RATIONAL HOMOGENEOUS VARIETIES

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These notes have been written for distribution to the participants to the summer school in Algebraic Geometry organized by the Scuola Matematica Universitaria in Cortona in the period 13-26 August 1995.

The aim was to describe the classification of rational homogeneous varieties and to provide the theorems of Borel-Weil and Bott.

Lie algebras are introduced starting from the definition and they are studied as far as it is necessary for the aim. At the other side, the knowledge of basic techniques of algebraic geometry (as sheaf cohomology) and algebraic topology was assumed.

I learned most of the material of these notes from many lectures and discussions with Vincenzo Ancona and Alan Huckleberry. I am sincerely grateful to both of them and also to all the participants to the course in Cortona, especially to Raffaella Paoletti and Anke Simon for careful proofreading and for their suggestions.

§1. Introduction

An algebraic variety is a quasiprojective variety over the field $\mathbf{C}$ of complex numbers.

Definition 1.1. *An algebraic group G is a set which is both a group and an algebraic variety such that the two structures are compatible in the sense that the map*

$$G \times G \to G$$
$$(x, y) \mapsto xy^{-1}$$

is an algebraic morphism.

In the category of algebraic groups the morphisms are algebraic maps that are group homomorphisms.

Basic example. *$G = GL(n)$ is the algebraic group of nonsingular $n \times n$ matrices with complex coefficients. $GL(n)$ is affine via the embedding*

$$GL(n) \to GL(n+1) \subset \mathbf{C}^{(n+1)^2}$$
$$g \mapsto \begin{pmatrix} g & 0 \\ 0 & \det\, g^{-1} \end{pmatrix}$$

Hence $GL(n)$ is described in $\mathbf{C}^{(n+1)^2}$ by $2n$ linear equations plus the equation $\det = 1$.

A closed (algebraic) subgroup of $GL(n)$ is called linear. In particular a linear group is affine. Also the converse is true (see [Borel]).

Definition 1.2. *An algebraic group G acts over an algebraic variety X if there is an algebraic morphism*

$$G \times X \to X$$
$$(g, x) \mapsto gx$$

satisfying the two conditions

$$1x = x \quad \forall x \in X$$
$$g_1(g_2 x) = (g_1 g_2)x \quad \forall x \in X \quad \forall g_1, g_2 \in G$$

Definition 1.3. *An algebraic variety is called homogeneous if there is an algebraic group acting transitively on it.*

Remark. *Every algebraic group is homogeneous by acting on itself. Every homogeneous variety is smooth.*

Examples.

- $GL(n+1)$ acts transitively on $\mathbf{C}^{n+1} \setminus \{0\}$ and on $\mathbf{P}^n$.
- $X = \mathbf{C}^n/\Gamma$ with Γ discrete subgroup of rank $2n$ satisfying the Riemann conditions is an algebraic group which is projective and is called an *abelian variety*.

We are interested mainly in projective varieties. We state now the basic results about these topics.

Theorem 1.4. *(Chevalley). A projective variety which is an algebraic group is an abelian variety (in particular it is an abelian group).*

Theorem 1.5. *(Borel-Remmert, 1962). A projective variety which is homogeneous is isomorphic to a product*

$$A \times X$$

where A is an abelian variety and X is rational homogeneous.

Rational homogeneous varieties are the main subject of this course. They can be completely classified. There are only finitely many rational homogeneous varieties (up to isomorphism) of fixed dimension n. The first result is

Theorem 1.6. *(Borel-Remmert). A rational homogeneous variety X is isomorphic to a product*

$$X = G_1/P_1 \times \ldots \times G_k/P_k$$

where G_i are simple groups and P_i are subgroups called parabolic.

Here simple means that there are no nontrivial (closed) normal connected subgroups.

Examples. *Projective spaces, grassmannians and smooth quadric hypersurfaces are the most known examples of rational homogeneous varieties. In the first two cases G is isomorphic to some $SL(n)$, while in the last case G is isomorphic to some $SO(n)$.*

Caution. *P_i's are not normal, so G_i/P_i is not a group but only a set of cosets.*

It is not difficult to prove the theorem 1.4 by using complex analytic techniques (see exercises ...), anyway it is worth to notice that it is a corollary of the following deep theorem.

Structure theorem for algebraic groups 1.7. *Let G be an algebraic group. Then there exists a (unique) normal connected affine subgroup N such that G/N is an abelian variety.*

The above structure theorem is commonly attributed to Chevalley. Anyway the first two complete proofs were published independently in 1956 by Barsotti and by Rosenlicht. The importance of the theorem is that every algebraic group can be obtained as an extension of two algebraic groups at the two "extremes", that is one affine and the other projective.

Remark. *Borel and Remmert proved something more than the theorem 1.5. In fact they proved that a compact Kähler manifold which is homogeneous is isomorphic to a product*

$$T \times X$$

where $T \simeq \mathbf{C}^n/\Gamma$ is a complex torus and X is rational homogeneous.

The theorems 1.5 and 1.6 open the way to the classification of projective homogeneous varieties. In fact Cartan in 1913 terminated the complete description of simple algebraic groups.

In this description it is useful an intermediate step, that is the study of semisimple groups and their Lie algebras.

Definition 1.8. *An algebraic group is called semisimple if it has no nontrivial (closed) normal connected solvable subgroups.*

Example. *$SL(n)$ is simple. $SL(n) \times SL(m)$ is semisimple but not simple (why?).*

Our first aim will be to show that the parabolic subgroups of a simple group can be completely described in terms of the Dynkin diagram of its Lie algebra.

We recall that after choosing a nondegenerate symmetric (resp. skewsymmetric) matrix Q (resp. J) we have the following definitions

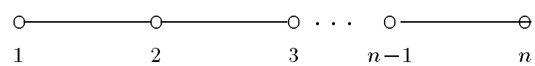
$$SO(n) := \{A \in SL(n) | AQA^t = Q\}$$

$$Sp(n) := \{A \in SL(n) | AJA^t = J\}$$

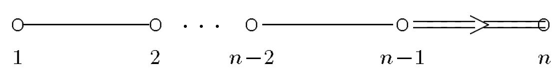
(in the second one n must be even).

The list of simple Lie groups consists in 4 families A_n, B_n, C_n, D_n and only 5 exceptional cases that are called E_6, E_7, E_8, F_4, G_2. The groups in the 4 families are called classical. A_n corresponds to $SL(n+1)$, B_n to $SO(2n+1)$, C_n to $Sp(2n)$ and D_n to $SO(2n)$. The Dynkin diagrams are the following:

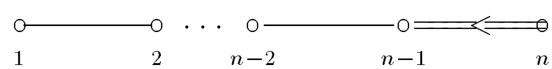
A_n $(n \geq 1)$:



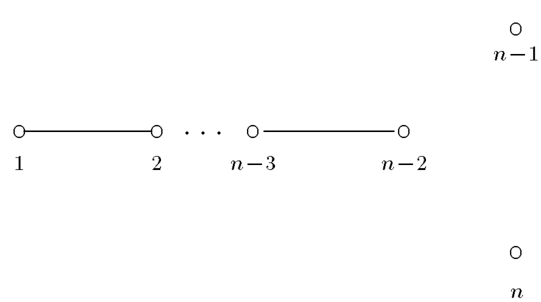
B_n $(n \geq 2)$:



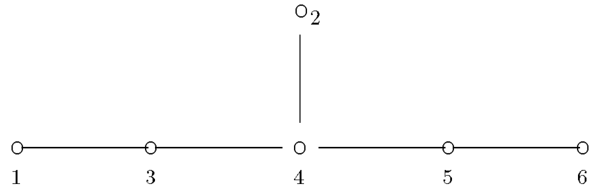
C_n $(n \geq 3)$:



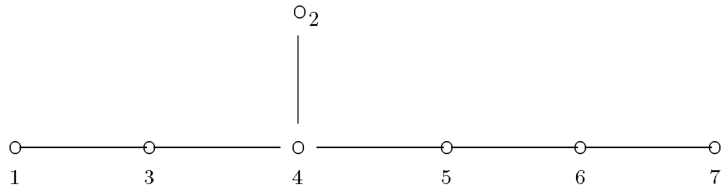
D_n $(n \geq 4)$:



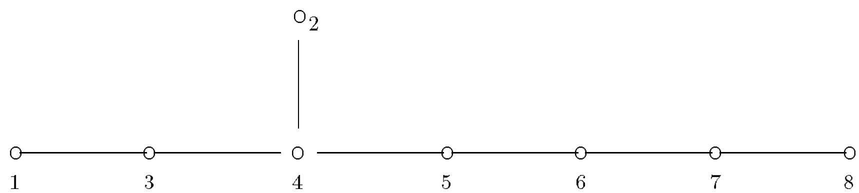
E_6 :



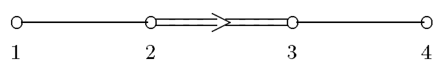
E_7 :



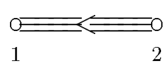
E_8 :



F_4 :



G_2 :



Our next job is to look at homogeneous bundles over a rational homogeneous variety G/P. They can be described in terms of representations $\rho: P \to GL(r)$. Call E_ρ the homogeneous bundle obtained by taking the quotient of $G \times \mathbf{C}^r$ via the relation $(g, v) \sim (g', v')$ if there exists $p \in P$ such that $g = g'p$ and $v = \rho(p^{-1})v'$. G acts on E_ρ, then it acts on the cohomology groups $H^i(G/P, E_\rho)$ too. The theorems of Borel-Weil and Bott describe (among other things) these last representations. In particular it follows that all the representations of a simple algebraic group G can be obtained as a space of sections H^0 of some line bundle, so we can look at representation theory from a geometric point of view.

§2. Grassmannians and flag manifolds

Let V be a vector space of dimension $n+1$ and consider $v \in V$, $v \neq 0$. Define

$$\phi_i \colon \wedge^i V \to \wedge^{i+1} V$$

by

$$\phi_i(\omega) := \omega \wedge v$$

Lemma 2.1 (Koszul complex of a vector). *The following sequence is exact*

$$0 \longrightarrow \wedge^0 V = \mathbf{C} \xrightarrow{\phi_0} \wedge^1 V \xrightarrow{\phi_1} \wedge^2 V \xrightarrow{\phi_2} \ldots \xrightarrow{\phi_n} \wedge^{n+1} V \longrightarrow 0$$

Proof It is evident that the above sequence is a complex. Choose a basis of V given by $e_1, \ldots, e_n, e_{n+1} = v$. Choose $\omega \in \wedge^k V$ such that $\phi_k(\omega) = \omega \wedge v = 0$. If $\omega = \sum_{i_1 < \ldots < i_k} a_{i_1 \ldots i_k} e_{i_1} \wedge \ldots \wedge e_{i_k}$ then each nonzero coefficient $a_{i_1 \ldots i_k}$ has $i_k = n+1$. Hence $\psi = \sum_{i_1 < \ldots < i_{k-1}} a_{i_1 \ldots i_k} e_{i_1} \wedge \ldots \wedge e_{i_{k-1}}$ satisfies $\phi_{k-1}(\psi) = \psi \wedge v = \omega$.

The theorem 2.1 admits the following generalization [Serre, Algèbre locale, multiplicités, LNM 11, Springer]. Let E be a vector bundle of rank n over X and consider $s \in H^0(X, E)$ such that $Z = \{x | s(x) = 0\}$ has pure codimension n. Define $\phi_i \colon \wedge^i E \to \wedge^{i+1} E$ by $\phi_i(\omega) = \omega \wedge s$ and the dual $\phi_i^t \colon \wedge^{i+1} E^* \to \wedge^i E^*$. Then the following sequence is exact

$$0 \longrightarrow \wedge^n E^* \xrightarrow{\phi_{n-1}^t} \wedge^{n-1} E^* \xrightarrow{\phi_{n-2}^t} \ldots \xrightarrow{\phi_1^t} E^* \xrightarrow{\phi_0^t} \mathcal{O}_X \longrightarrow \mathcal{O}_Z \longrightarrow 0$$

and it is called the Koszul sequence associated to s.

The Grassmannian

Let $\mathbf{P}^n = \mathbf{P}(V)$. Grassmannians parametrize the set of linear subspaces of dimension k in $\mathbf{P}^n$. The best way to give to this set the structure of a algebraic variety is the following definition.

Definition 2.2. *$Gr(k,n) = Gr(\mathbf{P}^k, \mathbf{P}^n)$ is defined as the subset of $\mathbf{P}(\wedge^{k+1} V)$ consisting of decomposable tensors.*

Theorem 2.3. *Gr(k,n) is a projective variety of dimension $(k+1)(n-k)$.*

In order to prove the theorem we have the following

Lemma 2.4.

i) If $\omega \in \wedge^{k+1} V$ then $\dim\{v \in V | \omega \wedge v = 0\} \leq k+1$.

ii) $\omega \in \wedge^{k+1} V$ is decomposable if and only if $\dim\{v \in V | \omega \wedge v = 0\} = k+1$.

Proof of lemma 2.4. By the theorem 2.1

$$\omega \wedge v = 0 \quad \Leftrightarrow \quad \exists \psi \text{ such that } \omega = \psi \wedge v$$

Hence if $v_1, \ldots, v_j$ are independent elements in $\{v \in V | \omega \wedge v = 0\}$ it follows that

$$\omega = \psi' \wedge v_1 \wedge \ldots \wedge v_j$$

(choose a basis containing $v_1, \ldots, v_j$!) and the result is obvious.

Proof of the theorem 2.3 Consider the morphism

$$\begin{aligned} \phi(\omega): V &\to \wedge^{k+2} V \\ v &\mapsto \omega \wedge v \end{aligned}$$

By the lemma $\omega \in Gr(k,n)$ if and only if $rk\ \phi(\omega) = n - k$. $rk\ \phi(\omega)$ is always $\geq n-k$ by the lemma 2.4 i), so the last condition is satisfied if and only if $rk\ \phi(\omega) \leq n-k$. The map

$$\begin{aligned} \wedge^{k+1} V &\to Hom(V, \wedge^{k+2} V) \\ \omega &\mapsto \phi(\omega) \end{aligned}$$

is linear, hence the entries of the matrix $\phi(\omega)$ are homogeneous coordinates on $\mathbf{P}(\wedge^{k+1} V)$ and $Gr(k,n)$ is defined by the vanishing of the $(n-k+1) \times (n-k+1)$ minors of this matrix.

The map $i: Gr(k,n) \to \mathbf{P}(\wedge^{k+1} V)$ is called the Plücker embedding. The equations that we have found define the Grassmannian as scheme but they do not generate the homogeneous ideal of $G = Gr(k,n)$. The ideal $I_{G,\mathbf{P}}$ is generated by quadrics that are called Plücker quadrics (see [Harris]).

It is useful to have a coordinate description of the Plücker embedding. A linear $\mathbf{P}^k \subset \mathbf{P}^n$ is determined by $k+1$ independent points $P_0, \ldots, P_k \in \mathbf{P}^k$. We can write down a $(k+1) \times (n+1)$ matrix A containing in the i-th row the coordinates of P_{i-1}. We get

$$A = \begin{pmatrix} x_{00} & \ldots & x_{0n} \\ \vdots & & \vdots \\ x_{k0} & \ldots & x_{kn} \end{pmatrix} \tag{2.1}$$

It is clear that this matrix has maximum rank $k+1$ and that two matrices A, A' determine the same subspace $\mathbf{P}^k$ if and only if there is $B \in GL(k+1)$ such that $A = BA'$. The point in $\mathbf{P}(\wedge^{k+1} V)$ given by the maximal minors of A is independent on the choice of $P_i's \in \mathbf{P}^k$ but depends only on the subspace $\mathbf{P}^k$. Conversely if $v_0 \wedge \ldots \wedge v_k$ is proportional to $w_0 \wedge \ldots \wedge w_k$ then $Span < v_0, \ldots, v_k >= Span < w_0, \ldots, w_k >$ (express $\{v_i\}$ in terms of a basis containing $\{w_j\} \ldots$).

In conclusion we have a biunivoc correspondence between points in $Gr(k,n)$ and linear subspaces $\mathbf{P}^k \subset \mathbf{P}^n$. The following construction shows that this correspondence is much more rich than a set correspondence.

Define the incidence variety $\mathcal{U} \subset Gr(k,n) \times \mathbf{P}^n$ given by $\{(g,x)|x \in g\}$ (really $\mathcal{U}$ is the projective bundle $\mathbf{P}(U)$ where U is the universal bundle on the Grassmannian). $\mathcal{U} \to Gr(k,n)$ satisfies the following universal property: for every subscheme $\mathcal{F} \subset S \times \mathbf{P}^n$ such that the projection $\mathcal{F} \to S$ is flat ($\mathcal{F}$ with this property is called a flat family) and $\mathcal{F}_s$ is a linear $\mathbf{P}^k$ for every $s \in S$ then there exists a unique morphism $\phi: S \to Gr(k,n)$ such that $\phi^*\mathcal{U} = \mathcal{F}$. This property says that the Grassmannians are *Hilbert schemes* (in fact they are the simplest Hilbert schemes). For an introduction to Hilbert schemes see ([Eis-Har]). It is interesting to remark that in order to construct the Hilbert schemes, the Grassmannians are needed as first step. We will see in connections with vector bundles other examples of the ubiquity of Grassmannians in modern geometry.

When $k = 0$ or $n-1$, $Gr(k,n)$ is isomorphic to the projective space $\mathbf{P}^n$. The simplest Grassmannian which is not a projective space is $Gr(1,3)$.

Exercise. *Let* $p_{ij} = \begin{vmatrix} x_i & x_j \\ y_i & y_j \end{vmatrix}$ *for* $0 \le i < j \le 3$ *be Plücker coordinates in the embedding* $Gr(1,3) \to \mathbf{P}^5$. *Prove that* $Gr(1,3)$ *is given by the smooth quadric with equation*

$$p_{01}p_{23} - p_{02}p_{13} + p_{03}p_{12} = 0$$

First hint: write a 4×4 matrix repeating twice the matrix $\begin{pmatrix} x_0 & \dots & x_3 \\ y_0 & \dots & y_3 \end{pmatrix}$.

Second hint: write a 4×4 skew-symmetric matrix with entries p_{ij} and computes its pfaffian.

Theorem 2.5. *$Gr(k,n)$ is a rational variety of dimension $(k+1)(n-k)$*

Proof The points in the open affine subset where $p_{01} \neq 0$ correspond to matrices

$$\begin{pmatrix} 1 & & & x_{0,k+1} & \dots & x_{0,n} \\ & \ddots & & \vdots & & \vdots \\ & & 1 & x_{k,k+1} & \dots & x_{k,n} \end{pmatrix}$$

It is easy to check that the above x_{ij}'s are exactly the maximal minors with k columns chosen among the first $k+1$. Hence $Gr(k,n) \cap \{p_{01} \neq 0\}$ is isomorphic to $\mathbf{C}^{(k+1)(n-k)}$

Theorem 2.6. *$Gr(k,n)$ is a homogeneous variety, in particular it is smooth.*

Proof $GL(n+1)$ acts transitively on the set of bases of any vector space of dimension $n+1$. In particular it acts transitively over $Gr(k,n)$

It is convenient to write explicitly the action of $GL(n+1)$ over $Gr(k,n)$ as the left matrix multiplication $g \cdot A^t$ where $g \in GL(n+1)$ and A^t is the transpose of the matrix (2.1) representing a point in the Grassmannian. If $P_0^k \in Gr(k,n)$ is spanned by the

$k+1$ points $(1,0,\ldots,0),(0,1,0,\ldots,0),\ldots,(0,\ldots,1,0,\ldots,0)$ then the isotropy subgroup $P=\{g\in GL(n+1)|g\cdot P_0^k=P_0^k\}$ has the block form

$$P=\{g\in GL(n+1)|g=\begin{pmatrix}M & * \\ 0 & N\end{pmatrix}, M\in GL(k+1), N\in GL(n-k)\}$$

$Gr(k,n)$ is so identified with the set of left lateral classes of $P\subset GL(n+1)$ which is denoted by $GL(n+1)/P$ (P is not normal!). It is sufficient to consider the action of $SL(n+1)=\{g\in GL(n+1)|\det\ g=1\}$ over $Gr(k,n)$ which is still transitive. Hence we can write also $Gr(k,n)=SL(n+1)/P'$ where

$$P'=\{g\in SL(n+1)|g=\begin{pmatrix}M & * \\ 0 & N\end{pmatrix}, M\in GL(k+1), N\in GL(n-k)\}$$

P' is called a parabolic subgroup, we will see that the correct notation for P' is $P(\alpha_{k+1})$ where α_{k+1} is the $k+1$-th simple root of $SL(n+1)$.

Schubert cycles [GH]

Fix a complete flag

$$P=\mathbf{P}^0_o\subset\mathbf{P}^1_o\subset\mathbf{P}^2_o\subset\ldots\subset\mathbf{P}^n$$

For any sequence of integers such that

$$n-k\geq a_0\geq a_1\geq\ldots\geq a_k$$

denote

$$W_{a_0,\ldots,a_k}=\{\mathbf{P}^k|\dim(\mathbf{P}^k\cap\mathbf{P}^j_o)=i \text{ for } n-k+i-a_i\leq j<n-k+i+1-a_{i+1}\}$$

These subsets are isomorphic to $\mathbf{C}^{(k+1)(n-k)-\sum a_i}$ and their closure are the subvarieties $\{\mathbf{P}^k\in Gr(k,n)|\dim(\mathbf{P}^k\cap\mathbf{P}_o^{n-k+i-a_i})\geq i\}$ that are called Schubert cycles.

Theorem. *$W_{a_0,\ldots,a_k}$ give a cell decomposition of $Gr(k,n)$.*

Theorem. *The Schubert cycles generate freely $H_*(Gr(k,n),\mathbf{Z})$ which has no torsion.*

Special care has to be reserved to $\overline{W_{1,\ldots,1}}$ (p times, the last $k-p$ entries are zero and are omitted). We get

$$\overline{W_{1,\ldots,1}}\ (p\text{ times }) =\{\mathbf{P}^k|\dim(\mathbf{P}^k\cap\mathbf{P}_o^{n-k+i-1})\geq i \text{ for } i=0,\ldots,p-1\}= \\ =\{\mathbf{P}^k|\dim(\mathbf{P}^k\cap\mathbf{P}_o^{n-k+p-2})\geq p-1\} \quad (2.2)$$

Flag Manifolds

The flag manifold $\mathbf{F}=F(k_1,\ldots,k_s,n)$ parametrizes all chains of linear subspaces $\mathbf{P}^{k_1},\ldots,\mathbf{P}^{k_s}$ in $\mathbf{P}^n$. We construct it as the incidence variety

$$\mathbf{F}\subset\mathbf{G}=Gr(k_1,n)\times\ldots Gr(k_s,n)$$

defined by

$$\mathbf{F}:=\{(V_1,\ldots,V_s)\in\mathbf{G}|V_1\subset\ldots\subset V_s\}$$

It is easy to prove that $\mathbf{F}$ is a variety.

