Geometry of Projective Varieties
Notes for PRAGMATIC 2006

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.... e com cinco ou seis retas é facil fazer um castelo

a Cledvane, a Giulia e a Luca

Preface

The aim of these notes is to furnish an introduction to some classical and recent results and techniques in projective algebraic geometry. We treat the geometrical properties of varieties embedded in projective space, their secant and tangent lines, the behaviour of tangent linear spaces, the algebro-geometric and topological obstructions to their embedding into smaller projective spaces, the classification in the extremal cases.

These are classical themes in algebraic geometry and the renewed interest at the beginning of the '80 of the last century came from some conjectures posed by Hartshorne, [**H2**], from an important connectedness theorem of Fulton and Hansen, [**FH**], and from its new and deep applications to the geometry of algebraic varieties, as shown by Fulton, Hansen, Deligne, Lazarsfeld and Zak, [**FH**], [**FL**], [**D2**], [**Z2**]. We shall try to illustrate these themes and results during the course and with more details through these notes.

There exists no introductory text on secant, tangent, dual varieties, Terracini Lemma, etc, and moreover, quite surprisingly, these notions are not well known today. Thus we were forced to recall their constructions at the beginning of the text and to prove their first properties. A more advanced reference on some topics presented here is [**Z2**], which influenced the presentation of many parts; these notes could be thought also as a natural preparation to portions of the above referred book.

Finally I apologize for the absence, only in the notes, of any figure as it should be the case in a text on geometry. It depends on my well known incompatibility with a normal use of this modern technology. I felt enough satisfied producing a document with an (automatic) index.

Ringraziamenti

Queste note sono basate su una versione preliminare scritta, fortunatamente a mano, durante il periodo passato all' Universitá di Roma Tor Vergata nell' ultimo trimestre 2002, insegnando un corso sull' argomento. Ringrazio Ciro (Ciliberto) per avermi invitato, per la splendida ospitalitá e per le innumerevoli e piacevoli discussioni, matematiche e non (e inoltre per quelle precedenti e successive); tutti i partecipanti per l' impegno e lo stimolo fornito; l' I.N.D.A.M. per aver finanziato il soggiorno e il viaggio.

Il CNPq e il PRONEX-Algebra Comutativa e Geometria Algebrica hanno finanziato negli ultimi cinque anni vari miei progetti di ricerca su questi argomenti, sia come borsista, sia con fondi diretti e esprimo qui la mia gratitudine per la fiducia concessa, spero almeno parzialmente ricambiata.

Sono riconoscente a Eduardo Esteves e agli organizzatori per avere avuto l' iniziativa di invitarmi ad offrire questo corso al 24° Coloquio Brasileiro de Matemática. Senza la loro richiesta non avrei mai intrapreso questo progetto.

Un ringraziamento incommensurabile a Cledvane per l' amore, la pazienza e l' incoraggiamento costanti e senza limiti, rafforzati ulteriormente in questo periodo in cui, con la scusa di scrivere queste note, mi sono frequentemente sottratto ad aiutarla nel difficile compito di convincere la nostra piccola Giulia che in certi orari sia meglio dormire. La nascita di Giulia, fonte di immensa allegria e continua felicitá, mi permette di scusarmi elegantemente con gli eventuali lettori per non avere avuto il tempo necessario per sottoporre il testo a una seria revisione, o almeno a una attenta rilettura.

Quanto scritto in queste note é frutto di quanto appreso da vari amici, colleghi e professori e in molte occasioni dai "classici", antichi e moderni. Innanzitutto dal mio amico e maestro Lucian (Bădescu), che per primo mi ha sapientemente incamminato verso questi temi sin dai tempi in cui ero studente attraverso gli illuminanti corsi [**B1**] e [**B2**], a cui in anteprima assistí in compagnia degli amici del Seminarul de Geometrie del giovedí . Quanto mi ha insegnato, non solo matematicamente, é tutt' oggi importantissimo per me e va ben oltre quanto queste parole possano esprimere.

Un ringraziamento particolare a Aron (Simis) e Fyodor (Zak). Il primo per avermi mostrato come l' algebra possa, talvolta, aiutare a comprendere meglio la geometria e per le innumerovoli discussioni sui piu' svariati temi. Il secondo per il suo grande spirito critico, per la sua visione dell' universo geometrico, che, sebbene qualche volta non condivisa, mi ha sempre influenzato; e anche per avermi spiegato e suggerito vari problemi molto interessanti.

Negli ultimi anni gli amici Alberto (Alzati), Ciro (Ciliberto) e Massimiliano (Mella) hanno avuto un ruolo fondamentale sia per l' amicizia sempre mostrata, sia per quanto mi hanno insegnato nelle nostre frequenti comunicazioni, non solo matematiche, e nei, purtroppo, piu' rari incontri. Queste frasi non potranno mai esprimere quanto sento. Senza il loro aiuto non avrei compiuto nessun passo, caso mai ne avessi fatto uno, nel tentativo di comprendere meglio una piccolissima parte dell' affascinante mondo proiettivo.

Recife, 30 agosto 2006 Francesco Russo

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Introduction

After the period in which new and solid foundations to the principles of algebraic geometry were rebuilt especially by Zariski, Grothendieck and their schools, at the beginning of the '70 a new trend began. There was a renewed interest in solving concrete problems and in finding applications of the new methods and ideas. One can consult the beautiful book of Robin Hartshorne *Ample subvarieties of algebraic varieties*, [**H1**], to have a picture of that situation. In [**H1**] many outstanding questions, such as the set-theoretic complete intersection of curves in $\mathbb{P}^3$ (still open), the characterization of $\mathbb{P}^N$ among the smooth varieties with ample tangent bundle (solved by Mori in [**Mo1**] and which cleared the path to the foundation of Mori theory, [**Mo2**]) were discussed and/or stated and a lot of other problems solved. In related fields we only mention Deligne proof of the Weil conjectures or, later, Faltings proof of the Mordell conjecture, which used the new machinery.

Let us quote a part of Zak's introduction to his fascinating book [**Z2**]: "*Among recent achievements in the field of multidimensional projective geometry we mention results of Hironaka, Matsumura, Ogus, and Hartshorne on formal neighborhoods and local cohomology, theorems of Barth, Goresky, and MacPherson on the topology of projective varieties, classification of Fano varieties given by Iskovskih, Mori, and others, and various versions of Schubert's enumerative geometry. One of the most important results of the last decade is the connectedness theorem of Fulton and Hansen* (cf. [**FH**], [**FL**])".

The interplay between topology and algebraic geometry returned to flourish. Lefschetz theorem and Barth-Larsen theorem, see subsection 2.1.1, also suggested that smooth varieties, whose codimension is small with respect to their dimension, should have very strong restrictions both topological, both geometrical. For example a codimension 2 smooth complex subvariety of $\mathbb{P}^N$, $N \geq 5$, has to be simply connected. If $N \geq 6$, there are no known examples of codimension 2 smooth varieties with the exception of the trivial ones, the complete intersection of two hypersurfaces, i.e. the transversal intersection of two hypersurfaces, smooth along the subvariety. In fact, at least for the moment, one is able to construct only these kinds of varieties whose codimension is sufficiently small with respect to dimension.

Based on these empirical observations and, according to Fulton and Lazarsfeld, "on the basis of few examples", Hartshorne was led to formulate two conjectures in 1974, [**H2**]. The first one is the following.

"*Let $X \subset \mathbb{P}^N$ be a smooth irreducible non-degenerate projective variety.*

If $N < \frac{3}{2}\dim(X)$, *i.e. if* $\operatorname{codim}(X) < \frac{1}{2}\dim(X)$*, then X is a complete intersection.*"

Let us quote Hartshorne: *While I am not convicted of the truth of this statement, I think it is useful to crystallize one's idea, and to have a particular problem in mind* ([**H2**]). The conjecture is sharp as the example of $\mathbb{G}(1,4) \subset \mathbb{P}^9$ shows.

It is not here the place to remark how many important results originated and still today arise from this open problem in the areas of vector bundles on projective space, of the study of defining equations of a variety and k-normality and so on. The list of these achievements is too long that we preferred to avoid citations, being confident that everyone has met sometimes a problem or a result related to it. It is quite embarrassing that the powerful methods of modern algebraic geometry did not yet produced a solution (or a counterexample).

The second problem posed by Hartshorne, also suggested by the fact that complete intersections are linearly normal and by some examples in low dimension, is the following. We recall that a nonsingular variety $X \subset \mathbb{P}^N$ is called *linearly normal* if $h^0(X, \mathcal{O}_X(1)) = N+1$, i.e., there is no $X' \subset \mathbb{P}^{N'}$, $N' > N$, such that X' is not contained in a hyperplane and can be isomorphically projected onto X.

"Let $X \subset \mathbb{P}^N$ be a smooth irreducible non-degenerate projective variety.

$$\textit{If } N < \frac{3}{2}\dim(X)+1, \textit{ i.e. if } \operatorname{codim}(X) < \frac{1}{2}\dim(X)+1, \textit{ then } X \textit{ is linearly normal."}$$

Let us quote once again Hartshorne point of view on this second problem: *Of course in settling this conjecture, it would be nice also to classify all nonlinearly normal varieties with $N = \frac{3}{2}n + 1$, so as to have a satisfactory generalization of Severi's theorem. As noted above, a complete intersection is always linearly normal, so this conjecture would be a consequence of our original conjecture, except for the case $N = \frac{3n}{2}$. My feeling is that this conjecture should be easier to establish than the original one* ([**H2**]). Once again the bound is sharp taking into account the example of the projected Veronese surface in $\mathbb{P}^4$.

The conjecture on linear normality was proved by Zak at the beginning of the '80's and till now it is the major evidence for the possible truth of the complete intersection conjecture. Moreover, Zak classified all the extremal cases showing that there are only 4 varieties analogous to the Veronese surface in $\mathbb{P}^4$, see chapter 3. These varieties were dubbed *Severi varieties* in honour of Francesco Severi, who first established the case $n = 2$ in [**Sev1**].

Many theorems in classical projective geometry deal with "general" objects, as the Bertini theorem on hyperplane sections, see theorem 1.5.2 here. A more refined version of this theorem says that if $f : X \to \mathbb{P}^N$ is morphism, with X proper and such that $\dim(f(X)) \geq 2$, and if $H = \mathbb{P}^{N-1} \subset \mathbb{P}^N$ is a general hyperplane, then $f^{-1}(H)$ is irreducible. The "Enriques-Zariski principle" says that "limits of connected varieties remain connected" and it is illustrated in the previous example because for an arbitrary $H = \mathbb{P}^{N-1} \subset \mathbb{P}^N$, one proves that $f^{-1}(H)$ is connected.

This last result is particularly interesting because, as shown by Deligne and Jouanolou, a small generalization of it proved by Grothendieck, [**Gr**] XIII 2.3, yields a simplified proof of a beautiful and interesting connectedness theorem of Fulton and Hansen in [**FH**], whose applications are deep and appear in different areas of algebraic geometry and topology as we survey in chapter 2. One of the most important is Zak's *theorem on tangencies*. In the simplest situation this theorem is formulated as follows. Let $X \subset \mathbb{P}^N$ be an irreducible n-dimensional projective variety over an algebraically closed field K that is not contained in any hyperplane, and let L be an m-dimensional linear subspace of $\mathbb{P}^N$ that is tangent to X along an r-dimensional subvariety $Y \subset X$ (this means that all (embedded) tangent spaces to X at the points of Y are contained in L, so that, in particular, $m \geq n$). Then $r \leq m - n$, see chapter 2.

In particular the classical theorem of Bertini can be improved and new statements appear; for example, for a nonsingular variety of dimension n, $X \subset \mathbb{P}^N$, each hyperplane section is reduced for $N < 2n$ and is normal for $N < 2n - 1$. Other applications furnished by Zak lead to the solution of many classical problems such as the finiteness of the Gauss map for smooth varieties or the fact that the dimension of the dual variety $X^* \subset \mathbb{P}^{N*}$ is not less than the dimension of $X \subset \mathbb{P}^N$.

The problems and results we exposed above and which are contained in these notes are examples of the themes treated in projective geometry. This means that we fix a variety, its embedding and we are studying the properties of this variety or of its projections onto smaller dimensional spaces. Thus only the different incarnations of the same variety embedded by a fixed very ample line bundle are studied, by considering various sublinear system of the complete one realizing it in projective spaces of different dimension. The existence of isomorphic projections onto smaller projective spaces translates into strong properties of the linearly normal embedded variety.

