the transition functions determine the bundle".*

If $g_{\alpha\beta}$ are transition functions for E and $h_{\alpha\beta}$ are transition functions for F then

$$\begin{pmatrix} g_{\alpha\beta} & \\ & h_{\alpha\beta} \end{pmatrix} \text{ are transition functions for } E \oplus F$$

(this can be taken as definition of $E \oplus F$)

$$(g_{\alpha\beta}^{-1})^t \text{ are transition functions for } E^* \text{ dual bundle}$$

$$g_{\alpha\beta} \otimes h_{\alpha\beta} \text{ are transition functions for } E \otimes F$$

If $T\colon GL(r) \to GL(r')$ is any representation we define $T(E)$ to be the bundle with transition functions $T(g_{\alpha\beta})$. This construction applies in particular to $T = \wedge^k$ and $T = S^k$.
If $f\colon X \to Y$ is a map and E is a bundle on Y with transition functions $g_{\alpha\beta}(y)$ then f^*E is the bundle on X with transition functions $g_{\alpha\beta}(f(x))$.
If X is smooth the bundle Ω^1_X of $1-forms$ can be defined as the bundle with transition functions given by the jacobian matrices obtained by change of local coordinates. The tangent bundle is $TX = (\Omega^1_X)^*$.
A vector bundle of rank 1 is called a line bundle. The set of line bundles has a natural structure of abelian group isomorphic to $H^1(X, \mathcal{O}^*)$ with the multiplication given by the tensor product and the inverse given by the dual bundle.
A section of E is an algebraic map

$$s\colon X \to E$$

such that $\pi \circ s = id_X$

10.8. Definition. *A vector bundle is called spanned if there are (global) sections $s_1, \ldots, s_k$ such that $\forall x \in X$ the vectors $s_1(x), \ldots, s_k(x)$ span the fiber $\pi^{-1}(x)$.*

To any vector bundle E we can associate a locally free sheaf of $\mathcal{O}_X$-modules $\mathcal{E}$ defined by

$$\mathcal{E}(U) := \{\text{sections of } E_{|U}\}$$

Conversely to any locally free sheaf $\mathcal{E}$ is associated a vector bundle with fiber $E_x \simeq \mathcal{E}_x/\mathcal{M}_x\mathcal{E}_x$ defined as the **Spec** of the symmetric algebra of $\mathcal{E}$ (see [Ha]).
For any coherent sheaf $\mathcal{E}$ the fiber $E_x \simeq \mathcal{E}_x/\mathcal{M}_x\mathcal{E}_x$ is a vector space whose dimension is called the rank of $\mathcal{E}$ at x.

10.9. Proposition. *$\mathcal{E}$ is locally free if and only if it has constant rank.*

Proof [Ha] A sheaf morphism between bundles $E \xrightarrow{f} F$ induces linear maps $E_x \xrightarrow{f_x} F_x$ for every $x \in X$. It follows from the definition of sheaf morphism that f is injective if and only if f_x is injective for generic $x \in X$. f is called a bundle morphism if it has constant rank $\forall x \in X$. In particular f is injective and it is a bundle morphism if and only if f_x is injective for any $x \in X$. When we write a exact sequence of sheaves or bundles

$$0 \to E \xrightarrow{f} F \to G \to 0$$

we assume it is a exact sequence of sheaf morphisms. In the sequence above assume E and F are locally free (bundles). Then f is always injective as sheaf morphism and it is a bundle morphism if and only if G is locally free. A version of Nakayama lemma states that a surjective sheaf morphism between bundles is also a bundle morphism, in particular its kernel is locally free.

10.10. Remark. *Let be given a bundle morphism $E \xrightarrow{f} F$. The* $\ker f$ *is locally free.*

10.7. Exercise. *Prove that for any vector space V of dimension r there is a canonical isomorphism $\wedge^k E \simeq \wedge^{r-k} E^* \otimes \wedge^r E$. The same isomorphism hold if E is a vector bundle of rank r.*

It is usual to identify a vector bundle E and the associated locally free sheaf $\mathcal{E}$. In particular the cohomology groups $H^q(X, E)$ are (by definition) the cohomology groups $H^q(X, \mathcal{E})$. Note that $H^0(X, E)$ is the space of global sections of E. In particular a vector bundle is spanned if and only if the evaluation map

$$H^0(X, E) \otimes \mathcal{O} \to E$$

is surjective.

10.3 Wedge power of a short exact sequence

Let A, B, C be vector spaces. Suppose we have a exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

Then for any integer $k \geq 1$ the following sequences are exact

$$0 \longrightarrow \wedge^k A \longrightarrow \wedge^k B \longrightarrow \wedge^{k-1} B \otimes C \longrightarrow \wedge^{k-2} B \otimes S^2 C \longrightarrow \ldots \longrightarrow S^k C \longrightarrow 0$$

$$0 \longrightarrow S^k A \longrightarrow S^{k-1} A \otimes B \longrightarrow S^{k-2} A \otimes \wedge^2 B \longrightarrow \ldots \longrightarrow \wedge^k B \longrightarrow \wedge^k C \longrightarrow 0$$

All the maps are natural. Obviously the second sequence is dual of the first one.

10.8. Exercise. *Prove from the above the following combinatorial identities*
$\binom{a}{k} = \sum_{j=0}^{k} (-1)^j \binom{a+c}{k-j} \binom{c+j-1}{j}$
$\binom{a+k-1}{k} = \sum_{j=0}^{k} (-1)^j \binom{a+c+k-j-1}{k-j} \binom{c}{j}$

Remark
There are analogous sequences exchanging symmetric with wedge powers.

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