Lemma 2.7. *The dimension of* $F(k_1, \ldots, k_s, n)$ *is* $\sum_{j=1}^{s}(k_{j+1} - k_j)(k_j + 1)$

Proof The fibers of the projection $F(k_1, \ldots, k_r, n) \to F(k_2, \ldots, k_r, n)$ are isomorphic to $Gr(k_1, k_2)$.

Examples. $F(0, 1, 2) \simeq \mathbf{P}(T\mathbf{P}^2)$ *is a 3-fold. In general* $F(0, n-1, n) \simeq \mathbf{P}(T\mathbf{P}^n)$

Definition 2.8. *The flag manifold* $F(0, 1, \ldots, n)$ *is called a complete flag manifold. It has a special role. Its dimension is* $\frac{n(n+1)}{2}$.

Theorem 2.9. *Every flag manifold is a rational variety.*

Proof As in the proof of lemma 2.7 every flag manifold can be expressed as a repeated locally trivial fibration with fibers Grassmannians. The result follows from the rationality of the Grassmannian (theorem 2.5).

Theorem 2.10. *Flag manifolds are homogeneous, in particular they are smooth.*

Proof The same proof of theor. 2.6 adapts to this case.

In particular the isotropy subgroup of a point in a complete flag manifold is given (in a convenient system of coordinates) by the subgroup B of upper-triangular matrices. Observe that B is solvable.

§3. Lie algebras and Lie groups

In this section it is more convenient to consider algebraic groups in the larger category of complex manifolds. The first two results are included mainly to motivate the definition of a Lie algebra.

Definition 3.1. *A complex manifold* G *which is also a group and such that the map*

$$G \times G \to G$$
$$(x, y) \mapsto xy^{-1}$$

is holomorphic is called a complex Lie group.

In the category of (complex) Lie groups the morphisms are holomorphic maps that are group homomorphisms. The reader can supply easily the notion of real Lie group.

On a complex Lie group G one can consider holomorphic vector fields. Among them it is convenient to consider the subclass of fields invariant by left translation (the left translation $L_g\colon G \to G$ is defined by $L_g(g') = gg'$). A left invariant vector field is characterized by the value that it assumes at the identity $e \in G$ (or at any other prescribed point). Then the vector space of left invariant vector fields is naturally isomorphic to T_eG.

A Lie group morphism $F: \mathbf{C} \to G$ is called a (complex) one-parameter subgroup. $\dot{F}(t)$ is a left invariant vector field along $Im\ F$. In particular the one-parameter subgroup F is uniquely characterized by $v = \dot{F}(0)$ (because of the uniqueness of the solution of the differential equation $\dot{X}(t) = L_{X(t)}v$).

Theorem 3.2. *There exists a unique holomorphic map*

$$\exp: T_e G \to G$$

taking 0 *to* e *with the property that for every* $v \in T_e G$ *the map*

$$\begin{aligned} \phi: \mathbf{C} &\to G \\ t &\mapsto \exp(tv) \end{aligned}$$

is the only Lie group morphism such that $\phi'(0) = v$. *In particular the differential of* exp *at the origin is the identity.*

Sketch of proof The solutions of differential equations of the first order depend well on the initial conditions.

Remark. *The reason for the terminology exp is that when* $G = GL(n)$ *then* $T_e G = M_n(\mathbf{C})$ *and*

$$F(t) = e^{tA} = \sum_k \frac{t^k A^k}{k!}$$

is the one-parameter subgroup such that $\dot{F}(0) = A$

Remark. *It holds*

$$det\ e^A = e^{tr\ A}$$

so that $T_e SL(n) = \{A \in M_n(\mathbf{C}) | tr\ A = 0\}$.

Corollary 3.3. *Let* G, H *be (connected) Lie groups and consider two Lie group morphisms* f_1, $f_2: G \to H$. *Then consider* $(df_1)_e$, $(df_2)_e: T_e G \to T_e H$. *The following is true:*

$$(df_1)_e = (df_2)_e \quad \Leftrightarrow \quad f_1 = f_2$$

Sketch of proof

$\Leftarrow$ obvious

$\Rightarrow$ Consider the following diagram for $i = 1, 2$

$$\begin{array}{ccc} T_e G & \xrightarrow{(df_i)_e} & T_e' H \\ \downarrow{\scriptstyle \exp} & & \downarrow{\scriptstyle \exp} \\ G & \xrightarrow{f_i} & H \end{array}$$

The diagram commutes because $\forall v \in T_eG$ the Lie group morphisms

$$\begin{aligned} \psi \colon \mathbf{C} &\to H \\ t &\mapsto f_i(\exp(tv)) \end{aligned}$$

$$\begin{aligned} \lambda \colon \mathbf{C} &\to H \\ t &\mapsto \exp(df_i)_e(tv) \end{aligned}$$

satisfy $\psi'(0) = (df_i)_e(v) = \lambda'(0)$. It follows $\psi(t) = \lambda(t)$, i.e. the diagram commutes. By the inverse function theorem the image of exp contains a neighborhood U of the identity where exp is invertible. Hence $f_1(v) = f_2(v)$ for $v \in U$. f_1 and f_2 are holomorphic, so that $f_1 = f_2$.

The previous corollary is a hint that T_eG encodes much information about G. In fact it encodes "everything" if we consider an additional structure on T_eG, that is the structure of Lie algebra. The correspondence

$$\text{Lie groups} \quad \leftrightarrow \quad \text{Lie algebras}$$

will be clear after the cor. 3.14 and the prop. 3.15.

It is well known that if X, Y are vector fields (i.e. derivations satisfying the Leibniz rule) then XY is no more a vector field but $[X, Y] := XY - YX$ is still a vector field. Moreover if X, Y are left invariant, then $[X, Y]$ is still left invariant.

We equip T_eG with the bracket $[\ ,\ ]$ which satisfies the three conditions

i) Bilinearity

ii) Skew-symmetry $\quad [X, Y] = -[Y, X] \quad \forall X, Y \in T_eG$

iii) Jacobi identity

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0 \quad \forall X, Y, Z \in T_eG$$

Definition 3.4. *A vector space V with a map*

$$[\ ,\ ] \colon V \times V \to V$$

satisfying i), ii) and iii) above is called a Lie algebra. In the category of Lie algebras the morphisms are vector space morphisms which preserve the bracket.

Definition 3.5. *Let G be a Lie group. The Lie algebra T_eG is called the Lie algebra associated to G and it is denoted by Lie G (it is common to use gothic letters for Lie algebras).*

Example. *The Lie algebra of $GL(n)$ is denoted $\mathcal{GL}(n)$ and consists of all the $n \times n$ matrices with bracket defined by $[X, Y] = X \cdot Y - Y \cdot X$ where $\cdot$ is the ordinary row-column product. This can be proved easily by using the theorem 3.11.*

Let V be a Lia algebra. A subalgebra $I \subset V$ is called a ideal if $\forall v \in V$, $i \in I$ we have $[v, i] \in I$. The quotient space V/I inherits a natural structure of quotient Lie algebra.

Exercise. *Check that the center*

$$\mathcal{Z}(V) := \{x \in V | [x, y] = 0 \quad \forall y \in V\}$$

is an ideal of V*.*

A Lie algebra V is called *solvable* if the derived series

$$V_1 := [V, V] \quad V_2 := [V_1, V_1] \quad \ldots \quad V_i := [V_{i-1}, V_{i-1}]$$

terminates to zero.

A Lie group G is solvable if and only if *Lie* G is solvable.

A first result is the following

Theorem 3.6. *Let* G *be a Lie group and let* $\mathcal{H} \subset$ *Lie* G *be a subalgebra. Then there exists a connected Lie subgroup* $H \subset G$ *such that Lie* $H = \mathcal{H}$*.*

Proof In [Warner] there is a proof using Frobenius theorem. Another proof is in [FuHa] prop. 8.41.

By using the exponential map it is not difficult to prove that

Proposition 3.7. *Let* $H \subset G$ *be a closed subgroup. Then* H *is normal if and only if the subalgebra Lie* H *is an ideal.*

Proof [NS] IX §3.

A Lie algebra V is simple if dim $V > 1$ and it contains no nontrivial ideals. A Lie algebra is semisimple if it has no nonzero solvable ideals.

Lie G is semisimple if and only if G is semisimple. The same is true for simple Lie algebras if dim $G > 1$. If I_1 and I_2 are solvable ideals then it is easy to check that $I_1 + I_2$ is a solvable ideal. It follows that there exists a unique maximal solvable ideal of a Lie algebra V which is called the radical and denoted by *rad* V. Hence V is semisimple if and only if *rad* $V = 0$

It is important to remark that there exists a purely algebraic definition of [,] in *Lie* G. In order to see this, we have to define the adjoint representation.

Definition 3.8. *The inner automorphism defined by an element* $g \in G$ *is called* ρ_g*, that is* $\rho_g(h) := ghg^{-1}$*. We get a morphism*

$$G \to Aut(G)$$
$$g \mapsto \rho_g$$

Definition of *Ad* **3.9.** *Consider the derivative at the identity of*

$$\rho_g : G \to G$$

that is

$$(d\rho_g)_e\colon Lie\ G \to Lie\ G$$

We define

$$Ad\colon G \to GL(Lie\ G)$$
$$g \to (d\rho_g)_e$$

Definition of ad **3.10.** *ad is the derivative at the identity of Ad, that is*

$$ad := (d\ Ad)_e\colon Lie\ G \to \mathcal{GL}(Lie\ G)$$

Ad is a representation of the group G, while ad is a representation of the Lie algebra $Lie\ G$.

Theorem 3.11.

$$ad(X)(Y) = [X, Y] \quad \forall X, Y \in Lie\ G$$

The theorem 3.11 gives a purely algebraic definition of the bracket [,]. Remark that ad is a Lie algebra morphism because of the Jacobi identity.

Remark. *(here char $K = 0$ is important!).*

$$Ker\ Ad = Z(G) \text{ center of the group [NS] IX §3}$$

$$Ker\ ad = \mathcal{Z}(Lie\ G) \text{ center of the Lie algebra}$$

Ado's Theorem 3.12. *Let $\mathcal{G}$ be a (finite-dimensional) Lie algebra. Then $\mathcal{G} \subset \mathcal{GL}(N)$ as subalgebra for some $N \in \mathbf{N}$.*

Proof [FuHa] appendix E

Lemma 3.13. *Let G be a Lie group. The universal covering $\tilde{G} \xrightarrow{\pi} G$ has a structure of Lie group such that π is a Lie group morphism.*

Sketch of proof Lift the map $\mu' := \mu \circ (\pi \times \pi)$ where μ is the multiplication map as in the following diagram

$$\begin{array}{ccc} \tilde{G} \times \tilde{G} & --\to & \tilde{G} \\ \Big\downarrow{\scriptstyle \pi \times \pi} & \searrow \mu' & \Big\downarrow{\scriptstyle \pi} \\ G \times G & \xrightarrow{\mu} & G \end{array}$$

The lifting exists by elementary topological arguments. You get $\tilde{\mu}\colon \tilde{G} \times \tilde{G} \to \tilde{G}$. By choosing $e' \in \pi^{-1}(e)$ we may suppose $\tilde{\mu}(e', e') = e'$. It is straightforward to check that $\tilde{G}$ verifies the group axioms with e' as identity.

Remark. *If $\tilde{G} \to G$ is a covering then Lie $\tilde{G} \simeq$ Lie G.*

Corollary 3.14. *Every Lie algebra is isomorphic to Lie G for some simply connected Lie group G.*

Proof By applying the theorems 3.12 and 3.6 there exists $G \subset GL(N)$ such that $Lie\ G = \mathcal{G}$

Now consider the universal covering

$$\tilde{G} \xrightarrow{\pi} G$$

and get $Lie\ \tilde{G} = Lie\ G = \mathcal{G}$.

Proposition 3.15. *Let G, H be Lie groups with G simply connected. Let*

$$\alpha\colon\ Lie\ G \to\ Lie\ H$$

be a linear map. The following are equivalent:

i) There exists $\beta\colon G \to H$ Lie group morphism such that $\alpha = (d\beta)_e$

ii) α is a Lie algebra morphism.

Proof

i) $\Rightarrow$ ii) It is standard (e.g. [Boothby, An introduction to diff. manifolds...] cor. IV 7.10)

ii) $\Rightarrow$ i) Consider

$$Lie\ (G \times H) = Lie\ G \oplus Lie\ H$$

and

$$\mathcal{J} := graph(\alpha) \subset Lie\ G \oplus Lie\ H$$

which is a subalgebra by the assumption. By the theorem 3.6 there exists a Lie subgroup $J \subset G \times H$ such that $Lie\ J = \mathcal{J}$. Consider

$$p_1\colon J \to G$$

Its differential at the identity

$$(dp_1)_e\colon \mathcal{J} \to Lie\ G$$

is an isomorphism. To check this claim it is enough to look at the two compositions

$$J \to G \times H \to G$$

$$\mathcal{J} \to Lie\ G \oplus Lie\ H \to Lie\ G$$

It follows that $p_1\colon J \to G$ is a diffeomorphism in a neighborhood of the identity. Since G is generated by any neighborhood of the identity (because G is connected) it follows that p_1 is surjective and hence it is a covering.

Since G is simply connected it follows that p_1 is an isomorphism. Hence

$$G \simeq J \xrightarrow{p_2} H$$

is the morphism β we looked for.

Corollary 3.16. *Let G_1, G_2 be simply connected. Then*

$$G_1 \simeq G_2 \quad \Leftrightarrow \quad Lie\ G_1 \simeq Lie\ G_2$$

ABC about representations

Definition 3.17. *A representation of a Lie group G in a vector space V is a Lie group morphism*

$$\rho: G \to GL(V)$$

Sometimes it is convenient to consider V as a G-module (which is equivalent!) by the rule $g \cdot v = \rho(g)v$. We will frequently interchange between the two languages.

Definition 3.18. *A representation of a Lie algebra $\mathcal{L}$ in a vector space V is a Lie algebra morphism*

$$\rho': \mathcal{L} \to \mathcal{GL}(V)$$

If ρ is a representation of G then $(d\rho)_e$ is a representation of $Lie\ G$ in the same vector space.

In the sequel we consider some properties of group representations, the reader can supply the analogous properties for Lie algebra representations.

Definition 3.19. *A morphism between two representations*

$$\rho_1: G \to GL(V_1)$$

and

$$\rho_2: G \to GL(V_2)$$

is a linear morphism $\phi: V_1 \to V_2$ such that

$$\phi(\rho_1(g)(v)) = \rho_2(g)(\phi(v)) \quad \forall g \in G, v \in V_1$$

Equivalently in terms of G-modules

$$\phi(g \cdot v) = g \cdot (\phi(v))$$

that is ϕ is G-equivariant.

Definition 3.20. *Let be given a G-module V. A subspace $W \subset V$ is called invariant if $G \cdot W \subset W$.*

Definition 3.21. *A G-module V is called irreducible if its invariant subspaces are only 0 and V.*

Definition 3.22. *A G-module V is called completely reducible if it is the direct sum of irreducible submodules.*

Example.

$$\mathbf{C} \to GL(2)$$
$$t \mapsto \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

is reducible but not completely reducible.

Schur Lemma 3.23. *If V_1 and V_2 are irreducible G-modules than any morphism $\phi: V_1 \to V_2$ is zero or it is a isomorphism.*

Proof ker ϕ and $Im\ \phi$ are both invariant subspaces. Then if $\phi \neq 0$ we have $\ker \phi = 0$ and $Im\ \phi = V_2$

Corollary 3.24. *If V is a G-module irreducible then any endomorphism $\phi: V \to V$ is equal to λI for some $\lambda \in \mathbf{C}$*

Proof Let λ be an eigenvalue for ϕ. Then $\phi - \lambda I$ is not a isomorphism and by the Schur lemma it must be zero.

Corollary 3.25. *If G is an abelian group, every irreducible G-module V has dimension 1.*

Proof Let $g_0 \in G$ be fixed. Then $g \cdot (g_0 \cdot v) = g_0 \cdot (g \cdot v)$. This means that

$$V \to V$$

$$v \mapsto g_0 v$$

is G-equivariant and by the corollary 3.24 there exists $\lambda(g_0) \in \mathbf{C}$ such that $g_0(v) = \lambda(g_0)v$ for every $v \in V$. This holds for every g_0, then every one-dimensional subspace is invariant.

Corollary 3.26. *Let $\rho: G \to GL(V)$ be an irreducible representation. Let $Z(G)$ be the center of G. Then $\rho(Z(G))$ consists of the scalar matrices λId.*

Proof The morphism

$$V \to V$$

$$v \mapsto zv$$

is G-equivariant $\forall z \in Z(G)$. Then apply the corollary 3.24.

Remark. *The corollary 3.26 in the case $G = GL(V)$ shows that $Z(GL(V)) = \mathbf{C}^*$.*

Action on $Hom(V, W)$.

Let V, W be G-modules. The induced action of G over $Hom(V, W) \simeq V^* \otimes W$ is

$$g \cdot f(v) = g[f(g^{-1}v)]$$

Proposition 3.27. *Let ρ, ρ' be two irreducible representations of G. Then*

i) $\rho \simeq \rho'$ if and only if $\rho^* \otimes \rho'$ contains with multiplicity one the one-dimensional trivial representation.

ii) $\rho \not\simeq \rho'$ if and only if $\rho^* \otimes \rho'$ does not contain the one-dimensional trivial representation.

Proof Let ρ act on V and let ρ' act on W. Then $\rho^* \otimes \rho'$ act on $Hom(V, W) \simeq V^* \otimes W$ as above. We have that $f \in Hom(V, W)$ is G-equivariant if and only if $g \cdot f = f$. Hence the subspace $Hom_G(V, W) \subset Hom(V, W)$ of G-equivariant morphisms is exactly the subspace where G acts trivially. The thesis follows from the Schur lemma and the corollary 3.24.

Proposition 3.28. *Let G simply connected and let*

$$\rho\colon Lie\ G \to \mathcal{GL}(V)$$

be a representation. Then there exists a representation

$$\rho'\colon G \to GL(V)$$

such that $\rho = (d\rho')_e$.

Proof Apply the prop. 3.15.

Proposition 3.29. *Let G be a Lie group and let $\rho\colon G \to GL(V)$ be a representation. Consider $d\rho\colon Lie\ G \to \mathcal{GL}(V)$. Then the following are equivalent:*

i) $W \subset V$ is invariant with respect to ρ

ii) $W \subset V$ is invariant with respect to $(d\rho)_e$.

Proof Let $Stab(W) \subset GL(V) = \{g \in GL(V) | gW \subset W\}$ and in the same way define $\underline{Stab}(W) \subset \mathcal{GL}(\mathcal{V})$

It is easy to check that $Lie\ Stab_\rho(W) = \underline{Stab}(W)$ (write down a basis of V containing a basis of W).

i)⇒ ii) By assumption ρ factors in the following diagram

$$\begin{array}{ccc} G & \xrightarrow{\rho} & GL(V) \\ & \searrow & \uparrow \\ & & Stab(W) \end{array}$$

Taking the derivatives we get

$$\begin{array}{ccc} Lie\ G & \xrightarrow{(d\rho)_e} & \mathcal{GL}(V) \\ & \searrow & \uparrow \\ & & \underline{Stab}(W) \end{array}$$

ii)⇒ i) Let $\tilde{G}$ be the universal covering of G. By assumption we have a commutative diagram

$$\begin{array}{ccc} Lie\ G & \xrightarrow{(d\rho)_e} & \mathcal{GL}(V) \\ & \searrow & \uparrow \\ & & Lie\ Stab(W) \end{array}$$

We get the following diagram

$$\begin{array}{ccc} \tilde{G} & \xrightarrow{\rho\circ\pi} & GL(V) \\ & \searrow \phi & \uparrow \\ & & Stab(W) \end{array}$$

where ϕ exists by the prop. 3.15 and the diagram commutes by the corollary 3.3. Then

$$\rho(G) = \rho \circ \pi(\tilde{G}) \subset Stab(W)$$

which means that W is invariant with respect to ρ.

§4. The Borel fixed point theorem

Remark. *Let* $\mathbf{P}^n = \mathbf{P}(V)$*, then* $Aut(\mathbf{P}^n) = PGL(V)$ *(see [Harris]). Of course* $PGL(V)$ *does not act over* V*.*

Theorem 4.1 (Blanchard). *Let* G *be an algebraic (connected) group acting over a projective variety* X *with* $H^1(X, \mathcal{O}) = 0$*. Then there exists a representation* $\rho\colon G \to PGL(V)$ *and an embedding* $X \subset \mathbf{P}(V)$ *such that the original action is induced by* ρ*. In particular the action is given by projective linear transformations.*

Proof By the assumption applied to the exponential sequence

$$0 \to \mathbf{Z} \to \mathcal{O} \xrightarrow{exp} \mathcal{O}^* \to 0$$

we get that $H^1(X, \mathcal{O}^*)$ injects in $H^2(X, \mathbf{Z})$ and then it is discrete. It follows that

$$g^*\mathcal{O}(1) \simeq \mathcal{O}(1) \quad \forall g \in G$$

In particular G acts over the hyperplane sections of X, then it acts over $\mathbf{P}(H^0(X, \mathcal{O}(1))^*)$. Then in the embedding given by the complete linear system the condition of the theorem is satisfied.

Remark. *The assumption* $H^1(X, \mathcal{O}) = 0$ *is necessary in the theorem 4.1 as it is shown by the example of plane cubic curves (e.g. every element of* $PGL(3)$ *acting on a plane cubic curve must preserve the flexes).*

Remark. *In the theorem 4.1 let* $\tilde{G}$ *be the universal covering of* G *acting on* X*. We get that the action is induced by a morphism* $\tilde{G} \to GL(V)$ *where* $V = H^0(X, \mathcal{O}(1))^*$*.*

There are two powerful results that we will use in the sequel. They are the theorem of Lie and the Borel fixed point theorem. The proof of both of them will be completed after the lemma 4.10.