Let us quote excerpts from Hilbert presentation of projective geometry in [**HCV**]:

"*..... we shall learn about geometrical facts that can be formulated and proved without any measurement or comparison of distances or of angles. It might be imagined that no significant properties of a figure could be found if we do without measurement of distances and angles and that only vague statements could be made. And indeed research was confined to the metrical side of geometry far a long time, and questions of the kind we shall discuss in this chapter arose only later, when the phenomena underlying perspective painting were being studied scientifically. Thus, if a plane figure is projected from a point onto another plane, distances and angles are changed, and in addition, parallel lines may be changed into lines that are not parallel; but certain essential properties must nevertheless remain intact, since we could not otherwise recognize the projection as being a true picture of the original figure. In this way, the process of projecting led to a new theory, which was called projective geometry because of its origins. Since the 19th century, projective geometry has occupied a central position in geometric research. With the introduction of homogeneous coordinates, it became possible to reduce the theorems of projective geometry to algebraic equations in much the same way that Cartesian coordinates allow this to be done for the theorems of metric geometry. But projective analytic geometry is distinguished by the fact that it is far more symmetrical and general than metric analytic geometry, and when one wishes, conversely, to interpret higher algebraic relations geometrically, one often transforms the relations into homogeneous form and interprets the variables as homogeneous coordinates, because the metric interpretation in Cartesian coordinates would be too unwieldy.*"

Varieties which could be projected isomorphically to projective spaces of lower dimension such that their codimension became *small*, are very special. First of all the projected manifold is not a complete intersection, being not linearly normal, so that the principles cited above say that near the bound there should be very few examples, satisfying strong restrictions and, at least experimentally, they are very few and could be classified; for examples most of them are homogeneous. To study projections one naturally deals with secant and tangent lines to the variety and with the varieties described by these lines in the ambient space.

We recall that a nonsingular variety $X \subset \mathbb{P}^N$ can be isomorphically projected to $\mathbb{P}^{N-1}$ if and only if $SX \neq \mathbb{P}^N$, where SX is the secant variety of X, i.e., the closure of the union of chords joining pairs of distinct points of X. Thus, the minimal number m such that X can be isomorphically projected to $\mathbb{P}^m$ is equal to the dimension of the variety SX. The relationship between embedded tangent spaces to X and SX is given by *Terracini lemma*, see chapter 1.

If $z \in SX$, $z \in < x, y >$, where $x, y \in X$ and $< x, y >$ is the chord joining x with y, then the (embedded) tangent space $T_z SX$ contains the (embedded) tangent spaces $T_x X$ and $T_y X$. Moreover, if the ground field has characteristic zero and z is a general point of SX, then

$$T_z SX = < T_x X, T_y X > .$$

From this it follows that if X can be isomorphically projected to $\mathbb{P}^m$, then for each pair of points $x, y \in X$ there exists an m-dimensional linear subspace of $\mathbb{P}^N$ which is tangent to X at the points x and y (if the characteristic of the ground field is equal to zero, then the converse is also true).

Along with the secant variety SX one can consider higher secant varieties $S^k X$, $k \geq 1$, where $S^k X$ is the c1osure of the union of k-dimensional linear subspaces spanned by generic collections of $k+1$ points of X. Zak established a connection between geometric characteristics of the varieties $S^k X$ for various k. In particular, for an arbitrary nonsingular variety $X \subset \mathbb{P}^N$ such that $< X >= \mathbb{P}^N$ he proved that

$$S^{[\frac{n}{\delta}]} X = \mathbb{P}^N,$$

where

$$\delta = \delta(X) = 2n + 1 - \dim(SX)$$

and $[\frac{n}{\delta}]$ is the largest integer not exceeding $\frac{n}{\delta}$, see chapter 4. This yields a bound for the maximal (for given n and $r \leq 2n$) number N for which there exist a variety $X \subset \mathbb{P}^N$ that can be isomorphically projected to $\mathbb{P}^r$. This bound is sharp; the varieties for which it is attained are called *Scorza varieties*, in honor of Gaetano Scorza (Severi varieties are special cases of Scorza varieties for $\delta = \frac{n}{2}$). Zak obtained a complete classification of Scorza varieties, viz., there exist three series of such varieties and one special sixteen-dimensional variety, see chapter 4. In other words, for a smooth variety $X \subset \mathbb{P}^r$ such that $\mathrm{codim}(X) \leq \dim(X)$ there exists a sharp bound for $h^0((X, \mathcal{O}_X(1)))$ in terms of $\dim(X)$ and r and Zak classified all varieties for which this bound is attained.

Scorza and Terracini together with classical algebraic Geometers, antique and modern, taught and teach to us also to experiment the "live rapport" with "the objects one studies" and showed us the "concrete intuition", described by Hilbert in his preface to the book "Geometry and the Imagination" [**HCV**]:

"*In mathematics, as in any scientific research, we find two tendencies present. On the one hand, the tendency toward abstraction seeks to crystallize the logical relations inherent in the maze of material that is being studied, and to correlate the material in a systematic and orderly manner. On the other hand, the tendency toward intuitive understanding fosters a more immediate grasp of the objects one studies, a live rapport with them, so to speak, which stresses the concrete meaning* *of their relations. As to geometry, in particular, the abstract tendency has here led to the magnificent systematic theories of Algebraic Geometry, of Riemannian Geometry, and of Topology; these theories make extensive use of abstract reasoning and symbolic calculation in the sense of algebra. Notwithstanding this, it is still as true today as it ever was that intuitive understanding plays a major role in geometry. And such concrete intuition is of great value not only far the research worker, but also far anyone who wishes to study and appreciate the results of research in geometry. In this book, it is our purpose to give a presentation of geometry, as it stands today, in its visual, intuitive aspects. With the aid of visual imagination we can illuminate the manifold facts and problems of geometry, and beyond this, it is possible in many cases to depict the geometric outline of the methods of investigation and proof, without necessarily entering into the details connected with the strict definitions of concepts and with the actual calculations.*

In this manner, geometry being as many-faceted as it is and being related to the most diverse branches of mathematics, we may even obtain a summarizing survey of mathematics as a whole, and a valid idea of the variety of its problems and the wealth of ideas it contains."

CHAPTER 1

Tangent cones, tangent spaces, tangent stars; secant, tangent and tangent star variety to an algebraic variety

1.1. Tangent cones to an algebraic variety and associated varieties

Let X be an algebraic variety, or more generally a scheme of finite type, over a fixed algebraically closed field K. Let $x \in X$ be a closed point. We briefly recall the definitions of *tangent cone to X at x* and of *tangent space to X at x*. For more details one can consult [**Mu**] or [**Sh**].

1.1.1. Definition. (**Tangent cone at a point**). Let $U \subset X$ be an open neighbourhood of x, let $i : U \to \mathbb{A}^N$ be a closed immersion and let U be defined by the ideal $I \subset K[X_1, \ldots, X_N]$. There is no loss of generality in supposing $i(x) = (0, \ldots, 0) \in \mathbb{A}^N$. Given $f \in K[X_1, \ldots, X_N]$ with $f(0, \ldots, 0) = 0$, we can define the "leading form" of f, f^*, as the lower degree homogeneous polynomial in its expression as a sum of homogenous polynomials in the variables X_i's. Let

$$I^* = \{\text{the ideal generated by the "leading form"} f^*, \text{ for all } f \in I\}.$$

Then

$$\mathcal{C}_x X := \operatorname{Spec}(K[X_1, \ldots, X_N]/I^*),$$

is called *the affine tangent cone to X at x*.

It could seem that it depends on the choice of $U \subset X$ and on the choice of $i : U \to \mathbb{A}^N$. It is not the case because if $(\mathcal{O}_x, m_x)$ is the local ring of regular functions of X at x, then it is immediate to see that

$$(k[X_1, \ldots, X_N]/I^*) \simeq \operatorname{gr}(\mathcal{O}_x) := \bigoplus_{n \geq 0} \frac{m_x^n}{m_x^{n+1}}.$$

This fact simply says that we can calculate $\mathcal{C}_x X$ by choosing an arbitrary set of generators of I and moreover that the definition is "local". It should be noticed that $\mathcal{C}_x X$ is a scheme, which can be neither irreducible nor reduced as the examples of plane cubic curves with a node and with a cusp show. We now get a geometrical interpretation of this cone and see some of its properties.

Since $\mathcal{C}_x X$ is "locally" defined by homogeneous forms, it can be naturally projectivized and thought as a subscheme of $\mathbb{P}^{N-1} = \mathbb{P}(\mathbb{A}^N)$. If we consider the blow-up of $x \in U \subset \mathbb{A}^N$, $\pi : Bl_x U \to U$, then $Bl_x U$ is naturally a subscheme of $U \times \mathbb{P}^{N-1} \subset \mathbb{A}^N \times \mathbb{P}^{N-1}$ and the exceptional divisor $E := \pi^{-1}(x)$ is naturally a subscheme of $x \times \mathbb{P}^{N-1}$. With these identifications one shows that $E \simeq \mathbb{P}(\mathcal{C}_x X) \subset \mathbb{P}^{N-1}$ as schemes, see [**Mu**], pg. 225. In particular, if X is equidimensional at x, then $\mathcal{C}_x X$ is an equidimensional scheme of dimension $\dim(X)$. Moreover, we deduce the following geometrical definition:

$$\mathcal{C}_x X = \overline{\bigcup_{y \in U} \lim_{y \to x} < y, x >}.$$

The cone $\mathcal{C}_x X$ can also be described geometrically in this way, see [**Sh**]. Let notation be as in the affine setting above and set

$$m = \min\{\mathrm{mult}_x(l \cap V(f)),\ l \text{ line through } x\ ,\ f \in I\}.$$

Then $\mathcal{C}_x X$ is swept out locally by the lines l through x such that $\mathrm{mult}_x(l \cap V(f)) > m$.

If $X \subset \mathbb{P}^N$ is quasi-projective, we define *the projective tangent cone to X at x*, indicated by $C_x X$, as the closure of $\mathcal{C}_x X \subset \mathbb{A}^N$ in $\mathbb{P}^N$, where $x \in U = \mathbb{A}^N \cap X$ is a suitable chosen affine neighbourhood. The same geometrical definition holds, remembering of the scheme structure,

$$C_x X = \overline{\bigcup_{y \in U} \lim_{y \to x} < y, x >} \subset \mathbb{P}^N.$$

We now recall the definition of tangent space to X at $x \in X$.

1.1.2. Definition. (**Tangent space at a point; Tangent variety to a variety**). Let notation be as in the previous definition. Given $f \in K[X_1, \ldots, X_N]$ with $f(0, \ldots, 0) = 0$, we can define the "linear term" of f, f^{lin}, as the degree one homogeneous polynomial in its expression as a sum of homogenous polynomials in the variables X_i's. In other words, $f^{lin} = \sum_{i=1}^N \frac{\partial f}{\partial X_i}(\mathbf{0}) X_i$. Let

$$I^{lin} = \{\text{the ideal generated by the "linear terms"} f^{lin},\ \text{for all } f \in I\}.$$

Then

$$\mathcal{T}_x X := \mathrm{Spec}(K[X_1, \ldots, X_N]/I^{lin}),$$

is called *the affine tangent space to X at x*.

Geometrically it is the locus of *tangent lines to X at x*, where a line through x is tangent to X at x if it is tangent to the hypersurfaces $V(f) = 0$, $f \in I$, i.e. if the multiplicity of intersection of the line with $V(f)$ at $(0, \ldots, 0)$ is greater than one. In particular this locus is a linear subspace of $\mathbb{A}^N$.

Since $I^{lin} \subseteq I^*$, we deduce the inclusion as schemes

$$\mathcal{C}_x X \subseteq \mathcal{T}_x X;$$

and that $\mathcal{T}_x X$ is the smallest *linear* subscheme of $\mathbb{A}^N$ containing $\mathcal{C}_x X$ as a subscheme (and not only as a set!). In particular for every $x \in X$ it holds $\dim(\mathcal{T}_x X) \geq \dim(X)$.

We recall that a point $x \in X$ is non-singular if and only $\mathcal{C}_x X = \mathcal{T}_x X$. Since $\mathcal{T}_x X$ is reduced and irreducible and since $\mathcal{C}_x X$ is of dimension $\dim(X)$, we have that $x \in X$ is non-singular if and only if $\dim(\mathcal{T}_x X) = \dim(X)$.

Once again there is an intrinsic definition of $\mathcal{T}_x X$

$$(K[X_1, \ldots, X_N]/I^{lin}) \simeq S(m_x/m_x^2),$$

where $S(m_x/m_x^2)$ is the symmetric algebra of the K-vector space m_x/m_x^2.

If $X \subset \mathbb{P}^N$ is a quasi-projective variety, we define *the projective tangent space to X at x*, indicated by $T_x X$, as the closure of $\mathcal{T}_x X \subset \mathbb{A}^N$ in $\mathbb{P}^N$, where $x \in U = \mathbb{A}^N \cap X$ is a suitable chosen affine neighbourhood. Then $T_x X$ is a linear projective space naturally attached to X and clearly $C_xX \subseteq T_xX$ as schemes. We also set, for a (quasi)-projective variety $X \subset \mathbb{P}^N$,

$$TX = \bigcup_{x \in X} T_xX,$$

the variety of tangents, or *the tangent variety of* X.

