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FOURIER-MUKAI TRANSFORMS, GENERIC VANISHING, AND REGULARITY

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Introduction. The Fourier-Mukai transform has become in recent years an ubiquitous tool in various branches of algebraic-geometry. From the point of view adopted here, it can best be described as the sheaf-theoretic analogue of a correspondence in cycle theory: it describes the hypercohomology of complexes on a scheme X, when tensored with sheaves varying over a parameter space Y.

On a different note, the study of generic vanishing for certain cohomology groups of line bundles, starting with the by-now classical Green-Lazarsfeld Generic Vanishing theorem ([GL1], [GL2], [EL]), is already well-known to be a crucial tool for investigating geometric problems on irregular varieties. Vanishing theorems of this sort were first proved using Hodge-theoretic methods.

It turns out that the Green-Lazarsfeld Generic Vanishing acquires a new perspective in a Fourier-Mukai setting provided by the Albanese map, where it eventually appears as a natural application of the Kodaira-Kawamata-Viehweg vanishing theorem, and extension of the Grauert-Riemenschneider vanishing theorem. This point of view was first highlighted by Hacon in [Hac], and subsequently extended by the speakers at this school in [PP5] to include a study of vanishing conditions for sheaves with respect to arbitrary Fourier-Mukai functors. In this framework, the generic vanishing of (hyper-)cohomology groups appears as the natural dual of another important notion, namely the Weak Index Theorem property.

An example of the interplay between the general notion of generic vanishing of hypercohomology and concrete geometric situations is the notion of M-regularity, introduced in [PP1] for sheaves on abelian varieties. It turns out that this notion is a strong form of the Generic Vanishing property, the explicit link between the two being provided by commutative algebra. M-regularity is implies various global generation properties of sheaves, in analogy with Castelnuovo-Mumford regularity for sheaves on projective spaces. This leads to several applications to the study of special subvarieties and linear series on abelian varieties, and also on more general irregular varieties and on some moduli spaces of vector bundles.

School topics. The aim of the course will be to provide an introduction to the use of Fourier-Mukai transforms in the study of irregular varieties and moduli spaces from a higher dimensional geometry point of view, and also to their use in specific geometric problems related to abelian varieties, for example the study of their defining equations, or detecting Jacobians. The lectures and research problems will roughly focus on the following subjects:

(1) Generalities on derived categories, Base-Change, and Fourier-Mukai functors ([Mu1], [Mu2], [Mum], [Hu]).

(2) Generalities on multiplier ideals ([La]).

(3) Sheaves satisfying Generic Vanishing and the Weak Index Theorem with respect to an arbitrary Fourier-Mukai functor ([PP5]).

(4) Generic vanishing theorems for irregular varieties and their relationship with standard vanishing theorems like Kawamata-Viehweg and its extensions ([GL1], [GL2], [Hac], [PP5], [La], [Ko1], [Ko2],[EV]).

(5) Generic vanishing index and the Fourier-Mukai transform as a k-th syzygy sheaf ([PP7], [EG]).

(5) M-regularity, generation of sheaves, and applications to abelian varieties ([Pa], [PP1], [PP2], [PP3], [PP7], [De]).

(6) Approaches to the geometric Schottky problem based on regularity and generic vanishing ([PP4], [PP6]).

Background for research problems. A number of research problems will be proposed to the participants at this school, according to their respective backgrounds. Some might fall outside of the topics listed above. As general prerequisites, we recommend familiarity with basic algebraic geometry as in [Ha], with abelian varieties, as in the introductory chapters in [Mum] and [BL], and with basic homological methods leading to the Serre-Grothendieck-Verdier duality theorem (for example the first three chapters in [Hu] cover this).

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