Theorem of Lie 4.2. *Let* $G \xrightarrow{\rho} GL(V)$ *be a representation of a solvable linear algebraic group* G*. Then there exists a basis of* V *such that* $\rho(g)$ *is in upper triangular form for every* $g \in G$*.*

Corollary 4.3. *Let* $\mathcal{G} \xrightarrow{\rho} \mathcal{GL}(V)$ *be a representation of a solvable Lie algebra* $\mathcal{G}$*. Then there exists a basis of* V *such that* $\rho(X)$ *is in upper triangular form for every* $X \in \mathcal{G}$*.*

Remark. *The statement of the theorem of Lie is equivalent to the existence of an eigenvector* v *for* ρ *(in fact consider the quotient representation* $V/ < v >$ *and make induction).*

Borel fixed point theorem 4.4. *Let* G *be a solvable linear algebraic group. Then any action of* G *on a projective variety* X *has a fixed point.*

The following particular case is more elementary

Exercise. *Prove that any action of a torus on a projective variety has a fixed point (see Fulton, Young tableaux 10.1)*

The theorems 4.2 and 4.4 are linked in the sense that each one implies (almost) the other via a simple (and instructive) proof.

Lie $\Rightarrow$ Borel fixed point (in weaker version) We will prove a slightly weaker statement, in fact we make the additional assumptions that $X \subset \mathbf{P}(V)$ and that the action is induced by a linear action over V. By the theorem 4.1 this is not restrictive when $H^1(X, \mathcal{O}) = 0$ (considering eventually the universal covering $\tilde{G}$). If $V_0 \subset V_1 \subset \ldots \subset V$ is the flag fixed by the Lie theorem then there exists i such that dim $X \cap V_i = 0$ and all the finitely many points in the intersection have to be fixed.

Borel fixed point $\Rightarrow$ Lie ρ induces a natural action on the complete flag manifold F of subspaces of V which is projective (§2). Then by the Borel fixed point theorem there is a complete flag fixed by ρ and in an adapted basis $\rho(g)$ is in upper triangular form.

The theorem of Lie can be proved by purely algebraic techniques (see for example [FuHa]).

We sketch now the proof of the Borel fixed point theorem. We recall the following basic theorem of algebraic geometry.

Theorem 4.5 (Chevalley). *Let $f: V \to W$ be a morphism between algebraic varieties. Then $f(V)$ contains a dense open set of $\overline{f(V)}$.*

Proof [Harris] 3.16

Remark. *The algebraic setting is necessary in the theorem 4.5 (think at the irrational line on the torus).*

Closed orbit lemma 4.6. *Let G be a algebraic irreducible group acting over a algebraic variety X. Then each orbit is a smooth variety which is open in its closure. Its boundary is a union of orbits of smaller dimension. In particular the orbits of minimal dimension are closed.*

Proof Let $M = G(x)$ be a orbit which we consider as the image of the morphism

$$G \to X$$
$$g \mapsto gx$$

By the theorem 4.5 M contains a dense open set of $\overline{M}$. Furthermore G leaves $\overline{M}$ invariant. Since the action over M is transitive, each point of M must be contained in a open set of $\overline{M}$. Hence M is open in $\overline{M}$. $\overline{M} \setminus M$ is left invariant too and the other statements follow easily.

Proposition 4.7. *Let G be a affine group and $H \subset G$ be a closed subgroup. Then there exists an injective homomorphism $\alpha: G \to GL(E)$ and a line $D \subset E$ such that $H = \{g \in G | \alpha(g)D = D\}$*

Proof [Borel] II 5.1

Remark 4.8. *The proposition 4.7 and the closed orbit lemma show that the set of cosets G/H has the structure of quasi-projective variety in $\mathbf{P}(E)$. This construction has nice properties ([Borel]) and G/H is called a homogeneous space.*

Proposition 4.9. *Let G be a affine group and $N \subset G$ be a normal closed subgroup. Then G/N is a affine group.*

Proof Apply the proposition 4.7 and find $\alpha: G \to GL(E)$ and a line $D \subset E$ such that $N = \{g \in G | \alpha(g)D = D\}$. Let O be the orbit of $[D]$ in $\mathbf{P}(E)$. As N is normal we get that it acts trivially over O. Then G/N embeds in the isotropy subgroup of O in $PGL(E)$ and then it is affine.

Another approach to the prop. 4.9 is the following. Let $G = Spec\ A$ and consider the natural action of N over A. Then the natural candidate for G/N is $Spec\ A^N$ where A^N consists of the elements of A fixed by N. The problem here is to show that A^N is a finitely generated $\mathbf{C}$-algebra. This fact is proved in [Hochster] II.4

Lemma 4.10. *The set of fixed points of an action is closed.*

Proof Consider

$$G \times X \xrightarrow{\phi} X \times X$$

$$(g, x) \mapsto (gx, x)$$

Let $\Delta \subset X \times X$ be the diagonal. If x is not fixed then $\exists g \in G$ such that $(g, x) \notin \phi^{-1}(\Delta)$. Then there exists a neighborhood U_x such that $\phi(g, U_x)$ is in the complement of Δ.

Proof of the Borel fixed point theorem We argue by induction on $d = \dim\ G$. If $d = 0$ then $G = \{e\}$, so assume $d > 0$. Then $N = DG$ (derived subgroup) is connected (exercise) and of smaller dimension, so that the set F of finite points of N in X is non empty and closed, hence it is a projective subvariety. As N is normal in G we claim that F is G-invariant. In fact let $n \in N$, $g \in G$, $f \in F$. There exists $n' \in N$ such that $ng = gn'$ so that

$$ng \cdot f = gn' \cdot f = g \cdot f$$

and this implies that $g \cdot f \in F$. By the closed orbit lemma there exists $x \in F$ such that the orbit $G(x)$ is closed. Let G_x be the isotropy subgroup of x.

We have $N \subset G_x$ so that G_x is normal and there is a bijective G-equivariant morphism $G/G_x \to G(x)$ that is an isomorphism.

G/G_x embeds in a affine space by the prop. 4.9, and it is projective because it is isomorphic to $G(x)$. This shows that G/G_x is a point, hence $G = G_x$ and x is the fixed point we looked for.

The automorphism group

If X is a algebraic variety then the group $Aut(X)$ of the automorphisms of X is an algebraic scheme. This is a general fact in algebraic geometry, in fact the set of morphisms between two algebraic varieties can be endowed with a structure of scheme.

It can happen that $Aut(X)$ has infinitely many components, anyway the connected component of the identity $Aut^0(X)$ is always a variety. In the complex analytic setting the fact that $Aut^0(X)$ is a Lie group is a deep theorem of Bochner and Montgomery.

Definition 4.11. *We say that G acts effectively on X if the morphism $G \to Aut(X)$ is injective. Taking the quotient by the kernel of the above map every action can be supposed effective.*

Remark 4.12. *The theorem of Blanchard 4.1 says that if X is a projective variety with $H^1(X, \mathcal{O}) = 0$ then $Aut^0(X)$ is linear algebraic.*

Remark. *The definition of homogeneous at page 1 can be reformulated by saying that X (irreduc.) is homogeneous if $Aut^0(X)$ acts transitively over X.*

The following theorem is simple but it is crucial for the classification of rational homogeneous varieties.

Theorem 4.13. *Let G be a linear group acting transitively and effectively over a variety X. Then G is semisimple.*

Proof Let H be a connected normal solvable subgroup $H \subset G$. By the Borel fixed point theorem there exists $x_0 \in X$ such that $hx_0 = x_0\ \forall h \in H$. Let $x \in X$, then there exists $g \in G$ such that $x = gx_0$. Consider that for every $h \in H$

$$hx = gg^{-1}hx = g(g^{-1}hg)x_0 = \text{ (because } H \text{ is normal)}$$

$$= gx_0 = x$$

Then H fixes every point, so that $H = \{e\}$ as we wanted.

Remark 4.14. *By the remark 4.12 if $H^1(X, \mathcal{O}) = 0$ the assumption that G is linear can be dropped from the theorem 4.13.*

Corollary 4.15. *Let X be a projective variety with $H^1(X, \mathcal{O}) = 0$. Then $Aut^0(X)$ is linear and semisimple.*

From the theorem 4.13 it follows that every rational homogeneous variety is isomorphic to G/P where G is semisimple. Two things remain to be understood.

First we want to know what are the possibilities of G. At the end of section 6 we will describe the Cartan classification of semisimple groups.

Given a semisimple group G, we want to know what subgroups $P \subset G$ define a projective variety G/P. At the end of the section 7 we will give the classification of the possible P's too. In particular it will follow from the theorem 10.1 that every variety isomorphic to G/P with G semisimple is rational.

Complete reducibility for finite and compact groups

We will not use the following proposition, but we include it because its proof is useful.

Proposition 5.1. *Let G be a finite group. Let V be a G module and $W \subset V$ be a submodule. Then there exists $W' \subset V$ submodule such that $V = W \oplus W'$.*

Proof We show that we can reduce V to be a unitary representation for a suitable Hermitian metric. Let H_0 be any Hermitian metric on the vector space V and define

$$H(v,w) := \frac{1}{|G|} \sum_{g \in G} H_0(gv, gw) \qquad \forall v, w \in V$$

It is easy to check that $H(gv, gw) = H(v,w) \quad \forall g \in G$ and that H is still a Hermitian metric.

Then the orthogonal subspace

$$W' := W^{\perp} = \{v | H(v,w) = 0 \quad \forall w \in W\}$$

satisfies our request.

Corollary 5.2. *Every (finite dimensional) representation of a finite group G is completely reducible.*

The proposition 5.1 can be generalized to all compact groups G by replacing

$$\frac{1}{|G|} \sum_{g \in G} f(g)$$

with

$$\int_G f(g) d\mu$$

where the volume form $d\mu$ is chosen to be translation invariant and such that $\int_G d\mu = 1$

Proposition 5.3. *Let G be a compact (real) group. Let V be a G module and $W \subset V$ be a submodule. Then there exists $W' \subset V$ submodule such that $V = W \oplus W'$.*

Proof As in the proof of prop. 5.1 let H_0 be any Hermitian metric on the vector space V and define

$$H(v,w) := \int_G H_0(gv, gw) d\mu$$

Then the orthogonal subspace

$$W' := \{v | H(v,w) = 0 \quad \forall w \in W\}$$

satisfies our request.

Example. *If $G = S^1$ parametrized by $(\cos\theta, \sin\theta)$ the integral in the proof of prop. 5.3 is* $\frac{1}{2\pi}\int_0^{2\pi} H(\theta v, \theta w) d\theta$

The unitary trick

The unitary trick of H.Weyl consists in restricting representations of a complex Lie group to a "big" real compact Lie subgroup. For example $\mathbf{C}^*$ contains S^1 with the property that $Lie\ S^1 \otimes_{\mathbf{R}} \mathbf{C} = Lie\ \mathbf{C}^*$ (why?).

Definition 5.4. *A complex Lie group G with the property that exists a compact real Lie group K such that $Lie\ K \otimes_{\mathbf{R}} \mathbf{C} = Lie\ G$ is called reductive.*

In particular a semisimple group is reductive (this is easy to verify for the classical groups, see the lemma 5.7 for $SL(n)$) and the unitary trick applies to this class of groups.

Theorem 5.5. *(Unitary trick). Let G be a reductive Lie group. Let V be a G-module and $W \subset V$ be a submodule. Then there exists $W' \subset V$ submodule such that $V = W \oplus W'$.*

Proof Restrict $\rho: G \to GL(V)$ to $\rho': K \to GL(V)$. By the prop. 5.3 there exists W' complementary subspace which is K-invariant. Then W' is $Lie\ K$-invariant for $d\rho'$, so that it is $Lie\ G$ invariant because G is reductive. By the prop. 3.29 we have the result.

Corollary 5.6. *Every representation of a reductive Lie group is completely reducible.*

We are interested in to the following special case. Let $SU(n)$ be the (real) Lie group of unitary matrices A (i.e. $A \cdot A^H = I$) with determinant 1. Its Lie algebra $\mathcal{SU}(n)$ consists of skew-hermitian matrices of trace zero.

Lemma 5.7.

i) $\mathcal{SL}(n) = \mathcal{SU}(n) \otimes_{\mathbf{R}} \mathbf{C}$

ii) $\mathcal{GL}(n) = \mathcal{U}(n) \otimes_{\mathbf{R}} \mathbf{C}$

In particular $SL(n)$ and $GL(n)$ are reductive.

Proof

$$A = \left(\frac{A - A^H}{2}\right) + i\left(\frac{-iA - iA^H}{2}\right)$$

Corollary 5.8. *Every representation of $SL(n)$ or $GL(n)$ is completely reducible.*

Definition 5.9. *The Lie group $G = \mathbf{C}^* \times \ldots \times \mathbf{C}^*$ (k-times) is called a torus. The reason for this terminology is that G is the complexification of a real torus. G should not be confused with the complex torus $\mathbf{C}^n/\Gamma$.*

Corollary 5.10.

i) Every representation of a torus is isomorphic to the direct sum of representation of dimension one.

ii) Every representation of dimension 1 of a torus $\mathbf{C}^* \times \ldots \times \mathbf{C}^*$ (k-times) has the form

$$\begin{aligned} \mathbf{C}^* \times \ldots \times \mathbf{C}^* &\to \mathbf{C}^* \\ (t_1, \ldots, t_k) &\mapsto t_1^{n_1} \ldots t_k^{n_k} \end{aligned}$$

where n_i are integers.

Proof i) follows from the corollary 5.6 and the fact that tori are commutative (see cor. 3.25). ii) follows from the fact that every algebraic (or holomorphic) group morphism between $\mathbf{C}^*$ and $\mathbf{C}^*$ has the form $t \mapsto t^n$ for some integer n.

Remark. *Note that the complex analytic setting is necessary in the coroll. 5.10. In fact the map $f: \mathbf{C}^* \to \mathbf{C}^*$ given by $f(\rho e^{i\theta}) = \rho^2 e^{3i\theta}$ is a real analytic group homomorphism.*

The corollary 5.10. states that representations of tori are "discrete". The same result is true for representations of general reductive groups. In the next section we will analyze the case of $SL(2)$ (see ther. 5.13 and coroll. 5.14) and then the case of a general semisimple group (through its Lie algebra).

Description of $\mathcal{SL}(2)$

$\mathcal{SL}(2)$ is important in its own and also because it is a "building block" that allows to construct all other semisimple Lie algebras. We will see that its Dynkin diagram is as simple as possible, in fact it consists of only one dot.

Fix the following basis of $\mathcal{SL}(2)$

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{5.1}$$

Check that

$$[H, X] = 2X \quad [H, Y] = -2Y \quad [X, Y] = H$$

Let V be a $\mathcal{SL}(2)$-module. By the corollary 5.6 it is completely reducible.

$\mathcal{H} :=< H >$ is an (abelian) subalgebra of dim 1.

Definition 5.11.

$$V_\alpha := \{v | H \cdot v = \alpha v\}$$

When $V_\alpha \neq 0$ then α is called a weight of V.

Lemma 5.12. *Let $v \in V_\alpha$.*

i) $H(X(v)) = (\alpha + 2)X(v)$

ii) $H(Y(v)) = (\alpha - 2)Y(v)$

Proof

$$H(X(v)) = X(H(v)) + [H, X](v) = \text{ because the representation preserves the bracket}$$

$$= X(\alpha v) + 2X(v) = (\alpha + 2)X(v)$$

This proves i). The proof of ii) is analogous.

The following theorem is fundamental:

Theorem 5.13. Integrality of weights *If V is a irreducible $SL(2)$-module then in (5.2) all the α are distinct integers that fill a sequence symmetric with respect to the origin (i.e. $-t, -t+2, \ldots, t$). Moreover* $\dim\ V_\alpha = 1$.

Proof It follows from the lemma 5.12 that for every α

$$X: V_\alpha \to V_{\alpha+2}$$
$$Y: V_\alpha \to V_{\alpha-2}$$
$$H: V_\alpha \to V_\alpha$$

If $V_{\alpha_0} \neq 0$ then

$$\sum_{n\in\mathbf{Z}} V_{\alpha_0+2n} \subset V$$

is a submodule, hence we have equality (V is irreducible). It is a well-known fact in linear algebra that the sum in the left side is direct, so that

$$V = \oplus_n V_{\alpha_0+2n} \tag{5.2}$$

Of course in (5.2) all terms are zero except finitely many. Let

$$m = \max\{\alpha_0 + 2n | V_{\alpha_0+2n} \neq 0\}$$

We want to prove that m is a nonnegative integer. In fact let $v \in V_m$

$$X(Y(v)) = [X,Y](v) + Y(X(v)) = H(v) + 0 = mv$$
$$X(Y^2(v)) = [X,Y]Y(v) + Y(X(Y(v))) = H(Y(v)) + Y(mv) = ((m-2)+m)Y(v)$$

and it is easy to prove by induction that

$$X(Y^k(v)) = ((m-2k+2) + (m-2k+4) + \ldots + m)Y^{k-1}(v) = k(m-k+1)Y^{k-1}(v) \tag{5.3}$$

It follows that when $Y^k(v) = 0$ for minimal k then $m = k-1 \in \mathbf{Z}_{\geq 0}$. By (5.3) it is evident that

$$v, Y(v), \ldots, Y^k(v), \ldots$$

span an invariant subspace of V and so they span V. This proves the theorem.

The above proof gives also a basis of V and we know exactly where each of H, X and Y takes each basis vector. Hence V is determined by the collection of weights, in particular it is determined by the m we started with (called maximal weight or highest weight).

We get

Corollary 5.14. *Every irreducible representation of $\mathcal{SL}(2)$ is a symmetric power S^mV of the standard representation $V \simeq \mathbf{C}^2$.*

S^mV has dimension $m+1$ and weights $-m, -m+2, \ldots, m-2, m$.

m is called the highest weight.

$-m$ is called the lowest weight.

Exercise. *Prove that the irreducible representations of $PGL(2)$ are exactly the even powers $S^{2n}V$*

§6. The Cartan decomposition

Let G be a semisimple Lie group, that is G does not contain nontrivial normal solvable subgroups. In this section we study the structure of *Lie* G, in order to have precise informations on G. It will follow that *Lie* G is a direct sum of simple Lie algebras $\mathcal{G}_i$ (i.e. dim $\mathcal{G}_i > 1$ and $\mathcal{G}_i$ does not contain nontrivial ideals). Moreover there is a complete list of all simple Lie algebras (as sketched in the introduction).

Definition 6.1. *A subalgebra $\mathcal{H} \subset \mathcal{G}$ is called abelian if*

$$[h_1, h_2] = 0 \quad \forall h_1, h_2 \in \mathcal{H}$$

Definition 6.2. *A subalgebra $\mathcal{H} \subset \mathcal{G}$ is called a Cartan subalgebra if*
i) $\mathcal{H}$ is abelian and $ad_{|\mathcal{H}}: \mathcal{H} \to \mathcal{GL}(\mathcal{G})$ acts diagonally
ii) $\mathcal{H}$ is maximal with respect to i)

We will see in a while that it is easy to check if a subalgebra satisfying i) satisfies also ii). The first nontrivial fact is the following existence theorem

Theorem 6.3. *In any semisimple Lie algebra $\mathcal{G}$ there exist Cartan subalgebras $\mathcal{H}$.*

Sketch of proof If $\mathcal{G} = \mathcal{SL}(n)$ then the subalgebra of diagonal matrices is a Cartan subalgebra. The same fact is true if $\mathcal{G}$ is a classical group. In general Cartan subalgebras can be found as the centralizer $\{X \in \mathcal{G} | [X, Y] = 0\}$ for a sufficiently general Y. For details see the appendix D of [FuHa].

Let now a Cartan subalgebra $\mathcal{H} \subset \mathcal{G}$ be fixed.

Definition 6.4. *For any $\alpha \in \mathcal{H}^*$ (dual of $\mathcal{H}$) denote*

$$\mathcal{G}_\alpha := \{X \in \mathcal{G} | ad(H)(X) = \alpha(H)X \quad \forall H \in \mathcal{H}\}$$

According to i) of the definition we get that $\mathcal{G}$ is decomposed as direct sum of the eigenspaces $\mathcal{G}_\alpha$

Theorem 6.5.

$$[\mathcal{G}_\alpha, \mathcal{G}_\beta] \subset \mathcal{G}_{\alpha+\beta}$$

Proof Let $X \in \mathcal{G}_\alpha$, $Y \in \mathcal{G}_\beta$, $H \in \mathcal{H}$. By the Jacobi identity

$$[H, [X, Y]] = -[X, [Y, H]] - [Y, [H, X]] = [X, \beta(H)Y] - [Y, \alpha(H)X] = (\alpha(H) + \beta(H))[X, Y]$$

Lemma 6.6.

$$\mathcal{H} = \mathcal{G}_0$$

Proof The inclusion $\subset$ is evident and if 0 appears among the α's then $\mathcal{H}$ could be enlarged still satisfying property i) by the theorem 6.5.

We get the decomposition

$$\mathcal{G} = \mathcal{H} \bigoplus (\oplus_{\alpha \in \mathcal{H}^*} \mathcal{G}_\alpha) \tag{6.1}$$

which is called the Cartan decomposition. The reader should notice the analogy with the case of $\mathcal{SL}(2)$.

Definition 6.7. *Any $\alpha \in \mathcal{H}^*$ such that $\mathcal{G}_\alpha \neq 0$ is called a root (except for 0 that it is not considered as a root). The set of roots is denoted by $\Phi \subset \mathcal{H}^*$. $\mathcal{G}_\alpha$'s are called the root spaces.*

Theorem 6.8. *If α is a root then $-\alpha$ is also a root.*

Proof [FuHa] D.13, in the exercises we will check directly this fact if $G = SL(n)$.

Now choose a direction in $\mathcal{H}^*$ irrational with respect to the lattice generated by the roots. This gives a decomposition

$$\Phi = \Phi^+ \cup \Phi^-$$

which is called an ordering of the roots. By the theorem 6.8 $-\Phi^+ = \Phi^-$.