At a non-singular point $x \in X \subset \mathbb{P}^N$, the equality $C_xX = T_xX$ says that every tangent line to X at x is "limit" of a secant line $< x, y >$ with $y \in X$ approaching x. For singular points this is not the case as one sees in the simplest examples of singular points of an hypersurface.

An interesting question is to investigate what are the limits of secant lines $< y_1, y_2 >$, $y_i \in X$, $y_1 \neq y_2$, when y_i, $i = 1, 2$, approaches a fixed $x \in X$. As we will immediately see for a non-singular point $x \in X$, every tangent line to X at x arises in this ways but for singular points this is not the case. These limits generate a cone, the tangent star cone to X at x, which contains but does not usually coincide with C_xX (or $\mathcal{C}_xX$). From now on we restrict ourselves to the projective setting since we will not treat local questions related to tangent star cones but the situation can be "localized". Firstly we introduce the notion of secant variety to a variety $X \subset \mathbb{P}^N$.

1.1.3. Definition. (**Secant varieties to a variety**). For simplicity let us suppose that $X \subset \mathbb{P}^N$ is a closed irreducible subvariety.

Let

$$S^0_X := \{((x_1, x_2), z) \ : \ z \in < x_1, x_2 >\} \subset (X \times X \backslash \Delta_X) \times \mathbb{P}^N.$$

The set is locally closed so that taken with the reduced scheme structure it is a quasi-projective irreducible variety of dimension $\dim(S^0_X) = 2\dim(X) + 1$. Recall that, by definition, it is a $\mathbb{P}^1$-bundle over $X \times X \backslash \Delta_X$, which is irreducible. Let S_X be its closure in $X \times X \times \mathbb{P}^N$. Then S_X is an irreducible projective variety of dimension $2\dim(X) + 1$, called *the abstract secant variety to* X. Let us consider the projections of S_X onto the factors $X \times X$ and $\mathbb{P}^N$,

$$\begin{tikzcd} & S_X \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ X \times X & & \mathbb{P}^N. \end{tikzcd}$$

The secant variety to X, SX, is the scheme-theoretic image of S_X in $\mathbb{P}^N$, i.e.

$$SX = p_2(S_X) = \overline{\bigcup_{x_1 \neq x_2\,,\, x_i \in X} < x_1, x_2 >} \subseteq \mathbb{P}^N,$$

which is an irreducible algebraic variety of dimension $s(X) \leq 2\dim(X) + 1$, the variety swept out by the secant lines to X. If equality (does not) holds, then X is said to be (*defective*) *non-defective*.

Let now $k \geq 1$ be a fixed integer. We can generalize the construction to the case of $(k+1)$-secant $\mathbb{P}^k$, i.e. to the variety swept out by the linear spaces generated by $k + 1$ independent points on X.

Define

$$(S^k_X)^0 \subset \underbrace{X \times \ldots \times X}_{k+1} \times \mathbb{P}^N$$

as the locally closed irreducible set

$$(S^k_X)^0 := \{((x_0, \ldots, x_k), z) \ : \ \dim(< x_0, \ldots, x_k >) = k \ , \ z \in < x_0, \ldots, x_k >\}.$$

Let S^k_X, *the abstract k-secant variety of X*, be

$$\overline{(S^k_X)^0} \subset \underbrace{X \times \ldots \times X}_{k+1} \times \mathbb{P}^N.$$

The closed set S^k_X is irreducible and of dimension $(k+1)\dim(X)+k$. Consider the projections of S^k_X onto the factors $\underbrace{X \times \ldots \times X}_{k+1}$ and $\mathbb{P}^N$,

$$\begin{tikzcd} & S^k_X \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ X \times \ldots \times X & & \mathbb{P}^N. \end{tikzcd}$$

The k-secant variety to X, S^kX, is the scheme-theoretic image of S^k_X in $\mathbb{P}^N$, i.e.

$$S^kX = p_2(S^k_X) = \overline{\bigcup_{x_i \in X \, , \, \dim(<x_0,\ldots,x_k>)=k} < x_0, \ldots, x_k >} \subseteq \mathbb{P}^N.$$

It is an irreducible algebraic variety of dimension $s_k(X) \leq (k+1)\dim(X)+k$. If equality (does not) holds, then X is said to be (*k-defective*)*not k-defective.*

We are now in position to define the last cone attached to a point $x \in X$. This notion was introduced by Johnson in [**Jo**] and further studied extensively by Zak. Algebraic properties of tangent star cones and of the algebras related to them are investigated in [**SUV**].

1.1.4. Definition. (**Tangent star at a point; Variety of tangent stars**, [**Jo**]). Let $X \subset \mathbb{P}^N$ be an irreducible projective variety.

The abstract variety of tangent stars to X, T^*_X, is defined by the following cartesian diagram

$$\begin{tikzcd} T^*_X \arrow[r, hook] \arrow[d, "p"'] & S_X \arrow[d, "p_2"] \\ \Delta_X \arrow[r, hook] & X \times X \end{tikzcd} .$$

The tangent star to X at x, T^*_xX, is defined by

$$T^*_xX := p_2(p^{-1}((x,x))) \subseteq \mathbb{P}^N.$$

It is a scheme which can be described geometrically as follows:

$$T^*_xX = \overline{\bigcup_{(x_1,x_2) \in X \times X \setminus \Delta_X} \lim_{x_i \to x} < x_1, x_2 >} \subset \mathbb{P}^N.$$

The variety of tangent stars to X is by definition

$$T^*X = p_2(T^*_X) \subseteq \mathbb{P}^N,$$

so that by construction

$$T^*X \subseteq SX;$$

moreover letting only one point varying we deduce

$$C_xX \subseteq T_x^*X.$$

It is also clear that the limit of a secant line is a tangent line, i.e. that

$$T_x^*X \subseteq T_xX.$$

By what we have defined and studied we deduce that for a point $x \in X \subset \mathbb{P}^N$, there is the following relation between the cones we attached at X:

$$C_xX \subseteq T_x^*X \subseteq T_xX.$$

Moreover a point $x \in X$ is non-singular if and only if $C_xX = T_x^*X = T_xX$. We immediately show in the following class of examples that at singular points strict inequalities can hold, i.e. at singular points there could exist tangent lines which are not limit of secant lines.

1.1.5. Example. (**Singular points for which** $C_xX \subsetneq T_x^*X \subsetneq T_xX$). Let $Y \subset \mathbb{P}^N \subset \mathbb{P}^{N+1}$ be an irreducible, non-degenerate variety in $\mathbb{P}^N$. Consider a point $p \in \mathbb{P}^{N+1} \setminus \mathbb{P}^N$ and let $X := S(p, Y)$ be *the cone over* Y *of vertex* p, i.e.

$$S(p, Y) = \bigcup_{y \in Y} < p, y > .$$

Then X is an irreducible, non-degenerate variety in $\mathbb{P}^{N+1}$. In fact, modulo a projective transformation, the variety X is defined by the same equations of Y, now thought as homogeneous polynomials with one variable more; in particular $\dim(X) = \dim(Y) + 1$.

The line $< p, y >$ is contained in X for every $y \in Y$, so that $X \subset T_xX$ and therefore $\mathbb{P}^N =< Y >\subset T_pX$. Since $p \in T_pX$, we get

$$T_pX = \mathbb{P}^{N+1}. \tag{1.1.1}$$

It follows from the definition of tangent cone to a variety that

$$C_pS(p, Y) = S(p, Y).$$

We also have that

$$S(p, SY) = SX. \tag{1.1.2}$$

Indeed, by projecting from p onto $\mathbb{P}^N$, it is clear that a general secant line to X projects onto a secant line to Y, proving $SX \subseteq S(p, SY)$. On the contrary if we get a general point $q \in S(p, SY)$, by definition it projects onto a general point $q' \in SY$, which belongs to a secant line $< p_1', p_2' >$, $p_i' \in Y$. The plane $< p, p_1', p_2' >$ contains the point q, while the lines $< p, p_i' >$, $i = 1, 2$, are contained in X by definition of cone; hence through q there pass infinitely many secant lines to X, yielding $S(p, SY) \subseteq SX$. The claim is proved.

The above argument proves the following general fact:

$$T_p^*S(p, Y) = S(p, SY).$$

Indeed by definition $T_p^*X \subseteq SX = S(p, SY)$ as schemes. On the other hand, by fixing two general points $p_1, p_2 \in X$, $p_1 \neq p_2$, $p_i \neq p$, the plane $< p, p_1, p_2 >$ is contained in T_p^*X as it is easily seen by varying the velocity of approaching p of two points $q_i \in < p, p_i >$. By the generality of the points p_i we get the inclusion $SX \subseteq T_p^* X$ as schemes and the proof of the claim.

As an immediate application one constructs example of irreducible singular varieties X with a point $p \in \mathrm{Sing}(X)$ for which

$$C_p X \subsetneq T_p^* X \subsetneq T_p X.$$

One can take as $Y \subset \mathbb{P}^4 \subset \mathbb{P}^5$ an irreducible, smooth, non-degenerate curve in $\mathbb{P}^4$ and consider the cone X over Y of vertex $p \in \mathbb{P}^5 \setminus \mathbb{P}^4$. Then $C_p X = S(p, Y) = X$, $T_p^* X = S(p, SY) = SX$ is an hypersurface in $\mathbb{P}^5$, because SY is an hypersurface in $\mathbb{P}^4$ (see 1.2.2 if you do not agree), while $T_p X = \mathbb{P}^5$. Every variety Y such that $SY \subsetneq \mathbb{P}^N$ (see the following exercise or take $N > 2\dim(Y) + 1$) will produce analogous examples.

1.1.6. Exercise. Let K be a(n algebraically closed) field. Recall that the linear combination of two (symmetric) matrixes of rank 1 has rank at most 2 and that every (symmetric) matrix of rank 2 can be written as the linear combination of two (symmetric) matrixes of rank 1.

Deduce the following geometrical consequences for the secant varieties of the varieties described below.

(1) Let $X = \nu_2(\mathbb{P}^2) \subset \mathbb{P}^5$ be the 2-Veronese surface in $\mathbb{P}^5$. Identify $\mathbb{P}^5$ with
$$\mathbb{P}(\{A \in M(3; K) \ : \ A = A^t\}),$$
and show that $X = \{[A] \ : \ \mathrm{rk}(A) = 1\}$. Prove that $SX = TX = V(det(A)) \subset \mathbb{P}^5$ is the cubic hypersurface given by the cubic polynomial $det(A)$. Show that if $x_1, x_2 \in X$, then $T_{x_1} X \cap T_{x_2} X \neq \emptyset$ ((if you have a lot of energy and not enough patience to wait for the next section, prove that if the points are general, then the intersection consists of a point). Prove that $\mathrm{Sing}(SX) = X$.

(2) Let $X = \mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^8$ be the Segre embedding of $\mathbb{P}^2 \times \mathbb{P}^2$ in $\mathbb{P}^8$. Identify $\mathbb{P}^8$ with
$$\mathbb{P}(\{A \in M(3; K)\}),$$
and show that $X = \{[A] \ : \ \mathrm{rk}(A) = 1\}$. Prove that $SX = TX = V(det(A)) \subset \mathbb{P}^8$ is the cubic hypersurface given by the cubic polynomial $det(A)$. Show that if $x_1, x_2 \in X$, then $T_{x_1} X$ and $T_{x_2} X$ intersect at least along a line (prove that if the points are general, then the intersection consists of a line). Take H be a general hyperplane in $\mathbb{P}^8$ and let $Y := X \cap H$. Then Y is a smooth, irreducible, non-degenerate 3-fold $Y \subset \mathbb{P}^7$ such that $SY \subseteq SX \cap H$ so that $\dim(SY) \leq 6$ (in fact one can prove that $SY = SX \cap H$ and hence that $\dim(SY) = 6$). Prove that given $y_1, y_2 \in Y$, then $T_{y_1} Y \cap T_{y_2} Y \neq \emptyset$ (consists of a point if the points are general). Prove that $\mathrm{Sing}(SX) = X$.

Let $p \in \mathbb{P}^9 \setminus \mathbb{P}^8$, let $Z = S(p, X) \subset \mathbb{P}^9$ and let $X' = X \cap W$, with $W \subset \mathbb{P}^9$ a general hypersurface, not an hyperplane, not passing through p. Then X' is a smooth, irreducible, non-degenerate 4-fold such that $SX' = SZ = S(p, SX)$. Conclude that $\dim(SX') = 8$ and use the fact that Z is a cone over X to deduce that two general tangent spaces to X' intersect.