Definition of the Killing form 6.9.

$$\begin{aligned} B: \mathcal{G} \times \mathcal{G} &\to \mathbf{C} \\ (X, Y) &\mapsto tr(ad(X) \circ ad(Y): \mathcal{G} \to \mathcal{G}) \end{aligned}$$

is called the Killing form. It is obviously bilinear and symmetric.

Exercise. *Check that for $\mathcal{G} = \mathcal{SL}(n)$ then $B(X, Y) = 2ntr(XY)$. Verify from this expression that B is nondegenerate.*

Lemma 6.10.
i) If $X \in \mathcal{G}_\alpha, Y \in \mathcal{G}_\beta$ then

$$ad(X) \circ ad(Y)(\mathcal{G}_\gamma) \subset \mathcal{G}_{\alpha+\beta+\gamma}$$

ii) Let $\mathcal{Q}_\alpha := \mathcal{G}_\alpha \oplus \mathcal{G}_{-\alpha}$. Then the decomposition

$$\mathcal{G} = \mathcal{H} \bigoplus (\oplus_{\alpha \in \Phi^+} \mathcal{Q}_\alpha)$$

is orthogonal with respect to the Killing form.

Proof i) is immediate from the theorem 6.5. ii) follows by i) because if $\alpha \neq -\beta$ then $ad(X) \circ ad(Y)(\mathcal{G}_\gamma)$ has zero component with respect to $\mathcal{G}_\gamma$ so the contribute to the trace is zero.

Lemma 6.11.

i)

$$B([X,Y],Z) = B(X,[Y,Z]) \quad \forall X,Y,Z \in \mathcal{G}$$

ii) For any ideal $\mathcal{I} \subset \mathcal{G}$ the orthogonal subspace

$$\mathcal{I}^{\perp} := \{X \in \mathcal{G} | B(X,Y) = 0 \quad \forall Y \in \mathcal{I}\}$$

is an ideal.

Proof i) is straightforward ([FuHa] 14.23). ii) is immediate from i).

The importance of the Killing form is stressed by the following theorem

Theorem 6.12.

i) **Cartan's criterion** *If $B(X,Y) = 0$ $\forall X,Y \in \mathcal{G}$ then $\mathcal{G}$ is solvable.*

ii) $\mathcal{G}$ is semisimple if and only if B is nondegenerate.

Proof [FuHa] appendix C.

Remark. *The theorem 6.8 follows from the theorem 6.12 ii). In fact from Lemma 6.10 i) if $-\alpha$ is not a root then $\mathcal{G}_\alpha$ is orthogonal to all $\mathcal{G}$.*

Theorem 6.13. *A semisimple Lie algebra $\mathcal{G}$ is a direct sum of simple Lie algebras.*

Proof For every ideal $\mathcal{I}$, $\mathcal{I} \cap \mathcal{I}^{\perp}$ is an ideal by the lemma 6.11 ii) and it is solvable by the Cartan criterion. Hence $G = \mathcal{I} \oplus \mathcal{I}^{\perp}$ and the result follows by induction.

Lemma 6.14. *The roots α span $\mathcal{H}^*$*

Proof Otherwise there is a nonzero $X \in \mathcal{H}$ such that $\alpha(X) = 0$ for all roots α. It follows $[X, \mathcal{G}_\alpha] = 0$ for any root α. Hence X is in the center of $\mathcal{G}$ which is zero because $\mathcal{G}$ is solvable.

Lemma 6.15 (find $\mathcal{SL}(2)$ inside $\mathcal{G}$). *Let $X \in \mathcal{G}_\alpha$, $Y \in \mathcal{G}_{-\alpha}$ such that $B(X,Y) \neq 0$ (they have to exist thanks to the lemma 6.10 and the theorem 6.12 ii)). Then $[X,Y]$, X and Y span a subalgebra $\mathcal{S}$ of $\mathcal{G}$ isomorphic to $\mathcal{SL}(2)$.*

Proof First we see that

$$[X,Y] \neq 0 \tag{6.2}$$

In fact $\forall H \in \mathcal{H}$ by the lemma 6.11

$$B(H,[X,Y]) = B([H,X],Y) = \alpha(H)B(X,Y)$$

We have the relations

$$[[X,Y],X] = \alpha([X,Y])X$$

$$[[X,Y],Y] = -\alpha([X,Y])Y$$

Note that $[X,Y] \in \mathcal{G}_0 = \mathcal{H}$ by the theorem 6.5 and the theorem 6.4. The crucial fact is that

$$\alpha([X,Y]) \neq 0 \tag{6.3}$$

Otherwise $\mathcal{S} \simeq ad\ \mathcal{S} \subset \mathcal{GL}(\mathcal{G})$ is a solvable subalgebra. By the Lie theorem there is a basis of $\mathcal{G}$ such that the elements of $ad\ \mathcal{S}$ are in upper triangular form, then $ad\ ([X,Y])$ is in strictly upper triangular form. But the elements of $ad\ \mathcal{H}$ are diagonalizable and this imply $ad\ [X,Y] = 0$ in contradiction with (6.2). Now adjusting by scalars we can find the same multiplication table of $\mathcal{SL}(2)$, in particular $\alpha([X,Y]) = 2$

Lemma 6.16. *Let α a root.*

i) For $k \in \mathbf{Z}$, $k \neq 1, -1$ then $k\alpha$ is not a root.

ii) $\dim\ \mathcal{G}_\alpha = 1$

Proof Pick up a Lie algebra $\mathcal{S}$ as in the lemma 6.15. Consider the adjoint action of $\mathcal{S}$ on $V := \mathcal{H} \bigoplus (\oplus_{k \in \mathbf{Z}} \mathcal{G}_{k\alpha})$ Now $\mathcal{S}$ acts trivially on $\ker\ \alpha$ and it acts irreducibly on $\mathcal{S}$ itself. By (6.3) $\mathcal{H} = \mathcal{G}_0 \subset V' := \ker\ \alpha \oplus \mathcal{S} \subset V$ and by the cor. 5.8 there exists a complement of V' as $\mathcal{S}$-module, but this complement has to be empty because we know by the cor. 5.14 that all the $\mathcal{SL}(2)$-modules have 0 among their weights. Hence $\mathcal{G}_{k\alpha} = \emptyset$ for $k \neq 1, -1$ which is i). Furthermore $V' = V$ which is ii).

Definition 6.17. *From the lemma 6.16 it follows that the subalgebra $\mathcal{S}$ defined in the lemma 6.15 is uniquely determined by α and we denote it by $\mathcal{S}_\alpha$. The element $H \in \mathcal{S}_\alpha$ such that $\alpha(H) = 2$ is denoted by H_α. X_α, Y_α and H_α have the multiplication table of X, Y and H in (5.1).*

Corollary 6.18. *Every one-dimensional representation of a semisimple Lie algebra $\mathcal{G}$ is trivial.*

Proof Restrict to the subalgebras $\mathcal{S}_\alpha$ and use the cor. 5.14.

We study now a arbitrary representation of $\mathcal{G}$. We will find a structure similar to the one obtained in the case of the adjoint representation. Let a Cartan subalgebra $\mathcal{H} \subset \mathcal{G}$ be fixed.

Definition 6.19. *Let $\rho: \mathcal{G} \to \mathcal{GL}(V)$ be a representation of $\mathcal{G}$. For any $\lambda \in \mathcal{H}^*$ denote*

$$V_\lambda := \{v \in V | \rho(H)(v) = \lambda(H)v \quad \forall H \in \mathcal{H}\}$$

Theorem 6.20.

$$\rho(\mathcal{G}_\alpha)V_\lambda \subset V_{\alpha+\lambda}$$

Proof Let $X \in \mathcal{G}_\alpha$, $v \in V_\lambda$, $H \in \mathcal{H}$.

$$\rho(H)\rho(X)v = \rho([H,X])v + \rho(X)\rho(H)v = \alpha(H)\rho(X)v + \lambda(H)\rho(X)v$$

hence $\rho(X)v \in V_{\alpha+\lambda}$.

Definition 6.21. *An element $\lambda \in \mathcal{H}^*$ such that*

$$V_\lambda \neq 0$$

is called a weight of ρ. V_λ's are called the weight spaces.

Theorem 6.22. *V is the direct sum of its weight spaces*

$$V = \oplus V_\lambda$$

Proof This is in the definition of Cartan subalgebras, so it is hidden in the existence theorem 6.3. If we know that representations of $\mathcal{G}$ are completely reducible we may suppose that V is irreducible and argue as follows. The sum $V' = \sum_\lambda V_\lambda$ is direct from elementary linear algebra. V' is a submodule of V by the theorem 6.20, hence $V = V'$.

Corollary 6.23 (Integrality of the weights). *Let $\rho: \mathcal{G} \to \mathcal{GL}(V)$ be a representation of $\mathcal{G}$. Let λ be a weight of ρ. Then*

$$\lambda(H_\alpha) \in \mathbf{Z}$$

for every root α

Proof Restrict ρ to $\mathcal{S}_\alpha$ and apply the theorem 5.13.

Definition 6.24.

$$\Lambda_W := \{\beta \in \mathcal{H}^* | \beta(H_\alpha) \in \mathbf{Z}\}$$

is called the weight lattice of $\mathcal{G}$

The corollary 6.23 can be reformulated by saying that all weights λ lie in the weight lattice.

Definition 6.25. *Λ_W contains the sublattice Λ_R generated by the roots. This is in general a sublattice of finite index.*

The Weyl group

We want to study the geometrical structure of the lattices Λ_W and Λ_R with respect to the Killing form. In order to draw the pictures one can find convenient to consider the real span of the roots $\mathbf{E}$. It is easy to check that B is positive definite when restricted to $\mathbf{E}$ [FuHa].

As B is nondegenerate it gives a isomorphism $\mathcal{H} \simeq \mathcal{H}^*$ and then B induces another nondegenerate form on $\mathcal{H}^*$ that we denote again by B.

Proposition 6.26. *The hyperplane*

$$\Omega_\alpha := \{\beta | \beta(H_\alpha) = 0\}$$

is the hyperplane orthogonal to α.

Proof The statement is equivalent to the dual assertion that H_α is orthogonal to $\ker\ \alpha$. This is proved as follows. If $H \in \ker\ \alpha$ then

$$B(H_\alpha, H) = B([X_\alpha, Y_\alpha], H) = \text{ by the lemma 6.11}$$

$$= B(X_\alpha, [Y_\alpha, H]) = B(X_\alpha, \alpha(H)Y_\alpha) = B(X_\alpha, 0) = 0$$

Definition 6.27. *The Weyl group is defined as the subgroup in $GL(\mathcal{H}^*)$ generated by the orthogonal reflections w_α with respect to Ω_α*

$$w_\alpha(\beta) = \beta - \frac{2B(\alpha,\beta)}{B(\alpha,\alpha)}\alpha$$

Lemma 6.28.

$$w_\alpha(\beta) = \beta - \beta(H_\alpha)\alpha$$

Proof It is enough to check that $\beta - \frac{1}{2}\beta(H_\alpha)\alpha \in \Omega_\alpha$. In fact $\beta(H_\alpha) - \frac{1}{2}\beta(H_\alpha)\alpha(H_\alpha) = 0$ because $\alpha(H_\alpha) = 2$ (see def. 6.17).

Corollary 6.29.

$$\beta(H_\alpha) = \frac{2B(\alpha,\beta)}{B(\alpha,\alpha)}$$

The most important property of the Weyl group for our purposes is resumed in the following theorem.

Theorem 6.30.

i) The set of weights of any representation of $\mathcal{G}$ is invariant under the action of the Weyl group.

ii) Let α be a root. If λ is a weight for some $\mathcal{G}$-module V then in the infinite sequence $\ldots - \alpha + \lambda, \lambda, \lambda + \alpha, \lambda + 2\alpha, \ldots$ the string of weights for V is a connected set. If λ' is the right extreme of this string, then the string has length $\lambda'(H_\alpha) + 1$. In other words, after replacing λ by $\lambda + k\alpha$ for convenient k the string of weights is

$$w_\alpha(\lambda) = \lambda - \lambda(H_\alpha)\alpha, \ldots, \lambda - \alpha, \lambda$$

Proof

1^{st} step In the case $\mathcal{G} = \mathcal{SL}(2)$ the Weyl group consists of only two elements and the statement is easy from the classification of $\mathcal{SL}(2)$-modules.

2^{nd} step Consider that $\oplus_{k\in\mathbf{Z}} V_{\lambda+k\alpha}$ is a $\mathcal{S}_\alpha$-submodule of V by the theorem 6.20 and apply the 1^{st} step.

Example. *The Weyl group for $\mathcal{G} = \mathcal{SL}(n)$ is the symmetric group Σ_n of order $n!$. ([FuHa] pag. 214)*

Definition 6.31. *The fundamental Weyl chamber $\mathcal{C}$ is the convex set*

$$\mathcal{C} = \{\gamma \in \mathcal{H}^* | B(\gamma, \alpha) \geq 0 \quad \forall \alpha \in \Phi^+\}$$

Theorem 6.32. *The Weyl group acts simply and transitively on the set of orderings of roots and likewise on the set of Weyl chambers.*

Proof [FuHa] appendix D

In the following let be fixed an ordering of the roots.

Definition 6.33. *Let $\rho: \mathcal{G} \to \mathcal{GL}(V)$ be a representation of $\mathcal{G}$. A vector $v \in V$, $v \neq 0$ is called a highest weight vector if satisfies the two properties:*

i) $\rho(\mathcal{G}_\alpha)(v) = 0 \quad \forall \alpha \in \Phi^+$

ii) v is a eigenvector for the action of $\mathcal{H}$. If $\rho(H)(v) = \lambda(H)v$ for $\lambda \in \mathcal{H}^*$ then λ is called a highest weight (or maximal weight).

Proposition 6.34. *Every representation of $\mathcal{G}$ admits a highest weight vector.*

Proof Choose $F \in \mathcal{H}^*$ such that $Ker\ F$ divides exactly Φ^+ from Φ^- and such that $\Phi^+ \subset \{F > 0\}$. Let λ be the weight such that $F(\lambda)$ is maximal. Then $\forall \beta \in \Phi^+$ we have $F(\lambda + \beta) = F(\lambda) + F(\beta) > F(\lambda)$ and it follows $V_{\lambda+\beta} = \emptyset$. Then any nonzero $v \in V_\lambda$ is a highest weight vector by the theorem 6.20.

Lemma 6.35. *Let V be the space of a representation of $\mathcal{G}$. Let v be a highest weight vector. The vector subspace spanned by $\mathcal{G}_{\beta_1} \cdots \mathcal{G}_{\beta_k} \cdot v$ for $\beta_i \in \Phi^-$ is an irreducible subrepresentation.*

Proof Let W_n be the subspace generated by $< \mathcal{G}_{\beta_1} \cdots \mathcal{G}_{\beta_k} \cdot v >$ for $\beta_i \in \Phi^-$ and $k \leq n$. If $X \in \mathcal{G}_\alpha$, $\alpha \in \Phi^+$, we want to prove

$$X \cdot W_n \subset W_n \tag{6.4}$$

by induction on n (the case $n = 0$ is trivial). In fact any element in W_n can be written as a sum of elements of the form $Y \cdot w$ where $Y \in \mathcal{G}_\beta$, $\beta \in \Phi^-$ and $w \in W_{n-1}$

$$X \cdot (Y \cdot w) = Y \cdot (X \cdot w) + [X, Y] \cdot w$$

As $X \cdot w \in W_{n-1}$ by the inductive hypothesis, then (6.4) is true. It follows that

$$W := \cup_n W_n$$

is a subrepresentation (clearly irreducible) as we wanted.

Theorem 6.36. *A representation of $\mathcal{G}$ is irreducible $\Leftrightarrow$ it admits a unique highest weight vector (up to scalars).*

$\Leftarrow$ immediate from the prop. 6.34 applied to the direct summands of the representation. $\Rightarrow$ In fact if w is another highest weight vector then we have

$$w \in < \mathcal{G}_{\beta_1} \cdots \mathcal{G}_{\beta_k} \cdot v > \tag{6.5}$$

and

$$v \in < \mathcal{G}_{\beta'_1} \cdots \mathcal{G}_{\beta'_t} \cdot w > \tag{6.6}$$

Subsituting (6.5) in (6.6) we get $k = t = 0$.

Remark 6.37. *We can define a lowest weight vector substituting Φ^- at the place of Φ^+ in the definition 6.33. The above results still hold with obvious modifications.*

Theorem 6.38 (Uniqueness theorem). *Let $\rho: \mathcal{G} \to GL(V)$ and $\rho': \mathcal{G} \to GL(V')$ be two irreducible representations. Let λ (resp. λ') be the highest weight of ρ (resp. ρ'). Then*

$$\rho \simeq \rho' \qquad \Leftrightarrow \qquad \lambda = \lambda'$$

Proof

$\Rightarrow$ trivial

$\Leftarrow$ Let $v \in V$ and $v' \in V'$ be the two highest weight vectors. Then $(v, v') \in V \oplus V'$ is highest weight vector with weight λ for $\rho \oplus \rho'$. Let $U \subset V \oplus V'$ the irreducible representation generated by (v, v') as in the lemma 6.35. The projections $\pi_1: U \to V$ and $\pi_2: U \to V'$ are both nonzero and by Schur lemma are isomorphisms. It follows $V \simeq U \simeq V'$.

Hence the highest weight λ determines a irreducible representation that we can denote by V_λ. In particular λ determines all the other weights. A careful analysis of what we have seen shows that the weights of V are exactly the weights that are congruent to λ modulo Λ_R and that lie in the convex hull of the set $w_\alpha(\lambda)$ for any $\alpha \in \Phi^+$ (see [FuHa] page 204).

Proposition 6.39. *The highest weight λ of a irreducible representation lie in the fundamental Weyl chamber $\mathcal{C}$.*

Proof Otherwise there exists $\alpha \in \Phi^+$ such that $B(\alpha, \lambda) < 0$. Then by the theorem 6.30 $\lambda' = w_\alpha(\lambda) = \lambda - \lambda(H_\alpha)\alpha$ is a weight too. By the corollary 6.29 $\lambda(H_\alpha) < 0$ so that λ cannot be the highest weight.

It is interesting to observe that $B(\alpha, \lambda') = -B(\alpha, \lambda) > 0$

Theorem 6.40 (Existence theorem). *$\forall \lambda \in \mathcal{C} \cap \Lambda_W$ there exists a irreducible representation V_λ of $\mathcal{G}$ with highest weight λ*

Proof In the setting of algebraic geometry, this theorem is a corollary of the theorem of Borel-Weil in the next section (see rem. 10.12). For an algebraic proof see [Hum]. For the classical groups the theorem can be checked explicitly, for example in the case $\mathcal{G} = \mathcal{SL}(n)$ it relies on the construction of the Schur functors.

Definition 6.41. *A root $\alpha \in \Phi^+$ (resp. Φ^-)is called simple if it cannot be expressed as a sum of two positive (resp. negative) roots.*

Exercise. *Find the simple roots in the case $\mathcal{G} = \mathcal{SL}(n)$.*

In the lemma 6.35 we could have considered $\mathcal{G}_{\beta_1} \cdots \mathcal{G}_{\beta_k} \cdot v$ only for β simple negative root.

Lemma 6.42. *If α_i, α_j are two distinct positive simple roots then*

$$B(\alpha_i, \alpha_j) \leq 0$$

(i.e. the angle between them is not acute)

Proof Otherwise $\alpha_i(H_{\alpha_j}) > 0$ by the corollary 6.29. Then by the corollary 6.30 ii) $\alpha_i - \alpha_j$ is a root. If it is positive we have $\alpha_i = \alpha_j + (\alpha_i - \alpha_j)$ and α_i is not simple. If it is negative we have $\alpha_j = \alpha_i + (\alpha_j - \alpha_i)$ and α_j is not simple.

Lemma 6.43. *The simple positive roots form a basis of E*

Proof Obviously the simple roots α_i span Λ_R and also $\mathcal{H}^*$ by the lemma 6.14. If there is a linear relation with real coefficients among them we can write

$$v = \sum_{i \leq k} n_i \alpha_i = \sum_{j > k} n_j \alpha_j$$

with $n_i,\ n_j \geq 0$. Then $B(v, v) = \sum n_i n_j B(\alpha_i, \alpha_j) \leq 0$ by the lemma 6.42. Hence $B(v, v) = 0$ which implies $v = 0$ because B is positive definite on the real span of the roots. Hence $n_i = n_j = 0$ because the summands of v lie on the same side of the hyperplane $\ker\ F$.

Corollary 6.44. *Let $\alpha_1, \ldots, \alpha_n$ be the simple positive roots. Then H_{α_i} for $i = 1, \ldots, n$ generate $\mathcal{H}$.*

Proof As in the proof of the prop. 6.26 we have $H_{\alpha_i} = [X_{\alpha_i}, Y_{\alpha_i}]$ and

$$B(H_{\alpha_i}, H) = \alpha_i(H) B(X_{\alpha_i}, Y_{\alpha_i}) \tag{6.7}$$

for every $H \in \mathcal{H}$. The isomorphism $\mathcal{H}^* \to \mathcal{H}$ induced by B is defined by

$$\alpha \mapsto T_\alpha$$

where $B(T_\alpha, H) = \alpha(H)$ for every $H \in \mathcal{H}$. From (6.7) $T_\alpha = \frac{H_\alpha}{B(X_\alpha, Y_\alpha)}$ is a multiple of H_α and this proves the result.

Definition 6.45. *The fundamental weights $\lambda_i \in \mathcal{H}^*$ are the dual basis (over* **R***) of H_{α_i} where α_i are simple roots. In other words the λ_i's are defined by the conditions*

$$\lambda_i(H_{\alpha_j}) = \delta_{ij} \tag{6.8}$$

Remark. *It can be shown that H_{α_i} are a basis of the complex vector space $\mathcal{H}$ so that λ_i are independent over* **C***.*

It follows that every element in Λ_W is an integral combination of λ_i and that the weights in the fundamental Weyl chamber $\mathcal{C}$ are exactly the integral combinations $\sum n_i \lambda_i$ with $n_i \geq 0$. Geometrically the λ_i's are the first weights met along the edges of $\mathcal{C}$.

Examples. In $\mathcal{SL}(2)$ we have

$$\alpha_1 = 2\lambda_1$$

In $\mathcal{SL}(3)$ we have

$$\alpha_1 = 2\lambda_1 - \lambda_2$$

$$\alpha_2 = -\lambda_1 + 2\lambda_2$$

Definition 6.46. *Let $\alpha_1, \ldots, \alpha_n$ be the simple positive roots of $\mathcal{G}$. The $n \times n$ matrix C with entries*

$$c_{ij} = \alpha_i(H_{\alpha_j})$$

is called the Cartan matrix of $\mathcal{G}$. Note that C has integral entries.

From (6.8) we have immediately that

$$\begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} = C \cdot \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix}$$

In particular

$$\det\ C = index(\Lambda_R, \Lambda_W)$$

From the corollary 6.29 we can write also

$$c_{ij} = \frac{2B(\alpha_i, \alpha_j)}{B(\alpha_j, \alpha_j)}$$

Dynkin diagrams

Any semisimple Lie algebra has a particular structure of the roots in $\mathcal{H}^*$. The set of vectors which satisfy the properties of the roots of a semisimple Lie algebra can be classified and conversely from any such set of roots it is possible to recover the corresponding Lie algebra. The technical tool to describe these sets of roots is the Dynkin diagram. Dynkin diagrams turn out to be a very general subject in mathematics (e.g. they have applications in the singularity theory).

Definition 6.47. *A root system is a finite set R spanning $\mathcal{H}^*$ (with a inner product B) such that*

i) $\alpha \in R \Rightarrow k\alpha \in R$ *iff* $k = \pm 1$

ii) $\forall \alpha \in R$ *the reflection* w_α *in the hyperplane* $\alpha^\perp$ *maps* R *to itself.*

iii) $\forall \alpha, \beta \in R$ *then* $\frac{2B(\beta,\alpha)}{B(\alpha,\alpha)} \in \mathbf{Z}$

A root system is called irreducible if it is not the orthogonal direct sum of two root systems.

Theorem 6.48. *The set of roots of a semisimple Lie algebra is a root system.*

Proof i) is lemma 6.16. ii) is theorem 6.30. iii) is cor. 6.23 joint with cor. 6.29.

The Dynkin diagram of a root system is given by assigning one dot for any simple root. We join two dots with a number of lines depending on the angle θ between the corresponding roots. Precisely

- no lines if $\theta = \pi/2$
- one line if $\theta = 2\pi/3$
- two lines if $\theta = 3\pi/4$
- three lines if $\theta = 5\pi/6$

It turns out that no other angle is possible. When two roots are joined by one line then they have the same length. If there are two or three lines we draw an arrow in the direction of the shorter root.

The following classification is a task of pure euclidean geometry

Theorem 6.49. *The Dynkin diagrams of irreducible root systems are in the list given in the introduction.*

Proof [FuHa] pag. 326

Theorem 6.50. *For each Dynkin diagram D appearing in the list of the theorem 6.49 there exist a unique simple Lie algebra such that its Dynkin diagram is D.*

Proof [FuHa] §21.3

The most important part of the theorem 6.50 is the uniqueness, in fact in the classical cases the existence can be shown directly.

Corollary 6.51.

i) Semisimple Lie algebras are all classified.

ii) Semisimple Lie groups are all classified. In particular the simply connected ones are all algebraic.

Proof The semisimple Lie algebras are all obtained as direct sum of simple Lie albegras by the theorem 6.13 and each summand correspond to $\mathcal{SL}(n)$, $\mathcal{SO}(m)$, $\ldots$ as in the list of Dynkin diagrams. For each of these $\mathcal{SL}(n) \oplus \mathcal{SO}(m) \oplus \ldots$ there is a unique semisimple simply connected Lie group $SL(n) \times Spin(m) \times \ldots$ which is algebraic by construction. All other semisimple Lie groups are obtained by covering. It is easy to show that the kernel of a covering morphism must be in the center. Note that we can get products $G_1 \cdot G_2$ where G_1, G_2 are simple but with finite intersection !

Irreducible representations of classical groups

We mention the irreducible representations of the classical groups that have the fundamental weights as highest weights. (see exercises...).

Complete tables can be found in [Tits, LNM 40].

$SL(n+1) = SL(V)$ has fundamental weights $\lambda_1, \ldots, \lambda_n$. The representation with highest weight λ_1 is the standard representation on V itself. λ_i corresponds to the wedge power $\wedge^i V = V_{\lambda_i}$. $k\lambda_1$ corresponds to $S^k V = V_{k\lambda_1}$. All the other representations can be constructed by means of the Young diagrams (exercises...).

$Sp(n) = Sp(V)$ has more representations (here dim $V = 2n$). We denote always by $\lambda_1, \ldots, \lambda_n$ the fundamental weights. We have always $V_{\lambda_1} = V$, but for example $\wedge^2 V = \mathbf{C} \oplus V_{\lambda_2}$ and in general

$$\wedge^k V = \wedge^{k-2} V \oplus V_{\lambda_k}$$

We have always $S^k V = V_{k\lambda_1}$.

In the case of $Spin(2n+1)$ the picture is similar to the one just given for $SL(n+1)$ with the remarkable exception of the last fundamental weight that defines the so called spin representation (see exercises...).

$Spin(2n)$ has even two half-spin representations. The geometrical interpretation of this fact relies on the study of linear subspaces in quadrics, that have a different behaviour if the dimension is even or odd (see [Ot]).

§7. Borel and parabolic subgroups

Consider a semisimple Lie algebra $\mathcal{G}$ with its Cartan decomposition (6.1). In the following has been fixed a Cartan subalgebra $\mathcal{H}$ and an ordering of the roots.

Proposition-Definition 7.1.

$$\mathcal{B} := \mathcal{H} \bigoplus (\oplus_{\alpha \in \Phi^+} \mathcal{G}_\alpha)$$

is a maximal solvable Lie subalgebra. A maximal solvable Lie subalgebra is called a Borel subalgebra.

Proof $\mathcal{B}$ is solvable by the theor. 6.5. If $\mathcal{B}' \supset \mathcal{B}$ is another solvable subalgebra then $\mathcal{B}'$ contains some $\mathcal{G}_{-\alpha}$ with $\alpha \in \Phi^+$ (check this fact by using that $\mathcal{B}'$ must be $ad_{|\mathcal{H}}$-invariant). Hence $\mathcal{B}' \supset \mathcal{S}_\alpha \simeq \mathcal{SL}(2)$ (see def. 6.17) which is simple and satisfies

$$[\mathcal{S}_\alpha, \mathcal{S}_\alpha] = \mathcal{S}_\alpha$$

It follows that $\mathcal{B}'$ cannot be solvable.

Remark. *$(\oplus_{\alpha\in\Phi^+}\mathcal{G}_\alpha)$ is a maximal nilpotent subalgebra of $\mathcal{G}$.*

Definition 7.2. *Let G such that $Lie\ G = \mathcal{G}$. A subgroup B such that $Lie\ B$ is a Borel subalgebra is called a Borel subgroup. In equivalent way B is a maximal connected solvable subgroup.*

Proposition 7.3. *Let $B \subset G$ be a Borel subgroup. Then B is closed and G/B is a projective variety. Moreover all Borel subgroups are conjugate.*

First proof B is closed by the maximality (in fact $\overline{DG} = D\overline{G}$, see [Borel]). By the prop. 4.7 we may choose $\rho: G \to GL(V)$ with a subspace $V_1 \subset V$ such that dim $V_1 = 1$ and $Stab(V_1) = B$. Then we apply Lie theorem to the action of B over V/V_1. We get a complete flag

$$0 \subset V_1 \subset V_2 \ldots \subset V$$

which has stabilizer again equal to B. Then G/B is the orbit of this flag in the complete flag manifold F of V. We want to prove that G/B is an orbit of minimal dimension, then it is closed (hence projective!) by applying the closed orbit lemma. First let the dimension of B be maximal among all the Borel subgroups. Consider an orbit G/R of a point $P \in F$. Since R leaves invariant the flag corresponding to P, it can be put in upper triangular form and it is solvable. By the choice of B we get $dim\ R \leq dim\ B$ so that $dim\ G/B \leq dim\ G/R$ and we have done. Now let B' be any other Borel subgroup and consider the natural action of B' over G/B. By the Borel fixed point theorem there exists a fixed point, that is $\exists g \in G$ such that

$$B'gB \subset gB$$

which implies $g^{-1}B'gB \subset B$. It follows $g^{-1}B'g \subset B$ and by the maximality of B' we get the equality. So all Borel subgroups are conjugate, and in particular they have the same dimension. Then G/B is projective for any Borel subgroup B.

In particular, up to choose a Cartan subalgebra (conjugate to $\mathcal{H}$), every Borel subgroup B satisfies

$$Lie\ B = \mathcal{H} \bigoplus (\oplus_{\alpha\in\Phi^+}\mathcal{G}_\alpha)$$

Remark. *The projective embedding of G/B which is described in the proof of the proposition 7.3 lie in a space of too large dimension and it is not the natural one. A natural embedding can be found by considering Ad as a G-action over $\mathbf{P}(\mathcal{G})$. Let α be the highest weight of Ad. Then G/B is the orbit of the point $[\mathcal{G}_\alpha] \in \mathbf{P}(\mathcal{G})$ (for details see [FuHa page 383).*

Exercise. *If H and H' are conjugate subgroups of G, prove that $G/H \simeq G/H'$.*

Definition 7.4. *A closed subgroup $P \subset G$ is called parabolic if it contains some Borel subgroup B.*

Theorem 7.5. *Let $P \subset G$ be a closed subgroup. P is parabolic iff G/P is a projective variety.*

Proof

$\Leftarrow$ Let B be a Borel subgroup. B acts naturally (by left translation) over G/P and by the Borel fixed point theorem there exists a fixed point, that is $\exists g \in G$ such that

$$BgP \subset gP$$

Then $g^{-1}BgP \subset P$. This implies $g^{-1}Bg \subset P$.

$\Rightarrow$ By the remark 4.8 G/P is quasiprojective, hence it is sufficient to check that it is compact. This follows from the projection $G/B \xrightarrow{\pi} G/P$ and the prop. 7.3.

Definition 7.6. *Let $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ be the set of simple (positive) roots of $\mathcal{G}$. Let $\Sigma \subset \Delta$. Let*

$$\Phi^-(\Sigma) := \{\alpha \in \Phi^- | \alpha = \sum_{\alpha_i \notin \Sigma} p_i \alpha_i\}$$

Proposition-Definition 7.7.

$$\mathcal{P}(\Sigma) := \mathcal{H} \bigoplus (\oplus_{\alpha \in \Phi^+} \mathcal{G}_\alpha) \bigoplus (\oplus_{\alpha \in \Phi^-(\Sigma)} \mathcal{G}_\alpha)$$

is a subalgebra. $P(\Sigma)$ is the subgroup such that Lie $P(\Sigma) = \mathcal{P}(\Sigma)$ (see the theorem 3.6)

Proof The statement is obvious from the theorem 6.5

Theorem 7.8 (Classification of parabolic subgroups). *Let G be semisimple and simply connected. Let P be a parabolic subgroup of G. There exist $g \in G$ and $\Sigma \subset \Delta$ such that*

$$g^{-1}Pg = P(\Sigma)$$

Proof By the theorem 7.3 we may choose $g \in G$ such that

$$P' := g^{-1}Pg \supset B$$

where *Lie* $B = \mathcal{B} := \mathcal{H} \bigoplus (\oplus_{\alpha \in \Phi^+} \mathcal{G}_\alpha)$.

Lie P' is a subspace of $\mathcal{G}$ containing $\mathcal{B} \supset \mathcal{H}$ and invariant under $ad_{|\mathcal{H}}$. Hence

$$Lie\ P' = \mathcal{H} \bigoplus (\oplus_{\alpha \in T} \mathcal{G}_\alpha)$$

for some subset $T \subset \Phi$.

We know that T contains all the positive roots. Moreover if α is a negative root in T and $\alpha = \beta + \gamma$ with β, γ negative roots too we have $-\beta$, $-\gamma \in T$. By 6.5 it follows that $\alpha - \beta = \gamma \in T$ and $\alpha - \gamma = \beta \in T$. Then $\Sigma := \Delta \setminus (-T)$ satisfies the condition of the theorem.

Corollary 7.9. *Let G be semisimple and simply connected. Let $G = G_1 \times \ldots \times G_k$ be the decomposition of G as the direct product of simple simply connected Lie groups (see the theorem 6.13). Let $P \subset G$ be a parabolic subgroup. Then there are parabolic subgroups $P_i \subset G_i$ such that $P = P_1 \times \ldots \times P_k$. In particular*

$$G/P \simeq G_1/P_1 \times \ldots \times G_k/P_k$$

Proof The root system of $\mathcal{G}$ is the orthogonal sum of the root systems of the $\mathcal{G}_i$'s.

Corollary 7.10. *Rational homogeneous varieties are classified. They are isomorphic to products of varieties $G/P(\Sigma)$ where G is simple and simply connected (and then in the list od Dynkin diagram) and Σ is a subset of the set of the simple roots of G.*

Corollary 7.11. *For each semisimple simply connected Lie group G, there are only finitely many projective varieties isomorphic to G/P for some $P \subset G$.*

§8. ABC about bundles

A vector bundle E of rank r over an algebraic variety X is by definition an algebraic variety E with a surjective morphism

$$\pi\colon E \to X$$

such that there exists an open covering $\{U_\alpha\}_{\alpha \in I}$ of X satisfying the two properties

i) there exist isomorphisms $\phi_\alpha\colon \pi^{-1}(U_\alpha) \longrightarrow U_\alpha \times \mathbf{C}^r$ making commutative the diagram

$$\begin{array}{ccc} \pi^{-1}(U_\alpha) & \xrightarrow{\phi_\alpha} & U_\alpha \times \mathbf{C}^r \\ \big\downarrow \pi & & \big\downarrow p_1 \\ U_\alpha & \xrightarrow{id} & U_\alpha \end{array}$$

ii) $\forall \alpha, \beta \in I$ the composition (restricted)

$$(U_\alpha \cap U_\beta) \times \mathbf{C}^r \xrightarrow{\phi_\beta^{-1}} \pi^{-1}(U_\alpha \cap U_\beta) \xrightarrow{\phi_\alpha} (U_\alpha \cap U_\beta) \times \mathbf{C}^r$$

has the form

$$\phi_\alpha \circ \phi_\beta^{-1}(x, v) = (x, g_{\alpha\beta}(x)v)$$

where

$$g_{\alpha\beta}\colon (U_\alpha \cap U_\beta) \to GL(r)$$

are algebraic.

i) means that the fibration is locally trivial and that each fiber $\pi^{-1}(x)$ is isomorphic to $\mathbf{C}^r$.

ii) means that the structure group of the bundle is linear.

$g_{\alpha\beta}$ are called the transition functions and satisfy the properties

$$g_{\alpha\beta}^{-1} = g_{\beta\alpha} \tag{8.1}$$

$$g_{\alpha\beta} \cdot g_{\beta\gamma} = g_{\alpha\gamma} \tag{8.2}$$

In equivalent way, given a covering $\{U_\alpha\}_{\alpha\in I}$ with a set of transition functions $g_{\alpha\beta}(x)$ satisfying (8.1) and (8.2) we can construct a vector bundle E as the quotient of the disjoint union

$$\coprod_\alpha (U_\alpha \times \mathbf{C}^r)$$

by the relation $\sim$ defined in the following way:

$$\forall (x,v) \in U_\alpha \times \mathbf{C}^r \quad (x',v') \in U_\beta \times \mathbf{C}^r$$

we have

$$(x,v) \sim (x',v') \text{ iff } x = x' \quad v = g_{\alpha\beta}(x)v'$$

Remark 8.1. *We can say synthetically that "the transition functions determine the bundle".*

If $g_{\alpha\beta}$ are transition functions for E and $h_{\alpha\beta}$ are transition functions for F then

$$\begin{pmatrix} g_{\alpha\beta} & \\ & h_{\alpha\beta} \end{pmatrix} \text{ are transition functions for } E \oplus F$$

this can be taken as definition of $E \oplus F$

$$(g_{\alpha\beta}^{-1})^t \text{ are transition functions for } E^* \text{ dual bundle}$$

$$g_{\alpha\beta} \otimes h_{\alpha\beta} \text{ are transition functions for } E \otimes F$$

If $T\colon GL(r) \to GL(r')$ is any representation we define $T(E)$ to be the bundle with transition functions $T(g_{\alpha\beta})$. This construction applies in particular to $T = \wedge^k$ and $T = S^k$.

If $f\colon X \to Y$ is a map and E is a bundle on Y with transition functions $g_{\alpha\beta}(y)$ then f^*E is the bundle on X with transition functions $g_{\alpha\beta}(f(x))$.

If X is smooth the bundle Ω^1_X of $1-forms$ can be defined as the bundle with transition functions given by the jacobian matrices obtained by change of local coordinates. The tangent bundle is $TX = (\Omega^1_X)^*$.

A vector bundle of rank 1 is called a line bundle. The set of line bundles has a natural structure of abelian group isomorphic to $H^1(X, \mathcal{O}^*)$ with the multiplication given by the tensor product and the inverse given by the dual bundle.

A section of E is an algebraic map

$$s\colon X \to E$$

such that $\pi \circ s = id_X$

Definition 8.2. *A vector bundle is called spanned if there are (global) sections $s_1, \ldots, s_k$ such that $\forall x \in X$ the vectors $s_1(x), \ldots, s_k(x)$ span the fiber $\pi^{-1}(x)$.*

To any vector bundle E we can associate a locally free sheaf of $\mathcal{O}_X$-modules $\mathcal{E}$ defined by

$$\mathcal{E}(U) := \{\text{sections of } E_{|U}\}$$

Conversely to any locally free sheaf $\mathcal{E}$ is associated a vector bundle with fiber $E_x \simeq \mathcal{E}_x/\mathcal{M}_x\mathcal{E}_x$ defined as the **Spec** of the symmetric algebra of $\mathcal{E}$ (see [Hart]).

It is usual to identify a vector bundle E and the associated locally free sheaf $\mathcal{E}$. In particular the cohomology groups $H^q(X, E)$ are (by definition) the cohomology groups $H^q(X, \mathcal{E})$. Note that $H^0(X, E)$ is the space of global sections of E. In particular a vector bundle is spanned if and only if the evaluation map

$$H^0(X, E) \otimes \mathcal{O} \to E$$

is surjective.

Example. *On $\mathbf{P}^n = \mathbf{P}(V)$ with homogeneous coordinates $(x_0, \ldots, x_n)$ we have the line bundles $\mathcal{O}(t)$ that on the standard covering given by $U_i = \{x | x_i \neq 0\}$ have transition functions $g_{ij} = (\frac{x_i}{x_j})^t$. In a more geometrical way, the bundle $\mathcal{O}(-1)$ is the "universal" bundle defined as the incidence variety*

$$\{(x, v) \in \mathbf{P}^n \times V | v \in [x]\}$$

endowed with the projection to the first factor. Then for $t \geq 0$ $\mathcal{O}(-t) := \mathcal{O}(-1)^{\otimes t}$ and $\mathcal{O}(t) := (\mathcal{O}(-1)^*)^{\otimes t}$. From the exponential sequence

$$0 \longrightarrow \mathbf{Z} \longrightarrow \mathcal{O} \overset{\exp}{\longrightarrow} \mathcal{O}^* \longrightarrow 0$$

we get $H^1(\mathbf{P}^n, \mathcal{O}^*) = \mathbf{Z}$ so that all the line bundles on $\mathbf{P}^n$ are isomorphic to $\mathcal{O}(t)$ for some integer t.

If F is a coherent sheaf, it is usual to denote $F \otimes \mathcal{O}(t)$ by $F(t)$. For $t \geq 0$ the space $H^0(\mathbf{P}^n, \mathcal{O}(t))$ consists of all homogeneous polynomials in $(x_0, \ldots, x_n)$ of degree t, or in equivalent way $H^0(\mathbf{P}^n, \mathcal{O}(t)) \simeq S^t V$ (the Borel-Weil theorem is a generalization of this last isomorphism). All the intermediate cohomology of $\mathcal{O}(t)$ is zero, that is

$$H^i(\mathbf{P}^n, \mathcal{O}(t)) = 0 \quad \text{for } 0 < i < n \quad \forall t \in \mathbf{Z}$$

The zero loci of sections of $\mathcal{O}(t)$ are exactly the hypersurfaces of degree t. The zero loci of a general section of $\mathcal{O}(n_1) \oplus \ldots \oplus \mathcal{O}(n_k)$ is called a complete intersection.

The geometrical definition of the universal bundle $\mathcal{O}(-1)$ gives immediately the exact sequence

$$0 \longrightarrow \mathcal{O}(-1) \longrightarrow \mathcal{O} \otimes V^* \longrightarrow Q \longrightarrow 0 \tag{8.3}$$

It is easy to identify the quotient bundle Q with the twisted tangent bundle $T\mathbf{P}^n(-1)$. (8.3) is called the Euler sequence.

Example. *Also on $Gr(k,n)$ we can define the universal bundle. Consider the incidence variety*

$$\{(x,v) \in Gr(k,n) \times \mathbf{C}^{n+1} | v \in [x]\}$$

endowed with the projection to the first factor. We get a vector bundle U on $Gr(k,n)$ of rank $k+1$ and an exact sequence

$$0 \longrightarrow U \longrightarrow \mathcal{O} \otimes V^* \longrightarrow Q \longrightarrow 0$$

where Q is a vector bundle of rank $n-k$ called the quotient bundle.

The computation of cohomology groups $H^i(Gr(k,n), U(t))$ and $H^i(Gr(k,n), Q(t))$ is more subtle and will be seen as a corollary of the Bott theorem.

Anyway many results can be seen geometrically. It is evident that U has no global sections (there is no point common to all $\mathbf{P}^k$!) while a section of U^* is given by a linear functional $F: V^* \to \mathbf{C}$ which restricts to any $k+1$-dimensional subspace $\mathbf{C}^{k+1} \subset V^*$. Then the zero locus of this section is given by

$$\{\mathbf{C}^{k+1} \subset V^* | \mathbf{C}^{k+1} \subset Ker\ F\}$$

which is isomorphic to $Gr(k, n-1)$.

Exercise. *Show that U^* is spanned.*

Exercise. *Show that the choice of a point in V^* gives a section of Q. The zero locus of this section is isomorphic to $Gr(k-1, n-1)$.*

Line bundles and embeddings in projective spaces

(see [GH] for more details)

A spanned line bundle L define a morphism

$$\begin{aligned} \phi_L : X &\to \mathbf{P}(H^0(X,L)^*) \\ x &\mapsto \{s \in H^0(X,L) | s(x) = 0\} \end{aligned}$$

In coordinates if $< s_0, \ldots, s_N >= H^0(X,L)$ then $\phi_L(x) = (s_0(x), \ldots, s_N(x)) \in \mathbf{P}^N$.