(3) Generalize the previous exercise and find the relation between $SX \subset \mathbb{P}^N$ and $SX' \subset \mathbb{P}^{N+1}$ for $X' \subset \mathbb{P}^{N+1}$ a general intersection of $Z = S(p,X) \subset \mathbb{P}^{N+1}$ with a general hypersurface $W \subset \mathbb{P}^{N+1}$, not passing through $p \in \mathbb{P}^{N+1} \setminus \mathbb{P}^N$.

1.2. Join of varieties

We generalize to arbitrary irreducible varieties $X, Y \subset \mathbb{P}^N$ the notion of "cone" or of "join" of linear spaces.

Let us remember that if $L_i \simeq \mathbb{P}^{N_i} \subseteq \mathbb{P}^N$, $i = 1, 2$, is a linear subspace, then

$$< L_1, L_2 >:= \bigcup_{x_i \in L_i\ ,\ x_1 \neq x_2} < x_1, x_2 >,$$

is a linear space called *the join of* L_1 *and* L_2. It is the smallest linear subspace of $\mathbb{P}^N$ containing L_1 and L_2. By Grassmann formula we have

$$\dim(< L_1, L_2 >) = \dim(L_1) + \dim(L_2) - \dim(L_1 \cap L_2), \tag{1.2.1}$$

where as always $\dim(\emptyset) = -1$. This shows that the dimension of the join depends on the intersection of the two linear spaces.

On the other hand, if $X \subset \mathbb{P}^N \subset \mathbb{P}^{N+1}$ is an irreducible subvariety and if $p \in \mathbb{P}^{N+1} \setminus \mathbb{P}^N$ is an arbitrary point, if we define as before

$$S(p, X) = \bigcup_{x \in X} < p, x >,$$

the cone of vertex p *over* X, then for every $z \in < p, x >$, $z \neq x$, $z \neq p$, we have by construction

$$T_z S(p, X) =< p, T_x X >=< T_p p, T_x X >, \tag{1.2.2}$$

i.e. the well known fact that the tangent space is constant along the ruling of a cone.

As we shall see in the next section, once we have defined the join of two varieties as the union of lines "joining" points of them, then we can "linearize" the problem looking at the tangent spaces and calculate the dimension of the "join" by looking at the affine cones over the varieties, exactly as in the proof of the formula 1.2.1. The dimension of the join of two varieties will depend on the intersection of a general tangent space of the first one with a general tangent space of the other one, a result known as Terracini Lemma, [**T1**]. Moreover a kind of property similar to the second tautological inequality in 1.2.2 will hold generically, at least in characteristic zero, see theorem 1.3.1.

1.2.1. Definition. (**Join of varieties; relative secant, tangent star and tangent varieties**). Let $X, Y \subset \mathbb{P}^N$ be closed irreducible subvarieties.

Let

$$S^0_{X,Y} := \{((x, y, z)\,, x \neq y :\ z \in < x, y >\} \subset X \times Y \times \mathbb{P}^N.$$

The set is locally closed so that taken with the reduced scheme structure it is a quasi-projective irreducible variety of dimension $\dim(S^0_{X,Y}) = \dim(X) + \dim(Y) + 1$. Let $S_{X,Y}$ be its closure in $X \times Y \times \mathbb{P}^N$. Then $S_{X,Y}$ is an irreducible projective variety of dimension $\dim(X)+\dim(Y)+1$, called *the abstract join of X and Y*. Let us consider the projections of $S_{X,Y}$ onto the factors $X\times Y$ and $\mathbb{P}^N$,

$$\begin{tikzcd} & S_{X,Y} \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ X\times Y & & \mathbb{P}^N. \end{tikzcd} \tag{1.2.3}$$

The join of X and Y, $S(X,Y)$, is the scheme-theoretic image of $S_{X,Y}$ in $\mathbb{P}^N$, i.e.

$$S(X,Y)=p_2(S_{X,Y})=\overline{\bigcup_{x\neq y,\, x\in X,\, y\in Y} <x,y>}\subseteq\mathbb{P}^N;$$

it is an irreducible algebraic variety of dimension $s(X,Y)\leq\dim(X)+\dim(Y)+1$, swept out by lines joining points of X with points of Y.

With this notation $S(X,X)=SX$ and $S(X,S^{k-1}X)=S^kX=S(S^lX,S^hX)$, if $h\geq 0$, $l\geq 0$, $h+l=k-1$. Moreover, for arbitrary irreducible varieties X, Y and Z, we have $S(X,S(Y,Z))=S(S(X,Y),Z)$.

When $Y\subseteq X\subset\mathbb{P}^N$ is an irreducible closed subvariety, the variety $S(Y,X)$ is usually *the relative secant variety of X with respect to Y*. Analogously, $T(Y,X)=\bigcup_{y\in Y}T_yX$. In this case by taking $\Delta_Y\subset Y\times X$ and by looking at 1.2.3, we can define $T^*_{Y,X}:=p_1^{-1}(\Delta_Y)\subseteq S_{Y,X}$ to be *the abstract relative tangent star variety* and finally

$$T^*(Y,X):=p_2(T^*_{Y,X})\subseteq S(X,Y) \tag{1.2.4}$$

to be *the relative tangent star variety.* If

$$T^*_y(Y,X)=p_2(p_1^{-1}(y\times y))=\overline{\bigcup_{(y_1,x_1)\in Y\times X\setminus\Delta_Y}\lim_{\substack{y_1\to y\\ x_1\to y}}<y_1,x_1>}\subset\mathbb{P}^N,$$

then $T^*(Y,X)=\bigcup_{y\in Y}T^*_y(Y,X)$. With this terminology, $T^*_y(y,X)=C_yX$ and $T^*_y(X,X)=T^*_yX$ for every $y\in X$. In particular $C_yX=T^*_y(y,X)\subseteq T^*_y(X,X)=T^*_yX$.

We furnish some immediate applications of the definition of join to properties of S^kX and to characterizations of linear spaces.

1.2.2. Proposition. ([**P2**]) *Let $X,Y\subset\mathbb{P}^N$ be closed irreducible subvarieties. The following holds:*

(1) *for every $x\in X$,*
$$Y\subseteq S(x,Y)\subseteq S(x,<Y>)\subseteq T_xS(X,Y);$$
(2) *if $S^kX=S^{k+1}X$ for some $k\geq 0$, then $S^kX=\mathbb{P}^{s_k(X)}\subseteq\mathbb{P}^N$;*
(3) *if $\dim(S^{k+1}X)=\dim(S^kX)+1$ for some $k\geq 0$, then $S^{k+1}X=\mathbb{P}^{s_{k+1}(X)}$ so that S^kX is a hypersurface in $\mathbb{P}^{s_{k+1}(X)}$;*
(4) *if $S^{k+1}X$, $k\geq 0$, is not a linear space, then $S^kX\subseteq\mathrm{Sing}(S^{k+1}X)$.*

Proof. By definition of join we get the inclusion $S(x,Y)\subseteq S(X,Y)$ and hence $T_xS(x,Y)\subseteq T_xS(X,Y)$. Moreover for every $y\in Y$, $y\neq x$, the line $<x,y>$ is contained in $S(x,Y)$ and passes through x so that it is contained in $T_xS(x,Y)$ and part 1) easily follows.

Let $z \in S^kX$ be a smooth point of S^kX. From part 1) we get

$$X \subseteq T_zS(S^kX, X) = T_zS^{k+1}X = T_zS^kX = \mathbb{P}^{s_k(X)}.$$

Thus $S^kX \subseteq < X > \subseteq T_zS^kX = \mathbb{P}^{s_k(X)}$ so that $S^kX =< X >= \mathbb{P}^{s_k(X)}$ since S^kX and T_zS^kX are both irreducible varieties of dimension $s_k(X)$.

To prove part 3), take a general point $z \in S^{k+1}X \setminus S^kX$. For general $x \in X$ we get $S^kX \subsetneq S(x, S^kX) \subseteq S(X, S^kX) = S^{k+1}X$. Thus for general $x \in X$ we get $S(x, S^kX) = S^{k+1}X$ since $s_{k+1}(X) = s_k(X) + 1$. In particular $z \in S(x, S^kX)$ for $x \in X$ general, i.e. there exists $y \in S^kX$ such that $z \in< x, y > \subset S^{k+1}X$. Thus a general point $x \in X$ is contained in $T_zS^{k+1}X$ so that

$$S^{k+1}X \subseteq < X > \subseteq T_zS^{k+1}X$$

yields $S^{k+1}X =< X >= \mathbb{P}^{s_{k+1}(X)}$ since $\dim(T_zS^{k+1}X) = s_{k+1}(X)$ by the generality of $z \in S^{k+1}X$.

Recall that in any case $S^{k+1}X \subseteq < X >$. Take $z \in S^kX$ and observe that by part 1) $S^{k+1}X \subseteq < X > \subseteq T_zS^{k+1}X$ so that $\dim(T_zS^{k+1}X) > \dim(S^{k+1}X)$, z is a singular point of $S^{k+1}X$ and part 4) follows. $\square$

To a non-degenerate irreducible closed subvariety $X \subset \mathbb{P}^N$ we can associate an ascending filtration of irreducible projective varieties, whose inclusion are strict by 1.2.2, and an integer $k_0 = k_0(X) \geq 1$:

$$X = S^0X \subsetneq SX \subsetneq S^2X \subsetneq \ldots \subsetneq S^{k_0}X = \mathbb{P}^N, \tag{1.2.5}$$

where k_0 is the least integer such that $S^kX = \mathbb{P}^N$.

The above immediate consequences of the definitions give also the following result, which was classically very well known, see for example [**P1**] footnote pg. 635, but considered as an open problem by Atiyah, [**At**] pg. 424. From the following corollary, an argument of Atiyah yields a proof of C. Segre and Nagata theorem about the minimal section of a geometrically ruled surface, see [**Ln**].

1.2.3. Corollary. ([**P1**]) *Let $C \subset \mathbb{P}^N$ be an irreducible non-degenerate projective curve. Then $s_k(C) = \min\{2k + 1, N\}$.*

Proof. For $k = 0$ it is true and we argue by induction. Suppose $S^kC \subsetneq \mathbb{P}^N$. By proposition 1.2.2 $s_k(C) \geq s_{k-1}(C)+2$ and the description $S^k(C) = S(C, S^{k-1}C)$ yields $s_k(C) \leq s_{k-1}(C)+2$ so that $s_k(C) = 2(k-1)+1+2 = 2k+1$ as claimed. $\square$

We define and study linear projections with the terminology just introduced and generalize in a suitable way the dimension formula 1.2.1, in characteristic zero, i.e. to the case of arbitrary cones over the variety X. In the next section we deal with the general case.

1.2.4. Definition. (**Linear projections and "linear" cones**) Let $L = \mathbb{P}^l \subset \mathbb{P}^N$ be a fixed linear space, $l \geq 0$, and let $M = \mathbb{P}^{N-l-1}$ be a linear space skew to L, i.e. $L \cap M = \emptyset$ and $< L, M >= \mathbb{P}^N$. Let $X \subseteq \mathbb{P}^N$ be a closed irreducible variety not contained in L and let

$$\pi_L : X \dashrightarrow \mathbb{P}^{N-l-1} = M,$$

be the rational map defined on $X \setminus (L \cap X)$ by

$$\pi_L(x) =< L, x > \cap M.$$

The map is well defined by Grassmann formula, 1.2.1. Let $X' = \overline{\pi_L(X)} \subset \mathbb{P}^{N-l-1}$ be the closure of the image of X by π_L. The whole process can be described with the terminology of joins. Indeed we have

$$X' = S(L, X) \cap M,$$

i.e. X' is the intersection of M with the *cone over* X *of vertex* L and moreover $S(L, X) = S(L, X')$. The projective differential of π_L is the projection of the tangent spaces from L, i.e. if $x \in X \setminus (L \cap X)$, then $d_{\pi_L}(T_x X) =< L, T_x X > \cap M \subseteq T_{\pi_L(x)} X'$ as it is easily seen eventually passing to (local) coordinates.

Suppose $L \cap X = \emptyset$, then we claim that $\pi_L : X \to X'$ is a finite morphism, which implies $\dim(X) = \dim(X')$. Being a morphism between projective varieties, it is sufficient to show that it has finite fibers. By definition for $x' \in X'$,

$$\pi_L^{-1}(x') =< L, x' > \cap X \subset < L, x' >= \mathbb{P}^{l+1}.$$

If there exists an irreducible curve $C \subset < L, x' > \cap X \subset < L, x' >$, then $\emptyset \neq L \cap C \subseteq L \cap X$, contrary to our assumption.

In particular for an arbitrary L, the dimension of X' does not depend on the choice of the position of M, except for the requirement $L \cap M = \emptyset$.