The projective embeddings defined by subspaces $V \subset H^0(X,L)^*$ correspond to projections of X from linear subspaces contained in $\mathbf{P}(H^0(X,L)^*)$ into projective spaces of smaller dimension.

A line bundle L is called very ample if ϕ_L is an embedding and is called ample if $L^{\otimes k}$ is very ample for some $k \in \mathbf{N}$.

Example. *On $\mathbf{P}^n$ the bundle $\mathcal{O}(t)$ is ample iff is very ample iff $t < 0$. $\mathcal{O}(t)$ is spanned iff $t \geq 0$.*

We have obviously

$$\phi_L^* \mathcal{O}(1) \simeq L$$

Vector bundles and embeddings in grassmannians

A spanned vector bundle E of rank r defines a morphism

$$\begin{aligned}\phi_E\colon X &\to Gr(\mathbf{P}^{r-1}, \mathbf{P}(H^0(X,E)^*)) \\ x &\mapsto \{s \in H^0(X,E) | s(x) = 0\}\end{aligned}$$

We have

$$E = \phi_E^* U^*$$

where U is the universal bundle on the grassmannian.

In equivalent way, if $h^0(X,E) = N$ we have a map to the dual grassmannian

$$X \xrightarrow{\phi'_E} Gr(\mathbf{P}^{N-r-1}, \mathbf{P}(H^0(X,E)))$$

and in this case

$$E = \phi_E'^* Q'$$

where Q' is the quotient bundle on $Gr(\mathbf{P}^{N-r-1}, \mathbf{P}(H^0(X,E)))$

One is tempted to define E to be very ample if ϕ_E is an embedding but this definition is too weak to have good properties. For example the bundle U^* on a grassmannian satisfies this definition (ϕ_U^* is the identity!) but its restriction to a line has a trivial summand.

The correct way is to define E (very) ample if $\mathcal{O}_{\mathbf{P}(E)}(1)$ is (very) ample on the projective bundle $\mathbf{P}(E)$ (see [Har66]).

Let J be a nondegenerate skew symmetric matrix of order $2n$. The symplectic group $Sp(n)$ consists of the matrices $A \in GL(2r)$ such that $AJA^t = J$. Another J' defines a conjugate subgroup. Its Lie algebra is given by

$$\mathcal{SP}(n) = \{A \in \mathcal{GL}(2r) | AJ + JA^t = 0\}$$

When we consider a symmetric matrix Q of order n at the place of J we define in the same way $O(n)$ orthogonal group which has two connected components depending on det $= \pm 1$. The connected component of the identity contains matrices with det $= 1$ and it is denoted by $SO(n)$. Its Lie algebra is $\mathcal{SO}(n)$

Definition 8.3. *A vector bundle E of rank r (even) is called symplectic if there is an atlas such that the transition functions take values in $Sp(r/2)$.*

Proposition 8.4. *E is symplectic if and only if there is an isomorphism*

$$\phi\colon E \to E^*$$

such that $\phi = -\phi^t$.

The reader can supply the analogous definition of orthogonal bundle.

Remark. *If E is symplectic then $\wedge^2 E$ contains $\mathcal{O}$ as direct summand. It is important to observe that if T is a representation of the symplectic group, then it is defined the bundle $T(E)$.*

Geometrical definition of Chern classes

There are several equivalent definitions of the Chern classes of a vector bundle E. The analytic definitions via the curvature is the more useful to prove formulas about the Chern classes. In the spirit of this course we sketch the geometrical definition of Chern classes of degeneracy loci that involves the map Φ_E in the grassmannian.

Let E be a spanned vector bundle of rank r over X. We denote by $s_1, \ldots, s_{r-p+1}$ $r-p+1$ generic sections of E. The subvariety

$$\{x \in X | s_1(x), \ldots, s_{r-p+1} \text{ are lin. dep.}\} \tag{8.4}$$

has codimension p and its homology class in $H_{2n-2p}(X, \mathbf{Z})$ does not depend on the sections (it is easy to check that even the rational equivalence class in the Chow ring is well defined). The Chern class $c_p(E) \in H^{2p}(X, \mathbf{Z})$ can be defined as the Poincaré dual of the class in (8.4).

If $p = r$ in (8.4) we get the zero locus of a generic section of E.

If $p = 1$ in (8.4) we get that $c_1(E) = c_1(\det\ E)$, furthermore c_1 of a line bundle associated to a divisor D is the class of D itself.

Proposition 8.5. *Let U be the universal bundle on $Gr(k, n)$. Then $c_p(U^*)$ is Poincaré dual to the Schubert cycle $\overline{W_{1,\ldots,1}}$ (p times) (see (2.2)).*

Proof The Poincaré dual of $c_p(U^*)$ is the degeneracy locus of $k-p+2$ sections of U^*. These sections are given by $k-p+2$ linear functional whose common kernel correspond to a linear $\mathbf{P}_o^{n-k+p-2}$. Then the result follows by (2.2).

Remark. *In the same way the Poincaré dual of $c_q(Q)$ (where Q is the quotient bundle) is $\overline{W}_q$. These are called special Schubert cycles.*

Corollary 8.6. *Consider the map*

$$X \xrightarrow{\Phi_E} Gr(k, n)$$

We have

$$\Phi_E^* c_p(U^*) = c_p(\Phi_E^* U^*) = c_p(E)$$

The corollary 8.6 is important because in order to show the equivalence of the geometrical definition of c_p with the analytic one it is sufficient to perform some standard curvature computations on $Gr(k, n)$ (see [GH] or [Kobayashi, Differential geometry of complex vector bundles]).

When E is not spanned there are two ways to supply the definition of Chern classes. The first one (as in [GH]) is to consider convenient C^∞ sections, in fact the Chern classes are C^∞-invariant. The second one is to tensor E with some ample line bundle L in order to get $E \otimes L$ spanned and then use the formula

$$c_k(E \otimes L) = \sum_{i=0}^{k} \binom{r-i}{k-i} c_i(E) c_1(L)^{k-i}$$

(of course one has to check that this definition is well posed!)

The Chern polynomial is the formal expression

$$c_E(t) := c_0(E) + c_1(E)t + c_2(E)t^2 + \ldots$$

If

$$0 \to E \to F \to G \to 0$$

is an exact sequence of vector bundles, the Whitney formula is

$$c_E(t) c_G(t) = c_F(t)$$

Sketch on principal bundles

We want to underline that the importance of transition functions stated in the remark 8.1 is a deep fact. The transition functions do not involve the fiber $\mathbf{C}^r$ but only the group $GL(r)$ acting on the fibers. This leads to the following generalization.

Let us consider an open covering $\{U_\alpha\}_{\alpha \in I}$ of X, a Lie group G and a set of holomorphic transition functions

$$g_{\alpha\beta}(x) \colon (U_\alpha \cap U_\beta) \to G$$

satisfying (8.1) and (8.2). Suppose moreover that we have a variety F and a representation

$$\rho \colon G \to Aut(F)$$

We can construct a bundle B over X with fiber F as the quotient of the disjoint union

$$\coprod_\alpha (U_\alpha \times F)$$

by the relation $\sim$ defined in the following way

$$\forall (x,f) \in U_\alpha \times F \quad (x',f') \in U_\beta \times F$$

we have

$$(x,f) \sim (x',f') \text{ iff } x = x' \quad f = \rho(g_{\alpha\beta})(x) f'$$

In particular we can take $F = G$ and ρ to be the left multiplication. The bundle P that we get is called a principal bundle. We say that the bundle B comes from the principal bundle P via ρ.

§9. Homogeneous bundles

We turn to the case of bundles over rational homogeneous varieties G/P (here G is a simply connected semisimple group and P is a parabolic subgroup).

It is natural to define a homogeneous bundle E if the action of G over G/P can be lifted to E. More precisely

Definition 9.1. *Let E be a bundle over G/P. E is called G-homogeneous (or simply homogeneous) if there exists an action of G over E such that the following diagram commutes*

$$\begin{array}{ccc} G\times E & \longrightarrow & E \\ \Big\downarrow & & \Big\downarrow \\ G\times G/P & \longrightarrow & G/P \end{array}$$

It is evident from this definition that the tangent bundles $T(G/P)$ are homogeneous.

Remark 9.2. *If $T\colon GL(r)\to GL(r')$ is any representation and E is a homogeneous bundle of rank r then $T(E)$ is a homogeneous bundle of rank r'. For example on $\mathbf{P}^n$ many homogeneous bundles can be constructed in this way beginning from the tangent bundle. We will se that in this way we get the so called irreducible bundles. There exist many other homogeneous bundles. An example is the following.*

Example 9.3. *Consider on $\mathbf{P}^n$ the evaluation map*

$$H^0(\mathbf{P}^n,\mathcal{O}(t))\otimes\mathcal{O}\xrightarrow{ev}\mathcal{O}(t)$$

The kernel $E_{n,t}:=\ker\ ev$ is a homogeneous bundle. Its fiber over $x\in\mathbf{P}^n$ can be identified with the space of hypersurfaces of degree t containing x. The simplest nontrivial case is the rank 5-bundle $E_{2,2}$ on $\mathbf{P}^2=\mathbf{P}(V^*)$ obtained in the sequence

$$0\longrightarrow E_{2,2}\longrightarrow S^2V\otimes\mathcal{O}\longrightarrow\mathcal{O}(2)\longrightarrow 0$$

This bundle is exceptional ,i.e. it has no deformations. The exceptional bundles on $\mathbf{P}^2$ (that are of course homogeneous) have been all classified in a beautiful paper of Drezet and LePotier [Ann. Sc. ENS 18, 193-243 (1985)]. The numerical invariants of exceptional bundles on $\mathbf{P}^2$ fill up a region with "fractal" boundary. Coming back by earth we want to show that $E_{2,2}$ appears in a sequence

$$0\longrightarrow S^2Q^*\longrightarrow E_{2,2}\longrightarrow Q\longrightarrow 0 \tag{9.1}$$

This is clear geometrically, looking at $S^2Q^*_x$ as the space of conics with a singularity at x. Otherwise, taking the second symmetric power of the dual Euler sequence

$$0\longrightarrow Q^*\longrightarrow V\otimes\mathcal{O}\longrightarrow\mathcal{O}(1)\longrightarrow 0$$

we get the exact sequence

$$0 \longrightarrow S^2Q^* \longrightarrow E_{2,2} \longrightarrow Q \longrightarrow 0$$

Then the result follows by applying the snake lemma in the diagram

$$\begin{array}{ccccccccc} & & & & S^2Q^* & & & & \\ & & & & \downarrow & & & & \\ 0 & \longrightarrow & E_{2,2} & \longrightarrow & S^2V \otimes \mathcal{O} & \longrightarrow & \mathcal{O}(2) & \longrightarrow & 0 \\ & & & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & Q & \longrightarrow & V \otimes \mathcal{O}(1) & \longrightarrow & \mathcal{O}(2) & \longrightarrow & 0 \end{array}$$

(9.1) is interesting because it gives a filtration of $E_{2,2}$ in irreducible homogeneous bundles. $E_{n,t}$ are all stable bundles (see [Pa]).

We will see in this section other two equivalent definitions of homogeneous bundle that are useful and more easy to handle.

Lemma 9.4. *$G \xrightarrow{\pi} G/P$ is a principal bundle with fiber P.*

Proof The only nontrivial statement is that π is a locally trivial fibration. In order to see this consider the derivative at the identity

$$Lie\ G \xrightarrow{d\pi} T_{[P]}G/P$$

$Lie\ P$ is a subalgebra of $Lie\ G$ that goes to zero under $d\pi$. Hence by dimensional reasons $Ker\ d\pi$ can be identified with $Lie\ P$. Choose any trasversal subspace such that $Lie\ G = Lie\ P \oplus V$ (direct sum of vector spaces only!) and consider $\exp\colon V \to G$. There exists a neighborhood $V' := \exp(U) \subset G$ where exp is invertible. In particular $\sigma := \pi_{|V'}\colon V' \longrightarrow \pi(V')$ is a diffeomorphism, so that $\forall x \in V'$ we have

$$xP = \sigma^{-1}(\pi(x))P$$

and we define the local trivialization

$$\begin{aligned} \pi^{-1}(\pi(V')) &\to \pi(V') \times P \\ x &\mapsto (\pi(x), x \cdot [\sigma^{-1}(\pi(x))]^{-1}) \end{aligned}$$

This local trivialization can be extended to all G/P by using the action of G.

Definition 9.5. *Let $\rho\colon P \to GL(r)$ be a representation. We can construct a vector bundle E_ρ on G/P as the bundle with fiber $\mathbf{C}^r$ coming from the principal bundle $G \xrightarrow{\pi} G/P$ via ρ.*

Remark 9.6. *In equivalent way, E_ρ can be defined as the quotient $G \times_\rho \mathbf{C}^r$ of $G \times \mathbf{C}^r$ via the equivalence relation $\sim$ where $(g, v) \sim (g', v')$ iff there exists $p \in P$ such that $g = g'p$ and $v = \rho(p^{-1})v'$.*

Remark. *The following are true*

$$E_{\rho_1} \oplus E_{\rho_2} \simeq E_{\rho_1 \oplus \rho_2}$$

$$\wedge^k E_\rho \simeq E_{\wedge^k \rho}$$

$$E_{\rho_1} \otimes E_{\rho_2} \simeq E_{\rho_1 \otimes \rho_2}$$

and so on.

Theorem 9.7 (Matsushima). *A vector bundle E of rank r over G/P is homogeneous if and only if there exists a representation $\rho: P \to GL(r)$ such that $E \simeq E_\rho$.*

Proof The proof is an exercise of running through the definitions. Given a homogeneous bundle $E \xrightarrow{\pi} G/P$ then the action of G restricted to P takes $\pi^{-1}(P)$ to $\pi^{-1}(P)$ and this is ρ. We can construct an isomorphism

$$E \longrightarrow E_\rho$$

that takes $e \in \pi^{-1}(gP)$ into $[g, g^{-1} \cdot e]$.

Conversely given $[(g,v)] \in E_\rho$ define the action of G over E_ρ as $g' \cdot [(g,v)] := (g'g, v)$

Lemma 9.8.

i) The space of sections $H^0(G/P, E_\rho)$ of E_ρ over G/P can be identified with

$$\{f: G \to \mathbf{C}^r | f(gp) = \rho(p^{-1}) f(g) \quad \forall g \in G, \quad \forall p \in P\}$$

ii) The action of G over $H^0(G/P, E_\rho)$ is given by

$$(g \cdot f)(g_1) := f(g^{-1} g_1)$$

Proof Straightforward check by using the remark 9.6.

Remark. *The tangent bundle comes from the representation of P with derivative*

$$ad: \mathcal{P} \to \mathcal{GL}(\mathcal{G}/\mathcal{P})$$

(recall that $\mathcal{G}/\mathcal{P} \simeq T_{[P]} G/P$).

Example. *On $\mathbf{P}^1 = SL(2)/P$ the bundles $\mathcal{O}(t)$ are obtained from the representations*

$$P \to \mathbf{C}^*$$

$$\begin{pmatrix} a & b \\ 0 & 1/a \end{pmatrix} \mapsto a^{-t}$$

A basis for $H^0(\mathbf{P}^1, \mathcal{O}(t))$ (for $t \geq 0$) is given by

$$f_i \begin{pmatrix} a & b \\ c & d \end{pmatrix} := a^{-i} c^{-t+i}$$

for $i = 0, \ldots, t$.

Let us denote the natural morphism $G \to Aut(G/P)$ by $g \mapsto \theta_g$. The following theorem shows that our definition of homogeneous bundle is equivalent to the one given in [C.Okonek, M.Schneider, H.Spindler, Vector bundles over complex projective spaces].

Theorem 9.9. *Let G semisimple and simply connected. A vector bundle E over $X = G/P$ is homogeneous if and only if*

$$\theta_g^* E \simeq E \quad \forall g \in G$$

Proof (A. Huckleberry) Let $Aut(E)$ be the connected component containing the identity of the algebraic group of automorphisms of E (considered as variety) and let $Aut_X(E)$ be the subgroup of automorphisms preserving each fiber and acting linearly on them. $Aut_X(E)$ is the group of invertible elements in $H^0(X, End\ E)$. Consider the algebraic group

$$F := \{g \in Aut(E) | g \text{ is linear on the fibers and induces on } X \text{ an element of } G\}$$

F can be obtained as an extension

$$0 \longrightarrow Aut_X(E) \longrightarrow F \longrightarrow G \longrightarrow 0$$

It is induced a surjective morphism

$$Lie\ F \xrightarrow{\phi} Lie\ G$$

It is easy to check that the image of a solvable algebra is still a solvable algebra. It follows $\phi(rad\ F) = 0$. By a theorem of Levi ([FuHa] theor. E.1) there exists a semisimple Lie algebra $\mathcal{S} \subset Lie\ F$ such that $\mathcal{S} \xrightarrow{\phi} Lie\ G$ is surjective. By the theor. 6.13 $\mathcal{S}$ is a sum of simple Lie algebras $\mathcal{S} \simeq \oplus_{i=1}^k \mathcal{S}_i$, hence there exists $j < k$ such that

$$\ker\ \phi \simeq \oplus_{i=1}^j \mathcal{S}_i \qquad Lie\ G \simeq \oplus_{i=j+1}^j \mathcal{S}_i$$

It follows by 3.6 that there exists $G' \subset F$ such that $Lie\ G' \simeq Lie\ G$. Hence G' acts over E, G is a covering of G' by the prop. 3.15 and also G acts over E as we wanted.

Remark. *It can be shown with the same tecnique that if G' is a reductive subgroup of G such that $\theta_{g'}^* E \simeq E$ $\forall g' \in G'$ then there exists $\tilde{G}'$ covering of G' that acts over E.*

Remark. *The theorem 9.9 does not hold if G is not simply connected. For example $PSL(2) = Aut(\mathbf{P}^1)$ acts on $\mathbf{P}^1 = PSL(2)/P'$ but $\mathcal{O}(t)$ has a $PSL(2)$-action (or equivalently is defined by a representation of P') if and only if t is even.*

Note that $K_{\mathbf{P}^1} = \mathcal{O}(-2)$ so that all the multiples of the canonical bundle are $Aut(\mathbf{P}^1)$-homogeneous.

§10. The theorem of Borel-Weil

Theorem 10.1.

i) A homogeneous projective variety X with $b_1(X) = 0$ is rational.

ii) Every homogeneous variety $G/P(\Sigma)$ is rational.

Proof Let X as in i). By the theor. 4.13 and the remark 4.14 X is isomorphic to G/P where $G := Aut^0(X)$ is semisimple. By the theorem 7.5 P is parabolic. By the theor. 7.8 there exists a subset Σ of the set of simple roots such that

$$Lie\ P = \mathcal{H} \oplus (\oplus_{\alpha>0}\mathcal{G}_\alpha) \oplus (\oplus_{\alpha\in\phi^-(\Sigma)}\mathcal{G}_\alpha)$$

where

$$\phi^-(\Sigma) = \{\alpha = \sum p_i \in \Phi^-\alpha_i | p_i = 0 \text{ if } \alpha_i \in \Sigma\}$$

So it is sufficient to prove ii). Let

$$\mathcal{U}^- := (\oplus_{\alpha\notin\phi^-(\Sigma)}\mathcal{G}_\alpha)$$

It is easy to check that $\mathcal{U}^-$ is a solvable subalgebra (it is even nilpotent). Consider the embedding

$$\mathcal{G} \xrightarrow{ad} \mathcal{GL}(\mathcal{G})$$

(The kernel of ad is given by the center which is zero because $\mathcal{G}$ is semisimple.) Considering that

$$ad^k(X)(Y) = [X, [X, [X, \ldots, [X, Y]]]]$$

and by the theorem 6.5

$$ad(\mathcal{G}_\alpha)(\mathcal{G}_\beta) \subset \mathcal{G}_{\alpha+\beta}$$

it follows that $ad(\mathcal{U}^-)$ consists of nilpotent endomorphisms. By the Lie theorem in a convenient basis of $\mathcal{G}$ we have that $\mathcal{U}^-$ is immersed in the subalgebra of strictly lower-triangular matrices. Then at level of Lie groups we have

$$U^- \subset G \subset GL(N)$$

such that the U^- is immersed in the subgroup of unipotent matrices. In particular

$$\mathcal{U}^- \xrightarrow{\exp} U^-$$

has inverse given by

$$log: U^- \to \mathcal{U}^-$$

$$A \mapsto \sum_{k=0}^{\infty}(-1)^k \frac{(A-I)^k}{k}$$

because the unipotency implies that this last sum is finite (and so it converges!). It follows that U^- is isomorphic to $\mathcal{U}^-$. Moreover from the matrix description we have $U^- \cap P = e$, so that the morphism $U^- \to G/P$ is injective and it is dominant too by dimensional reasons. Hence G/P is rational.

Remark. *The Bruhat decomposition generalize the description of the open cell $N \subset G/P$ that we have seen in the proof of the theorem 10.1. See [FuHa] §23.4.*

Corollary 10.2 (Ise). *Let G be semisimple and simply connected. Every line bundle on G/P is homogeneous.*

Proof By the rationality we have

$$H^i(G/P, \mathcal{O}) = 0 \quad \text{for } i > 0$$

Then the cohomology sequence associated to the exponential sequence

$$0 \longrightarrow \mathbf{Z} \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}^* \longrightarrow 0$$

implies

$$H^1(G/P, \mathcal{O}^*) \simeq H^2(G/P, \mathbf{Z})$$

so that $H^1(G/P, \mathcal{O}^*)$ is discrete and the G-action on it given by

$$L \mapsto g^* L$$

is trivial.

From the previous corollary it follows that the topological group $H^2(G/P, \mathbf{Z})$ is isomorphic to the group of one-dimensional representations of P. This group can be computed as in the prop. 10.4.