The relation $S(L, X) = S(L, X')$ allows us to calculate the dimension of the irreducible variety $S(L, X)$ for an arbitrary L. Exactly as in 1.2.2 for $z \in S(L, X) \setminus L$,

$$z \in < L, x >=< L, \pi_L(z) >=< L, x' >,$$

with $x \in X$ and $\pi_L(z) = \pi_L(x) = x' \in X'$. Since $S(L, X')$ is, modulo a projective transformation, the variety defined by the same homogeneous polynomials of X' now though as polynomial in $N + 1$ variables, we have

$$T_z S(L, X) =< L, T_{\pi_L(z)} X' > \supseteq < L, T_x X > . \tag{1.2.6}$$

Taking $z \in S(L, X)$ general and recalling that $L \cap M = \emptyset$ we deduce:

$$\dim(S(L, X)) = \dim(< L, T_{\pi_L(z)} X' >) = \dim(X') + l + 1. \tag{1.2.7}$$

Suppose till the end of the subsection $\mathrm{char}(K) = 0$. By generic smoothness, the differential map is surjective so that $T_{\pi_L(x)} X' = \pi_L(T_x X)$ for $x \in X$ general. In this case $\pi_L(x) = x' \in X'$ will be general on X' and finally

$$\dim(X') = \dim(T_{x'} X') = \dim(\pi_L(T_x X)) = \dim(X) - \dim(L \cap T_x X) - 1,$$

which combined with 1.2.7 gives the following generalization of 1.2.1:

$$\dim(S(L, X)) = \dim(L) + \dim(X) - \dim(L \cap T_x X), \tag{1.2.8}$$

$$x \in X \text{ general point.}$$

Moreover, we get the following refinement of 1.2.6

$$T_z S(L, X) =< L, T_x X >, \tag{1.2.9}$$

$$x \in X \ , \ z \in < L, x > \ \text{ general points.}$$

We have generalized the notion of cone over a variety lying in a skew space with respect to the vertex by taking $S(L, X)$ and shown that by projecting the variety X from the vertex L, we can find the description of it as an "usual" cone, $S(L, X')$.

Now we investigate under which condition a variety is a "cone", i.e. there exists a "vertex" $L \simeq \mathbb{P}^l \subseteq X$ such that $X = S(L, X) = S(L, X')$, if X' is the section with a general $\mathbb{P}^{N-l-1}$ skew with the "vertex" L. Clearly the "vertex" is not uniquely determined if we do not require some maximality condition. Let us begin with the definitions.

1.2.5. Definition. (**Cone**; **vertex of a variety**) Let $X \subset \mathbb{P}^N$ be a closed (irreducible) subvariety. The variety is a *cone* if there exists $x \in X$ such that $S(x, X) = X$. Geometrically this means that given $y \in X$, $y \neq x$, the line $< x, y >$ is contained in X. In particular $x \in \bigcap_{y \in X} T_y X$.

This motivates the definition of vertex of a variety. Given $X \subset \mathbb{P}^N$ an irreducible closed subvariety, *the vertex of* X, $\mathrm{Vert}(X)$, is the set

$$\mathrm{Vert}(X) = \{\, x \in X \;:\; S(x, X) = X \,\}.$$

In particular a variety X is a cone if and only if $\mathrm{Vert}(X) \neq \emptyset$; by definition $S(X, Y) = X$ if and only if $Y \subseteq \mathrm{Vert}(X)$.

We list some obvious consequences and leave to the interested reader the pleasure of showing that the hypothesis on the characteristic of the base field are necessary.

1.2.6. Proposition. *Let $X \subset \mathbb{P}^N$ be a closed irreducible variety of dimension $\dim(X) = n$. The following holds:*

(1)
$$\mathrm{Vert}(X) = \mathbb{P}^l \subseteq \bigcap_{x \in X} T_x X,$$
$l \geq -1$;

(2) *if* $\mathrm{codim}(\mathrm{Vert}(X), X) \leq 1$, *then* $\mathrm{Vert}(X) = X = \mathbb{P}^n \subset \mathbb{P}^N$;

(3) *if* $\dim(S(X, Y)) = \dim(X) + 1$, *then* $Y \subseteq \mathrm{Vert}(S(X, Y))$;

(4) *if* $\mathrm{char}(K) = 0$,
$$\mathrm{Vert}(X) = \bigcap_{x \in X} T_x X = \mathbb{P}^l \subseteq X,$$
$l \geq -1$;

(5) *suppose* $\mathrm{char}(K) = 0$ *and* $\emptyset \neq \mathrm{Vert}(X) \subsetneq X$, *then* $X = S(\mathrm{Vert}(X), X')$ *is a cone, where X' is the projection of X from* $\mathrm{Vert}(X)$ *onto a* $\mathbb{P}^{N-l-1}$ *skew to* $\mathrm{Vert}(X)$ $(\dim(X') = n - l - 1)$.

Proof. To prove 1) it is sufficient to show that, given two points $x_1, x_2 \in \mathrm{Vert}(X)$, the line $< x_1, x_2 >$ is contained in $\mathrm{Vert}(X)$, forcing $\mathrm{Vert}(X)$ irreducible and linear by proposition 1.2.2 part 2). Taken $y \in < x_1, x_2 > \setminus \{x_1, x_2\}$ and $x \in X \setminus \mathrm{Vert}(X)$, it is sufficient to prove that $< y, x > \subset X$. By definition the lines $< x_i, x >$ are contained in X and by varying the point $q \in < x_2, x > \subset X$ and by joining it with x_1 we see that the line $< x_1, q >$ is contained in X for every such q, i.e. the plane $\Pi_x = < x_1, x_2, x >$ is contained in X. Since y and x belong to Π_x, the claim follows.

If $\mathrm{Vert}(X) = X$, then $X = \mathbb{P}^n$ by part 1). If there exists $W = \mathbb{P}^{n-1} \subseteq \mathrm{Vert}(X) = \mathbb{P}^l \subseteq X$, i.e. if $l \geq n-1$, we can take $x \in X \setminus W$. Therefore $S(x, W) = \mathbb{P}^n$ and $S(W, x) \subseteq X$ forces $X = \mathbb{P}^n$.

To prove 3) take $y \in Y \setminus \mathrm{Vert}(X)$ and observe that for dimension reasons $S(y, X) = S(Y, X)$ and $S(y, S(X, Y)) = S(y, S(y, X)) = S(y, X) = S(Y, X)$ gives the desired conclusion.

Set $L = \bigcap_{x \in X} T_x X$ and assume char(K)=0. By 1.2.8 $\dim(S(L, X)) = \dim(X)$, yielding $X = S(L, X)$ and $L \subseteq \mathrm{Vert}(X)$, which proves part 4). Part 5) follows in a straightforward way. □

Later we will use the following result.

1.2.7. Corollary. *Let $X \subset \mathbb{P}^N$ be an irreducible non-degenerate variety of dimension $n = \dim(X)$. Assume* char(K)=0*, $N \geq n+3$ and $\dim(SX) = n+2$. If through the general point $x \in X$ there passes a line l_x contained in X, then X is a cone.*

Proof. Let $x \in X$ be a general point. Then $x \notin \mathrm{Vert}(X)$ and $x \notin \mathrm{Vert}(SX)$ since X is non-degenerate, so that $X \subsetneq S(l_x, X) \subseteq SX$. If $\dim(S(l_x, X)) = n+2$, then $S(l_x, X) = SX$. Since $S(l_x, SX) = S(l_x, S(l_x, X)) = S(l_x, X) = SX$, we would deduce $x \in l_x \subseteq \mathrm{Vert}(SX)$. In conclusion l_x is not contained in $\mathrm{Vert}(SX)$ and $\dim(S(l_x, X)) = n+1$. Then the general tangent space to X, $T_y X$, will cut l_x in a point $p_{x,y} := l_x \cap T_y X$. If this point varies with y, then two general tangent spaces $T_{y_1} X$ and $T_{y_2} X$ would contain l_x so that $< l_x, < T_{y_1} X, T_{y_2} X >> = < T_{y_1} X, T_{y_2} X >$ would force $S(l_x, SX) = SX$, i.e. $l_x \subseteq \mathrm{Vert}(SX)$. So the point remain fixed, i.e. $p \in \cap_{y \in X} T_y X = \mathrm{Vert}(X)$ and X is a cone by proposition 1.2.6. □

We end this section by putting in relation the projections of a variety and the dimension of its secant or tangent varieties.

If $L = \mathbb{P}^l \subset \mathbb{P}^N$ is a linear space and if $\pi_L : \mathbb{P}^N \setminus L \to \mathbb{P}^{N-l-1}$ is the projection onto a skew complementary linear space, then π_L restricts to a finite morphism $\pi_L : X \to \mathbb{P}^{N-l-1}$, as soon as $L \cap X = \emptyset$. In the idea that studying varieties whose codimension is small with respect to the dimension is easier (from some points of view but not from others!), we can ask when this finite morphism is one-to-one, or a closed embedding. Let us examine this conditions in the following proposition.

1.2.8. Proposition. *Let notation be as above. Then:*

(1) *the morphism $\pi_L : X \to \mathbb{P}^{N-l-1}$ is one-to-one if and only if $L \cap SX = \emptyset$;*
(2) *the morphism $\pi_L : X \to \mathbb{P}^{N-l-1}$ is unramified if and only if $L \cap TX = \emptyset$;*
(3) *the morphism $\pi_L : X \to \mathbb{P}^{N-l-1}$ is a closed embedding if and only if $L \cap SX = L \cap TX = \emptyset$.*

Proof. The morphism $\pi_L : X \to X' \subseteq \mathbb{P}^{N-l-1}$ is one-to-one if and only there exists no secant line to X cutting the center of projection: $< L, x > = < L, y >$ if and only if $< x, y > \cap L \neq \emptyset$. It is ramified at a point $x \in X$ if and only if $T_x X \cap L = \emptyset$ by looking at the projective differential of π_L. A morphism is a closed embedding if and only if it is one-to-one and unramified. □

We must state the following well known result, which only takes into account that for smooth varieties the equality $TX = T^*X$ furnishes $TX \subseteq SX$.

1.2.9. Corollary. *Let $X \subset \mathbb{P}^N$ be a smooth irreducible closed subvariety. If $N > \dim(SX)$, then X can be isomorphically projected into $\mathbb{P}^{N-1}$. In particular if $N > 2\dim(X)+1$, then X can isomorphically projected into $\mathbb{P}^{N-1}$.*

One could ask what is the meaning of $L \cap T^*X = \emptyset$. This means that π_L (or $d(\pi_L)$) restricted to T^*_xX is finite for every $x \in X$. This is the notion of J-unramified morphism, where J stands for Johnson [**Jo**], and it can be expressed in terms of affine tangent stars, see [**Z2**]. We take the above condition as the definition of J-unramified projection. In particular, if $L \cap SX = \emptyset$, then π_L is one-to-one and J-unramified and it is said to be *a J-embedding*. If the projection $\pi_L : X \to X' \subset \mathbb{P}^{N-l-1}$, then $\mathrm{Sing}(\pi_L(X)) = \pi_L(\mathrm{Sing}(X))$ so that X' does not acquire singularities from the projection.

It is clearly weaker than the usual notion of embedding and it is well behaved to study the projections of singular varieties. For example take $C \subset \mathbb{P}^4 \subset \mathbb{P}^5$ a smooth non-degenerate curve in $\mathbb{P}^4$ and let $p \in \mathbb{P}^5 \setminus \mathbb{P}^4$. If $X = S(p, C)$ is the cone over C, then $T_pX = \mathbb{P}^5$, 1.1.1, and X cannot be projected isomorphically in $\mathbb{P}^4$. Since $SX = S(p, SC)$, 1.1.2, is an hypersurface in $\mathbb{P}^5$, there exists a point $q \in \mathbb{P}^5 \setminus X$ such that $\pi_q : X \to X'$ is a J-embedding and $X' = S(\pi_q(p), C)$ is a cone over C of vertex $\pi_q(p) = p'$. In this example the morphism π_q is one-to-one and unramified outside the vertex of the cones and maps the tangent star at p, $T^*_pX = S(p, SC)$, m-to-one onto $\mathbb{P}^4$, where $m = \deg(S(p, SC)) = \deg(SC) = \binom{d-1}{2} - g$, $d = \deg(C)$, g the genus of C.

The conditions $L \cap S(Y, X) = \emptyset$, respectively $L \cap T^*(Y, X) = \emptyset$ or $L \cap T(Y, X) = \emptyset$, with $Y \subseteq X$, mean that π_L is one-to-one in a neighbourhood of Y, respectively is J-unramified in a neighbourhood of Y or unramified in a neighbourhood of Y.

1.3. Terracini Lemma and its first applications

As we have seen the definition of secant variety is the "join" of X with itself and it is not clear how to calculate the dimension of SX, see exercise 1.1.6, or more generally the dimension of $S(X, Y)$. In fact, the circle of ideas, which allowed Alessandro Terracini to solve the problem of calculating the dimension of SX, or more generally of S^kX, originated exactly from the study of examples like the ones considered in 1.1.6 and from the pioneering work of Gaetano Scorza, [**S1**] and [**S4**]. Let Terracini explain us this process, by quoting the beginning of [**T1**]:

É noto, [**dP**]*, che la sola V_2, non cono, di S_r, i cui S_2 tangenti si incontrano a due a due, é, se $r \geq 5$, la superficie di VERONESE; e che questa superficie,* [**Sev1**]*, é pure caratterizzata dall' essere, in un tale S_r, la sola, non cono, le cui corde riempono una V_4. Recentemente lo SCORZA,* [**S3**] *pg. 265, disse di aver ragione di credere, sebbene non gli fosse venuto fatto di darne una dimostrazione, che le V_3 di S_7, o di uno spazio piú ampio, le cui corde non riempiono una V_7 $<<$ rientrino $>>$ tra le V_3 a spazi tangenti mutuamente secantisi. Ora si puó dimostrare, piu' precisamente, che queste categorie di V_3 coincidono, anzi, piu' in generale, che: Se una V_k di S_r ($r > 2k$) gode di una delle due proprietá, che le corde riempiano una varietá di dimensione $2k-i$ ($i \geq 0$), o che due qualsiansi S_k tangenti si seghino in uno S_i, gode pure dell' altra. Questo teorema, a sua volta, non é se non un caso particolare di un teorema piú generale che ora dimostreremo, teorema che pone in relazione l' eventuale abbassamento di dimensione della varietá degli S_h $(h+1)$-seganti di una V_k immersa in uno spazio di dimensione $r \geq (h+1)k+h$, coll'* *esistenza di $h+1$ qualsiansi suoi S_k tangenti in uno spazio minore dell' ordinario.*

To calculate the dimension of $S(X,Y)$ in a simple way and to determinate the relation between $T_zS(X,Y)$, T_xX and T_yY, where $z \in < x, y >$, $z \neq x$, $z \neq y$, $x \neq y$, we recall the definition of affine cone over a projective variety $X \subset \mathbb{P}^N$.

Let $\pi : \mathbb{A}^{N+1} \setminus \mathbf{0} \to \mathbb{P}^N$ be the canonical projection. If $X \subset \mathbb{P}^N$ is a closed subvariety, we indicate by $C_{\mathbf{0}}(X)$ *the affine cone over* X, i.e. $C_{\mathbf{0}}(X) = \pi^{-1}(X) \cup \mathbf{0}$ is the affine variety cut out by the homogeneous polynomials in $N+1$ variables defining X. If $\mathbf{x} \neq \mathbf{0}$ is a point such that $\pi(\mathbf{x}) = x \in X$, then

$$\pi(\mathcal{T}_{\mathbf{x}} C_{\mathbf{0}}(X)) = T_xX.$$

Moreover, if $L_i = \pi(U_i)$, $i = 1, 2$, U_i vector subspace of $\mathbb{A}^{N+1}$, then by definition $< L_1, L_2 >= \pi(U_1 + U_2)$, where $+ : \mathbb{A}^{N+1} \times \mathbb{A}^{N+1} \to \mathbb{A}^{N+1}$ is the vector space operation. Therefore, thought as a morphism of algebraic varieties, the differential of the sum coincides with the operation, i.e.

$$d_{(\mathbf{x},\mathbf{y})} : \mathcal{T}_{(\mathbf{x},\mathbf{y})}(\mathbb{A}^{N+1} \times \mathbb{A}^{N+1}) = \mathcal{T}_{\mathbf{x}}\mathbb{A}^{N+1} \times \mathcal{T}_{\mathbf{y}}\mathbb{A}^{N+1} \to \mathcal{T}_{\mathbf{x}+\mathbf{y}}\mathbb{A}^{N+1}$$

is the sum of the corresponding vectors.

With the above notation we have

$$\overline{C_{\mathbf{0}}(X) + C_{\mathbf{0}}(Y)} = C_{\mathbf{0}}(S(X,Y)). \tag{1.3.1}$$

We are now in position to prove the so called Terracini Lemma. The original proof of Terracini relies on the study of the differential of the second projection morphism $p_2 : S_{X,Y} \to S(X,Y)$. Here we follow Ådlandsvik, [**Åd**], to avoid the "difficulty", if any, of writing the tangent space at a point $(x,y,z) \in S^0_{X,Y}$. When writing $z \in < x, y >$, we always suppose $x \neq y$.

1.3.1. THEOREM. (**Terracini Lemma**) *Let $X, Y \subset \mathbb{P}^N$ be irreducible subvarieties. Then:*

(1) *for every $x \in X$, for every $y \in Y$, $x \neq y$, and for every $z \in < x, y >$,*

$$< T_xX, T_yY > \subseteq T_zS(X,Y);$$

(2) *if* $\operatorname{char}(K) = 0$, *there exists an open subset U of $S(X,Y)$ such that*

$$< T_xX, T_yY >= T_zS(X,Y)$$

for every $z \in U$, $x \in X$, $y \in Y$, $z \in < x, y >$. In particular

$$\dim(S(X,Y)) = \dim(X) + \dim(Y) - \dim(T_xX \cap T_yY)$$

for $x \in X$ and $y \in Y$ general points.

PROOF. The first part follows from equation 1.3.1 and from the interpretation of the differential of the affine sum. The second part from generic smoothness applied to the affine cones over X, Y and $S(X,Y)$. □

Since we have quoted the original form given by Terracini, let us state it as an obvious corollary.

1.3.2. COROLLARY. ([**T1**]) *Let $X \subset \mathbb{P}^N$ be an irreducible subvariety of $\mathbb{P}^N$. Then:*

(1) *for every* $x_0, \ldots, x_k \in X$ *and for every* $z \in < x_0, \ldots, x_k >$,

$$< T_{x_0}X, \ldots, T_{x_k}X > \subseteq T_z S^k X;$$

(2) *if* $\mathrm{char}(K) = 0$, *there exists an open subset* U *of* $S^k X$ *such that*

$$< T_{x_0}X, \ldots, T_{x_k}X > = T_z S^k X$$

for every $z \in U$, $x_i \in X$, $i = 0, \ldots, k$, $z \in < x_0, \ldots, x_k >$. *In particular*

$$\dim(SX) = 2\dim(X) - \dim(T_x X \cap T_y X)$$

for $x, y \in X$ *general points.*

The first application we give is the so called *Trisecant Lemma*. Let us recall that a line $l \subset \mathbb{P}^N$ is said to be a *trisecant line to* $X \subset \mathbb{P}^N$ if $\mathrm{length}(l \cap X) \geq 3$.

1.3.3. Proposition. (**Trisecant Lemma**) *Let* $X \subset \mathbb{P}^N$ *be a non-degenerate, irreducible closed subvariety. Suppose* char(K)=0 *and* $\mathrm{codim}(X) > k$. *Then a general* $(k+1)$*-secant* $\mathbb{P}^k$, $< x_0, \ldots, x_k > = L = \mathbb{P}^k$, *is not* $(k+2)$*-secant, i.e.* $L \cap X = \{x_0, \ldots, x_k\}$ *as schemes. In particular, if* $\mathrm{codim}(X) > 1$, *the projection of* X *from a general point on it,* $\pi_x : X \dashrightarrow X' \subset \mathbb{P}^{N-1}$, *is a birational map.*

Proof. We claim that it is sufficient to prove the result for $k = 1$. Indeed X is not linear so that by taking a general $x \in X$ and projecting X from this point we get a non-degenerate, irreducible subvariety $X' = \pi_x(X) \subset \mathbb{P}^{N-1}$ with $\mathrm{codim}(X') = \mathrm{codim}(X) - 1 > k - 1$. If the general $L = < x_0, \ldots, x_k >$ as above were $k+2$-secant, by taking $x = x_k$, the linear space $< \pi_x(x_0), \ldots, \pi_x(x_{k-1}) > = \mathbb{P}^{k-1} = L'$ would be a general k-secant $\mathbb{P}^{k-1}$, which results to be $(k+1) = ((k-1)+2)$-secant. So we can assume $k = 1$ and we set $n = \dim(X)$.

Take $x \in X \setminus \mathrm{Vert}(X)$. Then a general secant line through x, $l = < x, y >$, is not tangent to X neither at x nor at y. If l is a trisecant line then necessarily it exists $u \in (l \cap X) \setminus \{x, y\}$. Consider the projection of X from x. Since $x \notin \mathrm{Vert}(X)$, if $X' = \pi_x(X) \subset \mathbb{P}^{N-1}$, then $\dim(X') = \dim(X)$ and $\pi_x(y) = \pi_x(u) = x'$ is a general smooth point of X'. By generic smoothness

$$< x, T_{x'}X' > = < x, T_y X > = < x, T_u X >$$

so that $T_y X$ and $T_u X$ are hyperplanes in $< x, T_{x'}X' > = \mathbb{P}^{n+1}$ so that

$$\dim(T_y X \cap T_u X) = n - 1.$$

Taking $z \in < x, y > = < y, u >$ general, we have a point in the set U specified in corollary 1.3.2 yielding $\dim(SX) = \dim(T_z SX) = \dim(< T_y X, T_u X >) = n + 1$. This implies $\mathrm{codim}(X) = 1$ by proposition 1.2.2 part 3). The last part follows from the fact that a generically one-to one morphism is birational if char(K)=0, being generically étale. $\square$

As a second application we reinterpret Terracini Lemma as tangency of tangent space to higher secant varieties at a general point along the locus described on X by the secant spaces passing through the point. To this aim we first define tangency along a subvariety and then the entry loci described above, studying their dimension.

1.3.4. Definition. (**Tangencies along a subvariety**) Let $Y \subset X$ be a closed (irreducible) subvariety of X and let $L = \mathbb{P}^l \subset \mathbb{P}^N$, $l \geq \dim(X)$, be a linear subspace.

The linear space L is said to be *tangent to* X *along* Y if for every $y \in Y$

$$T_y X \subseteq L,$$

i.e. if and only if $T(Y,X) \subseteq L$.

The linear space L is said to be J*-tangent to* X *along* Y if for every $y \in Y$

$$T_y^* X \subseteq L,$$

i.e. if and only if $T^*(Y,X) \subseteq L$.

Clearly if L is tangent to X along Y, it is also J-tangent to X along Y.

In the case $L = \mathbb{P}^{N-1}$, the scheme-theoretic intersection $L \cap X = D$ is a divisor, i.e. a subscheme of pure dimension $\dim(X) - 1$. By definition, for every $y \in D$, we have $T_y D = T_y X \cap L$ so that, if X is a smooth variety, $L = \mathbb{P}^{N-1}$ is tangent to X exactly along $\mathrm{Sing}(D) = \{y \in D \,:\, \dim(T_y D) > \dim(D)\}$.

We define the important notions of entry loci and k-secant defect and we study their first properties.

1.3.5. Definition. (**Entry loci and k-secant defect** δ_k) Let $X \subset \mathbb{P}^N$ be a closed irreducible non-degenerate subvariety. Let us recall the diagram defining the higher secant varieties $S^k X$ as join of X with $S^{k-1}X$:

$$\begin{tikzcd} & S^k_X \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ X \times S^{k-1}X & & \mathbb{P}^N. \end{tikzcd}$$

Let us define $\phi : X \times S^{k-1}X \to X$ to be the projection onto the first factor of this product.

Then, for $z \in S^k X$, *the k-entry locus of X with respect to z* is the scheme theoretic image

$$\Sigma^k_z = \Sigma^k_z(X) := \phi(p_2(p_1^{-1}(z))). \tag{1.3.2}$$

Geometrically, the support of Σ^k_z is the locus described on X by the $(k+1)$-secant $\mathbb{P}^k$ of X passing through $z \in S^k X$. If $z \in S^k X$ is general, then through z there passes an ordinary $(k+1)$-secant $\mathbb{P}^k$, i.e. given by $k+1$ distinct points on X and we can describe the support of Σ^k_z in this way

$$(\Sigma^k_z)_{\mathrm{red}} = \overline{\{x \in X \,:\, \exists x_1, \ldots, x_k \text{ distinct and } z \in < x, x_1, \ldots, x_k >\}}.$$

Moreover, by the theorem of the dimension of the fibers for general $z \in S^k X$, the support of Σ^k_z is equidimensional and every irreducible component contains ordinary $\mathbb{P}^k$'s since necessarily $\mathrm{codim}(X) > k$, see proposition 1.3.3. If $\mathrm{char}(K)$=0 and for general $z \in S^k X$ the scheme $p_1^{-1}(z)$ is smooth so that Σ^k_z is reduced.

To recover the scheme structure of Σ^k_z geometrically, one could define Π_z as the locus of $(k+1)$-secant $\mathbb{P}^k$'s through z and define $\Sigma^k_z = \Pi_z \cap X$ as schemes. For example if through $z \in SX$ there passes a unique tangent line l to X, then in this way we get $\Pi_z = l$ and $\Sigma_z = l \cap X$ the point of tangency with the double structure.