Lemma 10.3. *We have the decomposition*

$$Lie\ P = \mathcal{H} \bigoplus (\oplus_{\alpha > 0} \mathcal{G}_\alpha) \bigoplus (\oplus_{\alpha \in \phi^-(\Sigma)} \mathcal{G}_\alpha) =$$

$$= Lie\ S_P \bigoplus (\oplus_{i=1}^k [\mathcal{G}_{\alpha_i}, \mathcal{G}_{-\alpha_i}]) \bigoplus (\oplus_{\alpha \notin \phi^+(\Sigma)} \mathcal{G}_\alpha)$$

where S_P is semisimple and it is called the semisimple part of P.

Proposition 10.4. *Let $\Sigma = \{\alpha_1, \ldots, \alpha_k\}$ be a subset of the set of simple roots. Then*

$$H^2(G/P(\Sigma), \mathbf{Z}) \simeq Pic(G/P(\Sigma)) \simeq \mathbf{Z}^{\oplus k}$$

Proof Let $V = V_\lambda$ be a one-dimensional $Lie\ P$-module, where λ is the corresponding weight in $\mathcal{H}^*$. Looking at the decomposition of the lemma 10.3 we have that V restricted to $Lie\ S_P$ is trivial because it is trivial for every $\mathcal{SL}(2)$ inside (recall that the only one-dimensional representation of $\mathcal{SL}(2)$ is trivial).

By the theorem 6.20

$$\mathcal{G}_\alpha V_\lambda \subset V_{\alpha + \lambda}$$

and we get that the representation is trivial when restricted to $(\oplus_{\alpha\notin\phi^+(\Sigma)}\mathcal{G}_\alpha)$. Hence this representation is obtained from the abelian k-dimensional piece $\oplus_{i=1}^k[\mathcal{G}_{\alpha_i},\mathcal{G}_{-\alpha_i}]$ which is the Lie algebra of a torus $(\mathbf{C}^*)^k$. The result follows from the cor. 5.10.

We want now to classify all the irreducible representations of a parabolic subgroup P. Let $U\subset P$ be the subgroup such that

$$Lie\ U=(\oplus_{\alpha\notin\phi^+(\Sigma)}\mathcal{G}_\alpha)$$

(see the lemma 10.3). U is called the unipotent part of P. The subgroup R such that $Lie\ P=Lie\ U\oplus Lie\ R$ is reductive.

Proposition 10.5 (Ise). *A representation*

$$\rho: P\to GL(V)$$

is completely reducible if and only if $\rho_{|U}$ is trivial.

Proof $LieU$ is contained in the subalgebra

$$LieY:=(\oplus_{i=1}^k[\mathcal{G}_{\alpha_i},\mathcal{G}_{-\alpha_i}])\bigoplus(\oplus_{\alpha\notin\phi^+(\Sigma)}\mathcal{G}_\alpha)$$

which is solvable. From the Theorem of Lie4.2 there exists a basis in V such that $\rho(y)$ is upper triangular for every $y\in Y$.

Since $LieU=[LieY,LieY]$ we get that $d\rho(u)$ is strictly upper triangular for every $u\in LieU$.

It follows that there exists a nonzero $v\in V$ such that $\forall u\in U$

$$\rho(u)v=v$$

Let

$$F:=\{v\in V|\rho(u)v=v\quad\forall u\in U\}$$

By the above argument $0\neq F$. As U is normal, it is easy to check that F is P-invariant so that by the assumption $F=V$. This means that $\rho_{|U}$ is trivial.

Conversely, any representation of P which is trivial over U at the level of Lie algebras comes from a representation of

$$Lie\ S_P\bigoplus\mathcal{Z}$$

where $\mathcal{Z}=(\oplus_{i=1}^k[\mathcal{G}_{\alpha_i},\mathcal{G}_{-\alpha_i}])$, which is a direct sum of two Lie algebras (in the sense that are both ideals). Any such representation is the tensor product of a representation of $\mathcal{Z}$ (abelian Lie algebra of $(\mathbf{C}^*)^k$) and a representation of $Lie\ S_P$ which are both completely reducible.

Lemma 10.6.

$$\pi_1(SL(n)) = \pi_2(SL(n)) = 0$$

Proof By induction on n ($n = 1$ is trivial). Consider the natural $SL(n)$-action over $\mathbf{C}^n \setminus 0$. The isotropy subgroup is $H \simeq SL(n-1) \times \mathbf{C}^{n-1}$. Hence consider the long homotopy sequence associated to the fibration $SL(n) \to \mathbf{C}^n \setminus 0$ and get

$$\pi_2(H) \to \pi_2(SL(n)) \to \pi_2(\mathbf{C}^n \setminus 0) \to \pi_1(H) \to \pi_1(SL(n)) \to \pi_1(\mathbf{C}^n \setminus 0)$$

By the inductive assumption $\pi_1(H) = \pi_2(H) = 0$, hence the result.

Proposition 10.7. *Let G be a semisimple group. We have*

$$\pi_2(G) = 0$$

Proof For the classical groups the proof is similar to the one of lemma 10.6. In general we refer to [Brocker-tom Dieck, Representations of compact Lie groups, GTM 98, Springer], pag. 153 8.3, pag. 223 7.1.

Proposition 10.8. *Let G be semisimple and simply connected. Let $P \subset G$ be a parabolic subgroup. Then*

$$\pi_1(S_P) = 0$$

(see the lemma 10.3)

Proof Consider the long homotopy sequence associated to the fibration

$$G \to G/P$$

By the assumption and the prop. 10.7 it follows $\pi_1(P) \simeq \pi_2(G/P)$. By the prop. 10.4 and Hurewicz theorem it follows

$$\pi_1(P) \simeq \pi_2(G/P) \simeq H^2(G/P, \mathbf{Z}) \simeq \mathbf{Z}^{\oplus k}$$

There is a Levi decomposition $P = U \rtimes (S_P \cdot (\mathbf{C}^*)^k)$ ($\cdot$ means that a finite intersection is allowed). In a way similar to the proof of the theorem 10.1 we can show that U is isomorphic to a vector space, then it has the homotopy type of a point. Hence

$$\pi_1(S_P \cdot (\mathbf{C}^*)^k) \simeq \mathbf{Z}^{\oplus k}$$

. Let j be the covering map $S_P \times (\mathbf{C}^*)^k \to S_P \cdot (\mathbf{C}^*)^k$, then at the level of π_1's j_* is injective and the thesis follows.

Proposition 10.9 (Classification of irreducible bundles over G/P). *Let $\Sigma = \{\alpha_1, \ldots, \alpha_k\}$ be a subset of simple roots. Let $\lambda_1, \ldots, \lambda_k$ be the corresponding set of fundamental weights (see the def. 6.45). Then all the irreducible representations of $P(\Sigma)$ are*

$$V \otimes L_{\lambda_1}^{n_1} \otimes \ldots \otimes L_{\lambda_k}^{n_k}$$

where V is a representation of S_P and $n_i \in \mathbf{Z}$ (by the prop. 10.8 λ_i define representations of S_P).

Proof It follows from propositions 10.5, 10.8 and corollary 3.28.

Remark 10.10. *With the notations of the prop. 10.9, consider that the weight lattice of S_P is embedded in the weight lattice of G. If λ is the highest weight of a irreducible representation V_λ of S_P we will say that $\lambda + \sum n_i\lambda_i$ is the highest weight of the irreducible representation $V \otimes L_{\lambda_1}^{n_1} \otimes \ldots \otimes L_{\lambda_k}^{n_k}$ of $P(\Sigma)$. We underline that now the n_i's can be negative.*

The tensor product $L_{\lambda_1}^{n_1} \otimes \ldots \otimes L_{\lambda_k}^{n_k}$ corresponds to the twist by line bundles.

When $k = 1$ (parabolic subgroups corresponding to only one simple root) we get homogeneous rational varieties with $Pic = \mathbf{Z}$. For example grassmannians belong to this class. At the other extreme the parabolic subgroup corresponding to all the simple roots is the Borel subgroup B. Complete flag manifolds belong to this class. We have $S_B = 0$ (because B is solvable), hence all the irreducible representations of B have dimension 1 and define all line bundles.

Remark. *It is interesting to look at the projections*

$$G/B \xrightarrow{\pi} G/P$$

which are fibrations with fiber isomorphic to $P/B \simeq S_P/(B \cap S_P)$ which is again a rational homogeneous variety. π is flat. We will see that the irreducible homogeneous bundles on G/P are all isomorphic to $\pi_* L_\lambda$ where L_λ is a line bundle on G/B.

Chern classes of homogeneous bundles

Let E_ρ be a homogeneous bundle over G/P. The projection $G/B \xrightarrow{\pi} G/P$ is a fiber bundle, in particular

$$\pi^*: H^i(G/P, \mathbf{Z}) \to H^i(G/B, \mathbf{Z}) \tag{10.1}$$

is injective. This allows to compute the Chern classes $c_p(E_\rho)$ in the following way.

If μ_i for $i = 1, \ldots, r$ are all the weights of ρ then $\pi^* E_\rho$ admits a filtration with successive quotients given by the line bundles L_{μ_i} and by the Withney formula

$$c_{\pi^* E_\rho}(t) = \prod_{i=1}^{r} c_{L_{\mu_i}}(t) = \prod_{i=1}^{r} (1 + \mu_i t)$$

by identifying μ_i with the corresponding class in $H^2(G/B, \mathbf{Z})$. By (10.1) after eventual cancellations $c_p(E)$ must involve only the weights that are fundamental for P.

For example

$$c_1(E) = \sum_{i=1}^{r} \mu_i$$

$$c_2(E) = \sum_{i<j} \mu_i \mu_j$$

and so on.

Let $P = P(\alpha_1, \ldots, \alpha_k)$ and let $\alpha_1, \ldots, \alpha_n$ be all the roots (in some order). When ρ is irreducible $c_1(E_\rho)$ can be computed easily from the highest weight μ_1 of ρ. In fact all the other weights μ_i are obtained summing to μ_1 some roots in S_P. If $\mu_i = \sum_{j=1}^{n} q_{ij}\alpha_j$ for $i = 1, \ldots, r$ (here $q_{ij} \in \mathbf{Q}$) then for $j = 1$ to k q_{ij} does not depend on i so that

$$c_1(E_\rho) = \sum_{i=1}^{r} \mu_i = rq_{11}\alpha_1 + rq_{12}\alpha_2 + \ldots + rq_{1k}\alpha_k + \sum_{j=k+1}^{n} x_j\alpha_j = \sum_{i=1}^{k} y_i\lambda_i$$

where λ_i are the fundamental weights and y_i are the unknowns that can be determined by the square system

$$rq_{1j} = \sum_{i=1}^{k} y_i c^{ij} \quad j = 1, \ldots, k$$

where c^{ij} are the coefficients of the inverse of the Cartan matrix. C^{-1} is computed in [Hu1] pag. 69, table 1

Example. *Let E_{λ_2} the bundle defined over $Gr(k,n) = SL(n+1)/P(\alpha_{k+1})$ by the representation with highest weight λ_2. We want to compute $c_1(E_{\lambda_2})$ with the above technique (of course this could be done directly after observing that $E_{\lambda_2} = \wedge^2 U^*$!).*

We identify λ_{k+1} with the positive generator of $H^2(Gr(k,n), \mathbf{Z}) = \mathbf{Z}$

We have

$$r = rank\ E_{\lambda_2} = \binom{k+1}{2}$$

coefficient of α_{k+1} in the expression of $\lambda_2 = 2(n-k)/(n+1)$

coefficient of α_{k+1} in the expression of $\lambda_{k+1} = (n-k)(k+1)/(n+1)$

so that

$$c_1 = \frac{2\binom{k+1}{2}/(n+1)}{(k+1)/(n+1)} = k$$

The following theorem was already known to Cartan in 1913, but the modern translation in the language of bundles is known as Borel-Weil theorem.

Theorem 10.11 (Cartan, Borel-Weil). *Let G be semisimple and simply connected. Let $P \subset G$ be a parabolic subgroup. Let E_ρ be the homogeneous bundle defined by the irreducible representation ρ of P with highest weight λ (see the remark 10.10). Then*

$$H^0(G/P, E_\rho) \simeq G_\lambda$$

where G_λ is the irreducible representation of G with maximal weight λ.

Proof By the lemma 9.8 we have

$$H^0(G/P, E_\rho) \simeq \{f: G \to \mathbf{C}^r | f(gp) = \rho(p^{-1})f(g) \quad \forall p \in P, g \in G\} \tag{10.2}$$

Denote the corresponding representation by

$$\sigma: G \to GL(H^0(G/P, E_\rho))$$
$$(d\sigma)_e: \mathcal{G} \to \mathcal{GL}(H^0(G/P, E_\rho))$$

We prove first that σ is irreducible. Let $\phi \in H^0(G/P, E_\rho)$ be a lowest weight vector for σ (see remark 6.37). Let $U^- \subset G$ such that $Lie\ U^- = \oplus_{\alpha<0}\mathcal{G}_\alpha$.

We get

$$(d\sigma)_e(Lie\ U^-)\phi = 0$$

hence

$$\sigma(U^-)\phi = \phi$$

that is (see the lemma 9.8)

$$\phi(ug) = \phi(g) \quad \forall u \in U^-, g \in G \tag{10.3}$$

By (10.2)

$$\phi(gp) = \rho(p^{-1})\phi(g) \quad \forall p \in P, g \in G \tag{10.4}$$

We claim that $U^- \cdot P$ is dense in G. It is sufficient to check that $U^- \cdot B$ is dense where B is a Borel subgroup.

In fact

$$U^- \times B \to G$$
$$(u, b) \mapsto ub$$

is an injective map (not a group morphism!) of algebraic varieties of the same dimension over $\mathbf{C}$, hence it is dominant.

We get by using (10.3) and (10.4)

$$\phi(up) = \phi(p) = \rho(p^{-1})\phi(1) \quad \forall u \in U^-, p \in P$$

and this shows that ϕ is determined by $\phi(1)$. Then there exists a unique lowest weight vector (up to constant) and by the theor. 6.36 σ is a irreducible representation.

To conclude the proof, call σ_{low} the lowest weight of σ and ρ_{low} the lowest weight of ρ. It is sufficient to show that $\sigma_{low} = \rho_{low}$. By the definition of lowest weight (applied to the Lie groups)

$$\sigma(h)\phi = \sigma_{low}(h)\phi \quad \forall h \in H$$

that is

$$\phi(h^{-1}g) = \sigma_{low}(h)\phi(g)\forall h \in H, g \in G \tag{10.5}$$

By using (10.4) and (10.5)

$$\rho(h^{-1})\phi(1) = \phi(1 \cdot h) = \phi(h \cdot 1) = \sigma_{low}(h^{-1})\phi(1)$$

This means that σ_{low} is a weight for ρ with eigenvector $\phi(1)$. Moreover if $p \in S_P$ we have

$$\rho(p)\phi(1) = \phi(p^{-1}) = \sigma(p)\phi(1)$$

In particular if $p \in U^- \cap S_P$ we get

$$\rho(p)\phi(1) = \phi(1)$$

so that $d\rho(Lie\ (U^- \cap S_P))\phi(1) = 0$ and $\phi(1)$ is a lowest weight vector for ρ. It follows $\sigma_{low} = \rho_{low}$ as we wanted.

Remark 10.12. *In order to show the existence of a irreducible representation of G with prescribed highest weight $n_1\lambda_1 + \ldots + n_k\lambda_k$ (see the theor. 6.40) we can simply consider $H^0(G/B, L_{\lambda_1}^{n_1} \otimes \ldots \otimes L_{\lambda_k}^{n_k})$. Of course one have to define first L_{λ_i} but this can be done by considering the associated divisor which comes from the cellular decomposition of G/B. Conversely the Borel-Weil theorem 10.11 gives a interpretation of the fact that $G/P(\alpha_1, \ldots, \alpha_k)$ can be constructed as the unique closed orbit in the G-module with highest weight $\lambda_1 + \ldots + \lambda_k$. By the Borel fixed point theorem any closed orbit contains a fixed point for the action of B. The point in $\mathbf{P}(\mathcal{G}_{\lambda_1+\ldots+\lambda_k})$ corresponding to the highest weight vector can be the only point fixed by B. Then there is a unique closed orbit. This gives a geometrical interpretation of G/P as generalized flag manifolds. For example $Spin(2n+1)/P(\alpha_k)$ is the variety of linear $\mathbf{P}^{k-1}$ contained in the smooth quadric $Q_{2n-1} \subset \mathbf{P}^{2n}$ because the irreducible representation of the semisimple part of $P(\alpha_k)$ (which is $Spin(2n-1)$ with highest weight λ_k is the k-exterior power of the standard representation.*

More generally the unique closed orbit in $\mathbf{P}(V_{\sum_{i=1}^k n_i\lambda_i})$ is $G/P(\alpha_1, \ldots, \alpha_k)$ embedded with the line bundle $L_{\lambda_1}^{n_1} \otimes \ldots \otimes L_{\lambda_k}^{n_k}$.

Proposition 10.13. *Let G be semisimple and simply connected. Let $\lambda: B \to \mathbf{C}^*$. Let $P \subset G$ be a parabolic subgroup and let ρ the representation of P with highest weight λ (see the remark 10.10). Denote by π the projection $G/B \to G/P$. The following is true*

$$\pi_* L_\lambda \simeq E_\rho$$

Proof Let $P = P(\alpha_1, \dots, \alpha_k)$. Let

$$\lambda = \sum_{j=1}^{k} n_j \lambda_j + \sum_{j \neq \{1,\dots,k\}} n_j \lambda_j =: \lambda' + \lambda''$$

Then we have

$$L_\lambda \simeq L_{\lambda'} \otimes L_{\lambda''}$$

$L_{\lambda'}$ is the pullback $\pi^* L'_{\lambda'}$ of a line bundle on G/P. By the projection formula it follows

$$\pi_* L_\lambda = L'_{\lambda'} \otimes \pi_* L_{\lambda''}$$

The fiber of the bundle $\pi_* L_{\lambda''}$ is naturally isomorphic to

$$H^0(P/B, L_{\lambda''}) = H^0(S_P/(S_P \cap B), L_{\lambda''})$$

which is by the theorem 10.11 the representation ρ'' of S_P with highest weight λ''. It follows

$$\pi_* L_\lambda = L'_{\lambda'} \otimes E_{\rho''} = E_\rho$$

Remark 10.14. *The prop. 10.13 allows to describe geometrically homogeneous bundles too. For example consider the bundle E_{λ_1} over the grassmannian $SL(n+1)/P(\alpha_{k+1})$. The semisimple part of $P(\alpha_{k+1})$ is $S \simeq SL(k+1) \times SL(n-k) = SL(W^*) \times SL(V^*/W^*)$. The fiber of E_{λ_1} is isomorphic to $H^0(S/(S \cap B), V_{\lambda_1}) \simeq W^*$ and then it is isomorphic to the dual of the universal bundle. The weights are $\lambda_1, -\lambda_1+\lambda_2, \dots, -\lambda_k+\lambda_{k+1}$ (lowest weight). Correspondingly $E_{\lambda_k-\lambda_{k+1}}$ is the universal bundle. In analogous way E_{λ_n} is isomorphic to the quotient bundle.*

Lemma 10.15.

i) The bundle $L = L_{\lambda_1}^{n_1} \otimes \dots \otimes L_{\lambda_p}^{n_p}$ over G/B is spanned if and only if $n_i \geq 0 \quad \forall i$.

ii) Let E_ρ be the homogeneous bundle defined over G/P by the irreducible representation ρ of P with highest weight $\sum n_i \lambda_i$ (see the remark 10.10). Then E_ρ is spanned if and only if $n_i \geq 0 \quad \forall i$.

Proof i) follows from the theorem 10.11 because L is spanned if and only if $H^0(X, L) \neq 0$ (because of homogeneity).

ii) follows from i) and the prop. 10.13.

Theorem 10.16 (Borel-Weil). *Let $\alpha_1, \dots, \alpha_p$ be all the simple roots of G.*

i) The bundle $L = L_{\lambda_1}^{n_1} \otimes \dots \otimes L_{\lambda_p}^{n_p}$ over G/B is very ample if and only if it is ample if and only if $n_i > 0 \quad \forall i$.

ii) The line bundle $L = L_{\lambda_1}^{n_1} \otimes \dots \otimes L_{\lambda_k}^{n_k}$ over $G/P(\alpha_1, \dots, \alpha_k)$ is very ample if and only if it is ample if and only if $n_i > 0 \quad \forall i = 1, \dots, k$.

Proof We prove first ii) for the case $P = P(\alpha_i)$ (maximal parabolic). If $n_i < 0$ then the bundle has no sections, if $n_i = 0$ it is the trivial bundle. If $n_i > 0$ then by the lemma

10.15 the bundle is spanned and $\phi_{L^{n_i}_{\lambda_i}}(G/P(\alpha_i))$ is again a G-homogeneous variety $G/\tilde{P}$ with $P \subset \tilde{P}$, hence $P = \tilde{P}$ because P is maximal.

The general case of ii) (and then also i)) follows by using the standard projections. If some $n_i = 0$ then the bundle comes as a pullback from another G/P and cannot be ample (it is trivial on the fibers). If all $n_i > 0$ then the image of the associated map is $G/\tilde{P}$ and $\tilde{P} = P(\Sigma')$ with Σ' a subset of $\{\alpha_1, \ldots, \alpha_k\}$. If α_i is missing this would imply $\alpha_i = 0$.

Remark 10.17. *In the papers [Snow1] and [Snow2] D. Snow computes the k-ampleness (in the sense of Sommese) and gives necessary and sufficient conditions for the spannedness of a (possibly reducible) homogeneous bundle over G/P.*

Remark 10.18. *A classical formula of H. Weyl expresses the dimension of the irreducible representation G_λ of G with highest weight λ. In the setting of algebraic geometry this formula can be deduced from the Hirzebruch-Riemann-Roch formula and the theorem 10.11, in fact we will see in the next section that if E is a homogeneous irreducible spanned bundle then $H^i(E) = 0$ for $i > 0$. Set $\delta = \sum \lambda_i$ (sum of all the fundamental weights) and $(\ ,\ )$ be the Killing form. Then Weyl's formula is*

$$\dim\ G_\lambda = \prod_{\alpha \in \Phi+} \frac{(\lambda + \delta, \alpha)}{(\delta, \alpha)} \tag{10.6}$$

It is a useful exercise to verify (10.6) for $SL(2)$ and respectively $SL(3)$ by applying HRR over $\mathbf{P}^1$ and resp. over $F(0,1,2)$.

Note that by the theorem 10.11 the formula (10.6) allows to compute not only the space of sections but also the rank of a homogeneous irreducible bundle (consider the prop. 10.13 and apply the theorem 10.11 on the fibers).