Let us study the dimension of Σ_z^k for $z \in S^kX$ general. Before let us remark that if $x \in \Sigma_z^k$ is a general point in an irreducible component, $z \in S^kX$ general, then, as sets,

$$\phi^{-1}(x) = \dim(\{y \in S^{k-1}X \,:\, z \in < x, y >\} = < z, x > \cap S^{k-1}X \neq \emptyset$$

and $\dim(\phi^{-1}(x)) = 0$ because $z \in S^kX \setminus S^{k-1}X$ by the generality of z.

Then we define *the k-secant defect of* X, $1 \leq k \leq k_0(X)$, $\delta_k(X)$, as the integer

$$\delta_k(X) = \dim(\Sigma_z^k) = \dim(p_2(p_1^{-1}(z))) = s_{k-1}(X) + \dim(X) + 1 - s_k(X), \tag{1.3.3}$$

where $z \in S^kX$ is a general point.

For $k = 1$, we usually put $\Sigma_z = \Sigma_z^1$, $z \in SX$, and $\delta(X) = \delta_1(X) = 2\dim(X) + 1 - \dim(SX)$; for $k = 0$, $\delta_0(X) = 0$.

Let us reinterpret Terracini Lemma with these new definitions.

1.3.6. Corollary. (**Tangency along the entry loci**) *Let X be an irreducible non-degenerate closed subvariety. Let $k < k_0(X)$, i.e. $S^kX \subsetneq \mathbb{P}^N$, and let $z \in S^kX$ be a general point. Then:*

(1) *the linear space T_zS^kX is tangent to X along $(\Sigma_z^k)_{\mathrm{red}} \setminus \mathrm{Sing}(X)$;*
(2) $\delta_k(X) < \dim(X)$*;*
(3) $\delta_{k_0}(X) = \dim(X)$ *if and only if $s_{k_0-1}(X) = N - 1$, i. e. if and only if $S^{k_0-1}X$ is an hypersurface;*
(4)
$$s_k(X) = (k+1)(n+1) - 1 - \sum_{i=1}^{k} \delta_i(X) = \sum_{i=0}^{k} (\dim(X) - \delta_i(X) - 1);$$
(5) (cfr. 1.2.3) *if X is a curve,* $s_k(X) = 2k + 1$.

Proof. Part 1) is clearly a restatement of part 1) of corollary 1.3.2 when we take into account the geometrical properties of Σ_z^k, $z \in S^kX$ general, described in the definition of entry loci and the fact that the locus of tangency of a linear space is closed in $X \setminus \mathrm{Sing}(X)$, see also definition 1.5.8. Recall that if char(K)=0, the scheme Σ_z^k is reduced.

If $\dim(\Sigma_z^k) = \delta_k(X) = \dim(X)$, then a general tangent space to S^kX would be tangent along X and X would be degenerated.

With regard to 3), we remark that $\delta_{k_0}(X) = s_{k_0-1}(X) + \dim(X) + 1 - N$ so that $\dim(X) - \delta_{k_0}(X) = N - 1 - s_{k_0-1}(X)$.

Part 4) is an easy computation by induction, while part 5) follows from part 4) since for a curve $\delta_k(X) < \dim(X)$ yields $\delta_k(X) = 0$. □

1.3.7. Remark. The statement of part 1) cannot be improved. Take for example a cone $X \subset \mathbb{P}^5$ of vertex a point $p \in \mathbb{P}^5 \setminus \mathbb{P}^4$ over a smooth non-degenerate projective curve $C \subset \mathbb{P}^4$. If $z \in S(p, SC) = SX$ is general and if $z \in < x, y >$, $x, y \in X$, it is not difficult to see that $\Sigma_z(X) = < p, x > \cup < p, y >$. The hyperplane T_zSX is tangent to X at x and y by Terracini Lemma, so that it is tangent to X along the rulings $< p, x >$ and $< p, y >$ minus the point p. Since $T_pX = \mathbb{P}^5$, the hyperplane T_zSX is not tangent to X at p (neither J-tangent to X at p).

A phenomenon studied classically firstly by Scorza, [**S1**], [**S2**], [**S4**], and then by Terracini, [**T2**] is the case in which imposing tangency of a hyperplane at $k + 1$ general points, $k \geq 0$, of a variety $X \subset \mathbb{P}^N$ forces tangency along a positive dimensional variety, even if $\delta_k(X) = 0$. Indeed, Terracini Lemma says that if $\delta_k(X) > 0$, $k < k_0(X)$, than a hyperplane tangent at $k+1$ points, becomes tangent along the corresponding entry locus. The interesting and exceptional behaviour occurrs for varieties with $\delta_k(X) = 0$. The first examples are the tangent developable to a non-degenerate curve or cones of arbitrary dimension. Indeed they are 0-defective as every variety but by imposing tangency at a general point, we get tangency along the ruling passing through the point.

Varieties for which a hyperplane tangent at $k+1$, $k \geq 0$, general points is tangent along a positive dimensional subvariety are called *k-weakly defective varieties*, according to Chiantini and Ciliberto, [**CC**]. In [**CC**] many interesting properties of these varieties are investigated and a refined Terracini Lemma is proved, also putting in modern terms the classification of k-weakly defective irreducible surfaces obtained classically by Scorza, [**S2**], and Terracini, [**T2**]. Let us remark that, as shown in [**CC**], for every $k \geq 1$ there exist smooth varieties of dimension greater than one which are k-weakly defective but not k-secant defective.

As another application, we study the dimension of the projection of a variety from linear subspaces generated by general tangent spaces. Terracini Lemma says that we are projecting from a general tangent space to the related higher secant variety. As we have seen when the center of projection L cuts the variety it is difficult to control the dimension of the image of X under projection because we do not know a priori how a general tangent space intersects L. In the case of $L = T_z S^{k-1}X$ this information is encoded in the dimension of S^kX and of the defect $\delta_k(X)$ as we immediately see. In chapter **??** we shall see how the degree of the projections from $T_z S^k X$ is related to the number of $(k+2)$-secant $\mathbb{P}^{k+1}$ passing through a general point of $S^{k+1}X$, a problem dubbed as *Bronowski's conjecture*, [**B1**], and partially solved in [**CMR**]. Projections from tangent spaces, or more generally from $T_z S^k X$, were a classical tool of investigation, [**Ca**], [**E1**], [**S1**], [**S4**], [**B1**], [**B2**], and were recently used to study classical and modern problems, [**CC**], [**CMR**], [**CR2**].

1.3.8. Proposition. (**Projections from tangent spaces**) Let $X \subset \mathbb{P}^N$ be an irreducible, non-degenerate closed subvariety. Let $n = \dim(X)$ and suppose char(K)=0 and $N \geq s_k$, $k \geq 1$, where $s_k = s_k(X)$. Set $\delta_k = \delta_k(X)$. Let $x_1, \ldots, x_k \in X$ be k general points and let $L =< T_{x_1}, \ldots, T_{x_k} >$ and $\pi_k = \pi_L : X \to X' \subset \mathbb{P}^{N-s_{k-1}(X)-1}$. Then $\dim(L) = s_{k-1}(X) = s_{k-1}$ and, if $X'_k = \pi_k(X) \subset \mathbb{P}^{N-s_{k-1}-1}$, then

(1) $\dim(X'_k) = s_k - s_{k-1} - 1 = n - \delta_k$;
(2) suppose $N \geq (k+1)n+k$ and and $s_{k-1} = kn+k-1$, i.e. if $\delta_{k-1} = 0$. Then $s_k = (k+1)n + k$ (or equivalently $\delta_k = 0$) if and only if $\dim(X'_k) = n$; if and only if $\pi_k : X \dashrightarrow X'_k \subset \mathbb{P}^{N-kn+k}$ is dominant. In particular if $N = (k+1)n + k$ and if $s_{k-1} = kn + k - 1$, then $S^kX = \mathbb{P}^{(k+1)n+k}$ if and only if $\pi_k : X \dashrightarrow \mathbb{P}^n$ is dominant.

Proof. If $z \in< x_1, \ldots, x_k >$ is a general point, then z is a general point of $S^{k-1}X$ and by Terracini lemma $s_{k-1} = \dim(T_x S^{k-1}X) = \dim(< T_{x_1}, \ldots, T_{x_k} >)$. By equation 1.2.7 we get $\dim(X'_k) = \dim(S(T_z S^{k-1}X, X)) - s_{k-1} - 1 = s_k - s_{k-1} - 1 = n - \delta_k$. The other claims are only reformulations of part 1). $\square$

A complete description of $S^l X'_k$ in terms of higher secant varieties of X is possible and the dimensions $s_l(X'_k)$ are easily expressible as functions of the $s_m(X)$, i.e. $\delta_k(X'_k)$ is controlled by $\delta_m(X)$ and viceversa. These remarks and the possibility of constructing explicit rational maps reveals the importance of projections from tangent spaces.

We study via Terracini Lemma the tangent space to the entry locus of SX at a general point of it. As a minimal generalization we can define the projections onto the i-factor $\phi_i : X_1 \times X_2 \to X_i$ and for $z \in S(X_1, X_2)$, define $\Sigma_z(X_i) = \phi_i(p_2(p_1^{-1}(z)))$, where the morphism p_i's are the map used for the definition of the join. We remark that $\dim(\Sigma_z(X_1)) = \dim(\Sigma_z(X_2)) = \dim(X_1) + \dim(X_2) + 1 - \dim(S(X_1, X_2))$. With this notation we get the following result.

1.3.9. Proposition. *Let $X, Y \subset \mathbb{P}^N$ be closed irreducible subvarieties and assume* char(K)=0. *Suppose $S(X,Y) \supsetneq X$ and $S(X,Y) \supsetneq Y$ to avoid trivialities. If $z \in S(X,Y)$ is a general point, if $x \in \Sigma_z(X)$ is a general point and if $< z, x > \cap Y = y \in \Sigma_z(Y)$, then y is a smooth point of $\Sigma_z(Y)$,*

$$T_x\Sigma_z(X) = T_xX \cap < x, T_y\Sigma_z(Y) >= T_xX \cap < x, T_yY >,$$
$$T_y\Sigma_z(Y) = T_yY \cap < y, T_x\Sigma_z(X) >= T_yY \cap < y, T_xX >$$

and

$$T_xX \cap T_yY = T_x\Sigma_z(X) \cap T_y\Sigma_z(Y).$$

In particular for $z \in SX$ general point, X not linear, and for $x \in \Sigma_z(X)$ general point, we have that, if $< x, z > \cap X = y \in \Sigma_z(X)$, then y is a smooth point of $\Sigma_z(X)$,

$$T_x\Sigma_z(X) = T_xX \cap < x, T_y\Sigma_z(X) >= T_xX \cap < x, T_yY >$$

and

$$T_xX \cap T_yX = T_x\Sigma_z(X) \cap T_y\Sigma_z(X).$$

Proof. Let us remark that by assumption and by the generality of z and of x, we can suppose that $y \notin T_xX$ and that $x \notin T_yY$.

Take $S(z, \Sigma_z(X)) = S(z, \Sigma_z(Y))$. Then $\dim(S(z, \Sigma_z(X))) = \dim(\Sigma_z(X)) + 1$. If $u \in < z, x >=< z, y >$ is a general point, then $T_uS(z, \Sigma_z(X)) =< z, T_x\Sigma_z(X) >= \mathbb{P}^{\dim(S(z,\Sigma_z(X)))}$ because $z \notin T_xX$. In particular u is a smooth point of $S(z, \Sigma_z(X))$. By Terracini Lemma, we get $T_uS(z, \Sigma_z(X)) \supseteq < z, T_y\Sigma_z(Y) >$, which together with $z \notin T_yY$ yields $\dim(T_y\Sigma_z(Y)) = \dim(\Sigma_z(Y))$ so that $y \in \Sigma_z(Y)$ is a smooth point. Moreover,

$$T_x\Sigma_z(X) \subseteq T_uS(z, \Sigma_z(Y)) =< z, T_y\Sigma_z(Y) >=< x, T_y\Sigma_z(Y) >\subseteq < x, T_yY > .$$

Since $T_x\Sigma_z(X) \subseteq T_xX$, to conclude it is enough to observe that

$$\begin{aligned}\dim(T_xX \cap < x, T_yY >) &= \dim(X) + \dim(Y) + 1 - \dim(< T_xX, T_yX >)\\ &= \dim(\Sigma_z(X)) = \dim(T_x\Sigma_z(X)).\end{aligned}$$

The other claims follows from symmetry between x and y or are straightforward. □

1.4. Characterizations of the Veronese surface in $\mathbb{P}^5$ according to del Pezzo, Bertini and Severi and classification of algebraic varieties in $\mathbb{P}^N$, $N \geq \dim(X) + 3$ with $\dim(SX) = \dim(X) + 2$

In this section, as a beautiful application of the definitions and tools introduced in this chapter, we prove various characterizations of the Veronese surface in $\mathbb{P}^5$ among irreducible non-degenerate surfaces in $\mathbb{P}^N$, not cones, $N \geq 5$, having special geometrical properties. We also classify varieties in $\mathbb{P}^N$, $N \geq \dim(X) + 3$ with

$\dim(SX) = \dim(X) + 2$, a result due to Edwards for $\dim(X) \geq 3$, [**Ew**], and outlined and essentially solved by Scorza in [**S1**], as we shall see below. These results serve also as a motivation for the further generalizations of this classical material in the next chapters.