Exercises.

i) Compute $H^0(\mathbf{P}^2, S^2T\mathbf{P}^2)$.

ii) Compute rank and h^0 of E_{λ_1} over $Gr(1,3)$.

We will soon give a geometrical interpretation of homogeneous irreducible bundles, at least in the classical cases.

Remark. *We cannot avoid to mention an identification due to Borel of the cohomology ring $H^*(G/P, \mathbf{Z})$ with the quotient of the ring of polynomials on the Lie algebra $\mathcal{H}$ by the ideal generated by the polynomials invariant under the action of the Weyl group. When $G = SL(n)$ this identification reduces Schubert calculus on Grassmannians to computations with symmetric polynomials. For an account see [I.N. Bernstein, I.M. Gelfand, S.I. Gelfand, Schubert cells and cohomology of the spaces G/P Russian Math. Surveys 28 (1973)].*

§11. The theorem of Bott

The theorem of Bott generalizes the theorem of Borel-Weil, in the sense that it describes all higher cohomology groups

$$H^i(G/P, E_\rho)$$

where E_ρ is homogeneous irreducible.

Let G be a semisimple simply connected group and let $P \subset G$ be a parabolic subgroup. Let Φ be the set of roots of G. Let λ be a weight. Let E_λ be the homogeneous bundle arising from the irreducible representation of P with highest weight λ (see the rem. 10.10). In this section let us denote by (,) the Killing form B.

Definition 11.1. *λ is called singular if there exists $\alpha \in \Phi^+$ such that $(\lambda, \alpha) = 0$*

Definition 11.2. *λ is called regular of index p if it is not singular and if there exactly p roots $\alpha_1, \ldots, \alpha_p \in \Phi^+$ such that $(\lambda, \alpha) < 0$.*

Definition 11.3.

$$\delta = \sum_{i=1}^{n} \lambda_i$$

(sum of all the fundamental weights).

Theorem 11.4 (Bott).

i) If $\lambda + \delta$ is singular then

$$H^i(G/P, E_\lambda) = 0 \quad \forall i$$

ii) If $\lambda + \delta$ is regular of index p then

$$H^i(G/P, E_\lambda) = 0 \text{ for } i \neq p$$

Moreover

$$H^p(G/P, E_\lambda) = G_{w(\lambda+\delta)-\delta}$$

where $w(\lambda + \delta)$ is the unique element of the fundamental Weyl chamber of G which is congruent to $\lambda + \delta$ under the action of the Weyl group.

Corollary 11.5. *$h^i(G/P, E_\lambda)$ is nonzero for at most one value of i. In particular the dimension of the nonzero value can be computed from $\chi(E) = \sum(-1)^i h^i(E)$.*

Remark 11.6. *In order to prove the theorem of Bott we may suppose $P = B$. Look at the prop. 10.13 and the obvious vanishing $R^i\pi_*\mathcal{O} = 0$ for $i > 0$, in fact π is flat with rational fibers so that by the Leray sequence*

$$H^i(G/B, L_\lambda) \simeq H^i(G/P, \pi_* L_\lambda)$$

Remark. *If $\lambda \in \mathcal{C}$ then $\lambda + \delta$ is regular of index 0 and we get the Borel-Weil theorem 10.11.*

Proof of the theorem of Bott

The original proof has been simplified in several times. We present here the proof of M. Demazure [Dem2] which relies on a nice inductive argument assuming only the following vanishing on $\mathbf{P}^1$

$$H^i(\mathbf{P}^1, \mathcal{O}(-1)) = 0 \quad \forall i \tag{11.1}$$

Let us consider for $i = 1, \ldots, n$ the parabolic subgroups

$$P_{\alpha_i} := P(\alpha_1, \ldots, \hat{\alpha}_i, \ldots, \alpha_n)$$

so that the projection

$$G/B \xrightarrow{\pi} G/P_{\alpha_i}$$

has fibers isomorphic to

$$P_{\alpha_i}/B \simeq S_{P_{\alpha_i}}/(B \cap S_{P_{\alpha_i}}) \simeq SL(2)/(B \cap S_{P_{\alpha_i}}) \simeq \mathbf{P}^1$$

For simplicity in the rest of the proof we call $\alpha_i = \alpha$.

Denote by $V_{\lambda,\alpha}$ the irreducible representation of P_α with highest weight λ. As H-module $V_{\lambda,\alpha}$ is the direct sum of one-dimensional eigenspaces with weights (see theor. 5.13 and theor. 6.30 ii))

$$\lambda, \lambda - \alpha, \ldots, \lambda - \lambda(H_\alpha)\alpha = w_\alpha(\lambda)$$

Here w_α is the reflection with respect to the hyperplane orthogonal to α.

As B-module, $V_{\lambda,\alpha}$ has L_λ as quotient and $L_{w_\alpha(\lambda)}$ as submodule. Both L_λ and $L_{w_\alpha(\lambda)}$ correspond to line bundles.

More precisely we have the exact sequence

$$0 \longrightarrow K \longrightarrow V_{\lambda,\alpha} \longrightarrow L_\lambda \longrightarrow 0 \tag{11.2}$$

where $K = 0$ if $\lambda(H_\alpha) = 0$, $K = L_{w_\alpha}(\lambda)$ if $\lambda(H_\alpha) = 1$ and if $\lambda(H_\alpha) \geq 2$ we have also the sequence

$$0 \longrightarrow L_{w_\alpha(\lambda)} \longrightarrow K \longrightarrow V_{\lambda-\alpha,\alpha} \longrightarrow 0 \tag{11.3}$$

Lemma 11.7. *Let $\tau: B \to GL(V)$ be a representation and let $\mu: B \to \mathbf{C}^*$ be a morphism (i.e. $\mu \in \Lambda_W$). If τ can be extended to $\tilde{\tau}: P_\alpha \to GL(V)$ and if $\mu(H_\alpha) = -1$ then*

$$H^i(G/B, E_\tau \otimes L_\mu) = 0 \quad \forall i$$

Proof Consider the projection $\pi: G/B \to G/P_\alpha$. By the assumptions E_τ is trivial on each fiber of π and L_μ is a line bundle of degree -1. By (11.1) and the fact that π is flat we get

$$R^i\pi_*(E_\tau \otimes L_\mu) = 0 \quad \forall i$$

Hence the result follows.

Proposition 11.8. *Let α be a simple root such that $(\alpha, \lambda+\delta) \geq 0$. We have the isomorphisms of G-modules*

$$H^i(G/B, L_\lambda) \simeq H^{i+1}(G/B, L_{w_\alpha(\lambda+\delta)-\delta}) \quad \forall i$$

Proof Write down (11.2) and (11.3) with $\lambda+\delta$ at the place of λ

$$0 \longrightarrow K \longrightarrow V_{\lambda+\delta,\alpha} \longrightarrow L_{\lambda+\delta} \longrightarrow 0 \tag{11.4a}$$

$$0 \longrightarrow L_{w_\alpha(\lambda+\delta)} \longrightarrow K \longrightarrow V_{\lambda+\delta-\alpha,\alpha} \longrightarrow 0 \tag{11.4b}$$

with obvious modifications if $(\lambda+\delta)(H_\alpha) = 0$ or 1. By tensoring (11.4) by $L_\delta^* = L_{-\delta}$ we get

$$0 \longrightarrow \mathcal{M} \longrightarrow V_{\lambda+\delta,\alpha} \otimes L_\delta^* \longrightarrow L_\lambda \longrightarrow 0 \tag{11.5a}$$

$$0 \longrightarrow L_{w_\alpha(\lambda+\delta)-\delta} \longrightarrow \mathcal{M} \longrightarrow V_{\lambda+\delta-\alpha,\alpha} \otimes L_\delta^* \longrightarrow 0 \tag{11.5b}$$

By the lemma 11.7 we have

$$H^i(G/B, V_{\lambda+\delta,\alpha} \otimes L_\delta^*) = 0 \quad \forall i$$

In fact $-\delta(H_\alpha) = -1$. In the same way

$$H^i(G/B, V_{\lambda+\delta-\alpha,\alpha} \otimes L_\delta^*) = 0 \quad \forall i$$

Then the result follows from the cohomology sequences associated to (11.5).

Lemma 11.9. *The Weyl group is generated by reflection with respect to simple roots.*

Proof [FuHa] D.27 pag. 493

Definition 11.10. *If $w \in W$ denote by $l(w)$ the minimum length of an expression of w as product of reflections with respect to simple roots.*

Corollary 11.11. *Let $\lambda \in \Lambda_W$, $\lambda+\delta \in \mathcal{C}$ (fundamental Weyl chamber). Then*

$$H^i(G/B, L_\lambda) \simeq H^{i+l(w)}(G/B, L_{w(\lambda+\delta)-\delta})$$

Proof Let

$$w = w_{\alpha_1} \circ \ldots \circ w_{\alpha_k}$$

Apply inductively the proposition 11.8. The only thing to observe is that the condition $(\alpha, \lambda+\delta) \geq 0$ is true at each step. In fact at the first step $(\alpha_k, \lambda+\delta) \geq 0$ because $\lambda+\delta \in \mathcal{C}$.

At the second step we have to check that

$$(\alpha_{k-1}, w_{\alpha_k}(\lambda+\delta)) \geq 0 \tag{11.6}$$

As the Killing form is invariant under the action of the Weyl group (11.6) is equivalent to

$$(w_{\alpha_k}(\alpha_{k-1}), \lambda+\delta) \geq 0$$

which it is true because w_{α_k} permutes all the positive roots different from α_k. By continuing in this way the corollary is proved.

Lemma 11.12. *If* $\lambda + \delta \in \mathcal{C}$ *then*

$$H^i(G/B, L_\lambda) = 0 \quad \text{for } i > 0$$

Proof By the Bruhat decomposition there exists an element $w \in W$ such that $l(w) = \dim G/B$ (see [FuHa] pag. 397). Then apply the corollary 11.11

We conclude now the proof of the theorem of Bott 11.4. In order to prove i) consider that by assumption $(\alpha, \lambda + \delta) = 0$ for any root α. There exists $w \in W$ such that

$$w(\lambda + \delta) \in \mathcal{C}$$

It follows

$$(w(\alpha), w(\lambda + \delta)) = 0$$

for any root α, then there exists a simple root β such that

$$(\beta, w(\lambda + \delta)) = 0$$

It follows

$$(\beta, w(\lambda + \delta) - \delta) = -1$$

By the lemma 11.7 (with τ trivial) we get

$$H^i(G/B, L_{w(\lambda+\delta)-\delta}) = 0 \quad \forall i$$

Apply the corollary 11.11 with $w(\lambda + \delta) - \delta$ at the place of λ and w^{-1} at the place of w, then

$$w^{-1}[w(\lambda + \delta) - \delta + \delta] - \delta = \lambda$$

and it follows

$$H^i(G/B, L_\lambda) = 0 \quad \forall i$$

as we wanted.

In order to prove ii) consider that by the lemma 11.12 we have

$$H^i(G/B, L_{w(\lambda+\delta)-\delta}) = 0 \quad \forall i > 0$$

Applying again the corollary 11.11 with $w(\lambda + \delta) - \delta$ at the place of λ and w^{-1} at the place of w, then

$$H^i(G/B, L_\lambda) = 0 \quad \forall i \neq l(w^{-1}) = l(w)$$

and

$$H^{l(w)}(G/B, L_\lambda) \simeq H^0(G/B, L_{w(\lambda+\delta)-\delta}) \simeq G_{w(\lambda+\delta)-\delta}$$

(the last isomorphism by the Borel-Weil theorem).

The proof of the theorem of Bott is so concluded, up to the check that $l(w)$ is exactly the index of regularity of $\lambda + \delta$. This form allows more handy computations. This fact can be checked by analyzing the euclidean structure of the space of the roots (e.g. the case of $G = SL(n)$ is a simple exercise) and we leave it to the reader.

Exercise. *If $Pic(G/P) = \mathbf{Z}$ we have a well defined line bundle $\mathcal{O}(1)$. Prove that*

$$H^i(G/P, \mathcal{O}(t)) = 0 \quad \forall t \in \mathbf{Z} \quad \forall i : 0 < i < \dim\ G/P$$

Varieties with this property are called arithmetically Cohen-Macaulay (this is equivalent to the homogeneous coordinate ring being Cohen-Macaulay)

§12. Stability of homogeneous bundles

Let $\alpha_1, \ldots, \alpha_n$ be a fundamental system of roots of a simple simply connected Lie group G and let $\lambda_1, \ldots, \lambda_n$ be the fundamental weights. Let $P = P(\Sigma) \subset G$ be a parabolic subgroup.

Lemma 12.1. *Let*

$$\lambda_i = \sum_j c^{ij} \alpha_j$$

(c^{ij} are the entries of the inverse of the Cartan matrix). Then

$$c^{ij} > 0 \quad \forall i, j > 0$$

Proof The inverse of C is computed explicitly in [Hum] page 69 but the computation is long. A direct proof is as follows.

Let

$$S_1^i = \{j \in [1, n] | c^{ij} < 0\}$$

$$S_2^i = \{j \in [1, n] | c^{ij} > 0\}$$

$$S_3^i = \{j \in [1, n] | c^{ij} = 0\}$$

$$\Lambda_1 = -\sum_{j \in S_1^i} c^{ij} \alpha_j$$

$$\Lambda_2 = \sum_{j \in S_2^i} c^{ij} \alpha_j$$

Then

$$\lambda_i = \Lambda_2 - \Lambda_1$$

By the lemma 6.42 we have $(\Lambda_1, \Lambda_2) \leq 0$. It follows

$$0 \leq (\lambda_i, \Lambda_1) = (\Lambda_2, \Lambda_1) - (\Lambda_1, \Lambda_1) \leq 0$$

Then

$$0 \geq (\Lambda_2, \Lambda_1) = (\Lambda_1, \Lambda_1) \geq 0$$

These inequalities imply that $(\Lambda_1, \Lambda_1) = 0$. It follows $\Lambda_1 = 0$ so that $S_1^i = \emptyset$.

If $k \in S_3^i$ we get

$$0 \leq (\lambda_i, \alpha_k) = \sum_{j \in S_2^i} c^{ij}(\alpha_j, \alpha_k) \leq 0$$

Therefore

$$(\alpha_j, \alpha_k) = 0 \quad \forall j \in S_2^i, k \in S_3^i$$

As G is simple we get $S_3^i = \emptyset$.

Theorem 12.2. *Let ρ be a irreducible representation of P with maximal weight λ. It follows*

$$\lambda = 0 \Leftrightarrow E_\rho = \mathcal{O} \Leftrightarrow H^0(G/P, E_\rho) = \mathbf{C}$$

$$\lambda = \sum p_i \alpha_i \text{ with } p_i > 0 \Leftrightarrow h^0(G/P, E_\rho) \geq 2$$

Proof Immediate from the Borel-Weil theorem and the lemma 12.1.

The reductive part of $P = P(\Sigma)$ is

$$\mathcal{H} \oplus \sum_{\alpha \in \Phi^+(\Sigma)} (g_\alpha \oplus g_{-\alpha})$$

Denote by Z the center of the reductive part, then we have

$$Lie\ Z = \{h \in \mathcal{H} | \alpha(h) = 0 \quad \forall \alpha \in \Phi^+(\Sigma)\} = \oplus_{\alpha \notin \Phi^+(\Sigma)} [\mathcal{G}_\alpha, \mathcal{G}_{-\alpha}] \tag{12.1}$$

Theorem 12.3(Ramanan). *Let ρ be a irreducible representation of P. Then E_ρ is a simple bundle.*

Proof Observe that $E_\rho \otimes E_\rho^* \simeq E_{\rho \otimes \rho^*}$. By the coroll. 3.26 $\rho \otimes \rho^*$ is trivial over Z. By the prop. 10.5 $\rho \otimes \rho^*$ is completely reducible. Let λ be the highest weight of a irreducible summand of $d(\rho \otimes \rho^*)_e$. We have

$$\lambda_{|Lie\ Z} = 0$$

By (12.1) λ is a linear combination of the simple roots in the complement of Σ. By the theorem 12.2 and the lemma 12.1 $h^0(E_\lambda) = 0$ only if $\lambda = 0$. Hence $H^0(E_\rho \otimes E_\rho^*) = \mathbf{C}^k$ where k is the number of times that the trivial representation appears as a direct summand in $\rho \otimes \rho^*$. The result follows by applying the prop. 3.27.

Theorem 12.4. *Let ρ_1, ρ_2 be two completely reducible representations of P. Then*

$$E_{\rho_1} \simeq E_{\rho_2} \quad \Leftrightarrow \quad \rho_1 \simeq \rho_2$$

Proof As in the proof of the theorem 12.3 $H^0(G/P, E_{\rho_1} \otimes E_{\rho_2}^*) = \mathbf{C}^r$ where r is the number of times that the trivial representation appears as a direct summand in $\rho_1 \otimes \rho_2^*$. By the prop. 3.27 and the theorem 12.3 it follows the result.

Definition 12.5. *Let H be an ample divisor over a variety X of dimension d. We set*

$$\mu_H(E) := \frac{c_1(E) \cdot H^{d-1}}{rk\ E}$$

μ_H is called the slope with respect to H.

The main property of the slope that we will use is the following. If

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

is an exact sequence of sheaves then

i) if $\mu_H(A) \neq \mu_H(C)$ then $\mu_H(B)$ is contained in the open interval with extremes $\mu_H(A)$ and $\mu_H(C)$.

ii) if $\mu_H(A) = \mu_H(C)$ then $\mu_H(A) = \mu_H(B) = \mu_H(C)$.

Definition 12.6. *A bundle E over X is called H-(semi)stable if for every subsheaf F such that $0 \subset_{\neq} F \subset_{\neq} E$ we have*

$$\mu_H(F) < \quad (\leq) \quad \mu_H(E)$$

Theorem 12.7 (Rohmfeld).

i) If a homogeneous bundle $E = E_\rho$ is not H-semistable then there exists a homogeneous subbundle F induced by a subrepresentation of ρ such that

$$\mu_H(F) > \mu_H(E)$$

Hence in order to check the semistability it is sufficient to check a finite number of inequalities involving the possible invariant subbundles for the given representation.

ii) Moreover F can be chosen to be a direct sum $\oplus F_i$ where all F_i are homogeneous H-stable bundles with the same slope and the same rank (not necessarily invariant for the given representation).

The theorem 12.7 can be reformulated in the following form

Criterion of stability 12.8.

i) If $\mu_H(F) \leq \mu_H(E_\rho)$ for any homogeneous subbundle invariant for ρ then E_ρ is H-semistable.

ii) If E_ρ is indecomposable and $\mu_H(F) < \mu_H(E_\rho)$ for any homogeneous subbundle invariant for ρ then E_ρ is H-stable.

Corollary 12.9 (Ramanan). *If ρ is irreducible then E_ρ is stable.*

Proof By the theorem 12.3 E_ρ is simple, then indecomposable. Then apply 12.8 ii).

In order to prove the theorem 12.7 we need some preliminary results.

Lemma 12.10. *Let $H_i = L_{\lambda_i}$ for $i = 1, \ldots r$ be the ample generators of $Pic(G/P)$. Let E be a bundle over G/P. There exists $N \in \mathbf{Z}$ such that if $H^0(E \otimes H_1^{n_1} \otimes \ldots \otimes H_r^{n_r}) \neq 0$ then $n_i \geq N \quad \forall i$.*

Proof By the theorem A of Serre there exists m such that $E \otimes H_1^m \otimes \ldots \otimes H_r^m$ is a subbundle of the trivial bundle $\mathcal{O} \otimes \mathbf{C}^k$ for some k. Then the statement is trivial.

Lemma 12.11. *Let E be a bundle over G/P. The set $\{\mu_H(F) | F$ is a subsheaf of $E\}$ admits a maximum μ_0.*

Proof If $F \subset E$ is a subsheaf of rank f then $(\wedge^f F)^{**}$ is a line subbundle of $\wedge^f E$ and $\wedge^f E(-c_1(F))$ has a section. The result follows by the lemma 12.10 applied to $\wedge^f E$ for $1 \leq f \leq rk\ E$.

Lemma 12.12. *Let E_ρ be a homogeneous bundle over G/P. Let $\mathcal{A} = \{F \subset E_\rho | \mu(F) = \mu_0$ and the rank of F is minimum$\}$. Then the elements of A are finitely many and homogeneous and we get*

$$\oplus_{F \in \mathcal{A}} F \subset E$$

The bundle $\oplus_{F \in \mathcal{A}} F$ is invariant for ρ.

Proof Let $F_1 \in \mathcal{A}$ and $F_2 \subset E_\rho$ such that $\mu(F_2) = \mu_0$. We claim that $F_1 \cap F_2 = 0$ or $F_1 \subset F_2$. In fact if $0 \neq F_1 \cap F_2 \neq F_1$ then

$$\mu_H(F_1 \cap F_2) < \mu_H(F_1) < \mu_H((F_1 + F_2)/F_2) \tag{12.2}$$

Moreover $\mu_H(F_2) \geq \mu_H(F_1 + F_2)$ so that from the sequence

$$0 \longrightarrow F_2 \longrightarrow F_1 + F_2 \longrightarrow (F_1 + F_2)/F_2 \longrightarrow 0$$

it follows

$$\mu_H(F_2) \geq \mu_H(F_1 + F_2) \geq \mu_H((F_1 + F_2)/F_2) \tag{12.3}$$

Putting together (12.2) and (12.3) it follows

$$\mu_H(F_1) < \mu_H(F_2)$$

which is a contradiction.

In particular if $F_1, F_2 \in \mathcal{A}$ then $F_1 \cap F_2 = 0$ or $F_1 = F_2$. Therefore $\oplus_{F \in \mathcal{A}} F$ is really a finite direct sum and has to be ρ-invariant by its definition. Hence $\oplus_{F \in \mathcal{A}} F$ is a vector bundle so that any $F \in \mathcal{A}$ is a vector bundle. In order to prove that every $F \in \mathcal{A}$ is homogeneous divide $\mathcal{A}$ into classes of isomorphic bundles. G is connected and acts trivially on the set of these classes. It follows the lemma.

End of the proof of the theorem 12.7

If E is not semistable then $\mu_0 > \mu_H(E)$. The bundle $\oplus_{F \in \mathcal{A}} F$ defined in the lemma 12.12 has slope μ_0.

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