The proof we propose here is the most "elementary" we are aware of since it not based on any result involving dual varieties, contact loci, flatness and so on. We essentially use the previous results and the elementary fact that for an irreducible curve, not a line, supposing char(K)=0, a general tangent line to the curve at a point is tangent to it only at that point. This is an easy property which is immediately reduced to the analogous statement for plane curves by a linear projection. For plane curves it simply says that the dual curve of a plane curve has only a finite number of singular points. We followed a suggestion of Gaetano Scorza in [**S1**], footnote at page 197: *"Non mi sembra inutile far notare come partendo da un' osservazione analoga a quella del testo si possa arrivare alla dimostrazione del teorema del prof. Del Pezzo [n.d.A.: e del Prof. Edwards] in modo abbastanza rapido e semplice".*

1.4.1. Theorem. (**Characterizations of the Veronese surface**) *Let $X \subset \mathbb{P}^N$, $N \geq 5$, be a non-degenerate irreducible surface, not a cone. Then $N = 5$ and X is projectively equivalent to the Veronese surface in $\mathbb{P}^5$, $\nu_2(\mathbb{P}^2) \subset \mathbb{P}^5$, if and only if one of the following equivalent conditions holds:*

(1) *if $x, y \in X$ are general points, then $T_xX \cap T_yX \neq \emptyset$ ([**dP**]);*
(2) $\dim(SX) = 4$ ([**Sev1**])*;*
(3) *X contains a two dimensional family of irreducible conics* ([**Be**], pg. 392)*.*

First of all, by Terracini Lemma if 1) holds, then $\dim(SX) \leq 4$ but since X is non-degenerate part 3) of proposition 1.2.2 implies $\dim(SX) = 4$. By Terracini Lemma 2) implies that $T_xX \cap T_yX$ consists of a point. Also condition 3) implies 1) (or 2)). Indeed, there exists at least a conic $C_{x,y}$ passing through the general points x and y so that $p_{x,y} := T_xC_{x,y} \cap T_yC_{x,y} \subseteq T_xX \cap T_yX$ and in fact equality holds. So it will sufficient to show that if $T_xX \cap T_yX = p_{x,y}$ is a point, then X is projectively equivalent to the Veronese surface, which is Del Pezzo's theorem, [**dP**] and [**Be**], pg. 394. During the proof of the preliminaries lemma the apparently more general fact that X contains a two dimensional linear system of Cartier divisors of selfintersection 1, which are conics in the fixed embedding, is seen to be a consequence of condition 1). This is essentially also Bertini's proof that 3) characterizes the Veronese surface, see [**Be**], pg. 392. So all the equivalences and the necessary tools will be established.

Let us recall that if $X \subset \mathbb{P}^N$ is an irreducible projective non-degenerate variety of dimension $n = \dim(X)$, then $\dim(SX) \geq n + 1$ and that equality implies $N = n + 1$, see proposition 1.2.2. Hence if $\mathrm{codim}(X) > 1$, $\dim(SX) \geq n + 2$. Suppose $\dim(SX) = n + 2$. If $N = n + 2$, then $SX = \mathbb{P}^N$ and there is no particular restriction on X and clearly there exist infinitely many examples. If $N > n + 2$ the complete classification of varieties with $\dim(SX) = n + 2$ is contained in the following theorem of Scorza-Edwards.

1.4.2. Theorem. (Scorza, [**S1**], Edwards,[**Ew**]) *Let $X \subset \mathbb{P}^N$ be an irreducible projective variety of dimension $n = \dim(X) \geq 3$. Assume $N \geq n + 3$ and that $\dim(SX) = n + 2$. Then either X is a cone over a curve or $N = n + 3$ and X is* *a cone over the Veronese surface in* $\mathbb{P}^5$*. On the contrary such varieties enjoy those geometrical properties.*

Equivalently, $X \subset \mathbb{P}^N$*,* $N \geq n+3$*, is a cone over a curve or a cone over the Veronese surface in* $\mathbb{P}^5$ *if and only if it contains an irreducible two dimensional family of divisors which are quadric hypersurfaces in the fixed embedding. The general member of this family is a reducible quadric if and only if* X *is a cone over a curve.*

Once again, it is clear that if through two general point there passes a quadric hypersurface of dimension $n-1$, then for a general point $z \in SX$,

$$2n+1-\dim(SX) = \dim(\Sigma_z(X)) \geq n-1,$$

yields $\dim(SX) \leq n+2$ and hence equality by the non-degenerateness of $X \subset \mathbb{P}^N$ and by the hypothesis $N \geq n+3$. The other implication will follow once again by the next lemma of Scorza.

1.4.3. Lemma. (**Scorza Lemma, [S1], footnote pg. 197**) *Let* $X \subset \mathbb{P}^N$ *be an irreducible non-degenerate projective variety of dimension* $n = \dim(X)$*. Suppose* $N \geq n+3$ *and* $\dim(SX) = n+2$*. Then:*

(1) *the closure of the general fiber of the tangential projection of* X *from a general point* $x \in X$ *onto the irreducible curve* $C_x \subset \mathbb{P}^{N-n-1}$*,* $\pi_x : X \dashrightarrow C_x \subset \mathbb{P}^{N-n-1}$ *is either a* $\mathbb{P}^{n-1}$ *or an irreducible quadric hypersurface of dimension* $n-1$*. The first case occurs if and only if* X *is a cone over a curve. Moreover,* X *contains a two dimensional family of quadric hypersurfaces, whose general member is the union of two* $\mathbb{P}^{n-1}$ *if and only if* X *is a cone over a curve.*

(2) *If* X *is not a cone over a curve, then* C_x *is an irreducible conic, so that* $N = n+3$ *and the general fiber of* $\pi_x : X \dashrightarrow C_x \subset \mathbb{P}^2$ *is an irreducible quadric hypersurface of dimension* $n-1$*.*

Proof. Let $x, y \in X$ general points. The image of the tangential projection $\pi_x : X \dashrightarrow X'_x = C_x \subset \mathbb{P}^{N-n-1}$, or of $\pi_y : X \dashrightarrow X'_y = C_y \subset \mathbb{P}^{N-n-1}$, is an irreducible non-degenerate curve by proposition 1.3.8. Moreover π_x is defined at y, respectively π_y is defined at x, since being general points they do not belong to $\mathrm{Vert}(X)$. Let F_y denote the closure of the irreducible component of the fiber of π_x passing through y, respectively F_x denote the closure of irreducible component of the fiber of π_y passing through x. By generic smoothness they are reduced varieties of dimension $n-1$ since they are generically smooth irreducible varieties of dimension $n-1$. Moreover, by definition of π_x, respectively π_y, we have $F_y \subset < T_xX, y > \cap X$, respectively $F_x \subset < T_yX, x > \cap X$.

If $F_x \subseteq T_xX \cap X$, then by the generality of x and y, $F_y \subseteq T_yX \cap X$ so that $F_x \cup F_y \subset \Pi_{x,y} := < T_xX, y > \cap < T_yX, x > = \mathbb{P}^n$.

Suppose that F_x is not contained in $T_xX \cap X$ so that also F_y is not contained in $T_yX \cap X$. Let $\mathcal{C}_x = S(T_xX, C_x)$ be the cone over C_x of vertex T_xX and let $\mathcal{C}_y = S(T_yX, C_y)$ be the cone over C_y of vertex T_yX. By the generality of x, respectively y, the point $\pi_x(y) \in C_x$, respectively $\pi_y(x) \in C_y$, is a general point on C_x, respectively on C_y, so that the tangent space

$$< T_xX, T_{\pi_x(y)}C_x > = < T_xX, T_yX > = < T_{\pi_y(x)}C_y, T_yX >$$

is tangent to $\mathcal{C}_x$, respectively $\mathcal{C}_y$, exactly along $< T_xX, \pi_x(y) > \setminus T_xX = < T_xX, y > \setminus T_xX$, respectively $< T_yX, \pi_y(x) > \setminus T_yX = < T_yX, x > \setminus T_yX$. Recall that a general tangent line to a curve is tangent to it only at one point. Since $X \subseteq \mathcal{C}_x \cap \mathcal{C}_y$, the locus $Y \subset X$ of smooth points of $X \setminus ((T_xX \cap X) \cup (T_yX \cap X)$ at which the linear space $< T_xX, T_yX >$ is tangent is contained in the linear space $\Pi_{x,y} =< T_xX, y > \cap < T_yX, x >= \mathbb{P}^n$. In our hypothesis, F_x, respectively F_y, has an open dense set in common with Y, yielding that $F_x \cup F_y$ is contained in $\Pi_{x,y}$.

More generally, using the same argument, we get that the closure of the fibers $\pi_x^{-1}(\pi_x(y))$ and of $\pi_y^{-1}(\pi_y(x))$ are both contained in $\Pi_{x,y}$. In conclusion, in any case $F_x \cup F_y \subseteq \overline{\pi_y^{-1}(\pi_y(x))} \cup \overline{\pi_x^{-1}(\pi_x(y))} \subseteq \Pi_{x,y} = \mathbb{P}^n$.

Since the secant line $< x, y >$ is general, it is not a trisecant line. The line $< x, y >$ is contained in $\Pi_{x,y}$ so that

$$2 \geq \deg(\overline{\pi_x^{-1}(\pi_x(y))} \cup \overline{\pi_y^{-1}(\pi_y(x))}) \geq \deg(F_x \cup F_y) \geq 2,$$

where the last inequality holds since it cannot clearly be $F_x = F_y = \mathbb{P}^{n-1}$, the line $< x, y >$ being a proper secant line. In conclusion, either $\overline{\pi_y^{-1}(\pi_y(x))} = F_x = F_y = \overline{\pi_x^{-1}(\pi_x(y))}$ is an irreducible quadric hypersurface of dimension $n-1$ passing through x and y or $F_x = \overline{\pi_y^{-1}(\pi_y(x))}$ and $F_y = \overline{\pi_x^{-1}(\pi_x(y))}$ are $\mathbb{P}^{n-1}$'s intersecting in the linear space $L_{x,y} = T_xX \cap T_yX = \mathbb{P}^{n-2}$. In the last case the linear space $L = L_{x,y}$ does not vary by moving y in X because otherwise the linear spaces F_y would describe a $\mathbb{P}^n$ contained in X. Then $\mathbb{P}^{n-2} = L \subseteq \mathrm{Vert}(X) = \mathbb{P}^l$, $l \leq n-2$, forces $\mathrm{Vert}(X) = \mathbb{P}^{n-2}$ so that X is a cone over a curve by proposition 1.2.6. On the contrary if X is a cone over a curve, clearly the $\mathbb{P}^{n-1}$'s passing through two general points $x, y \in X$ are contracted by π_y, respectively π_x, so that $F_x \cup F_y$ is a reduced quadric hypersurface.

To prove part 2) it suffices to remark that if X is not a cone over a curve, then by the previous analysis two general points on X are connected by an irreducible quadric hypersurface, which dominates C_x, so that, since $X \subset \mathbb{P}^N$ is non-degenerate, $C_x \subset \mathbb{P}^{N-n-1}$ is an irreducible non-degenerate conic, yielding $N - n - 1 = 2$. □

As a corollary of Scorza Lemma we get the information about the entry locus of a variety $X \subset \mathbb{P}^N$, $N \geq n+3$ with $\dim(SX) = n+2$, the original key observation of Severi for $n = 2$ in his proof of the characterization of the Veronese surface, see [**Sev1**].

1.4.4. Corollary. ([**Sev1**]) *Let $X \subset \mathbb{P}^N$, $N \geq n+3$, be an irreducible non-degenerate projective variety of dimension n such that $\dim(SX) = n+2$. Let $z \in SX$ be a general point and let notation as in lemma 1.4.3. Then $\Sigma_z(X) = F_x \cup F_y$ is a quadric hypersurface which is reducible if and only if X is a cone over a curve.*

Proof. Let notation as in the above lemma. Then if $z \in < x, y >$ is general, by corollary 1.3.6 $T_zSX =< T_xX, T_yX >$ is tangent to X along $\Sigma_z(X) \setminus \mathrm{Sing}(X)$. By Scorza Lemma $\Sigma_z(X) \setminus \mathrm{Sing}(X) \subset \Pi_{x,y} = \mathbb{P}^n$ so that $\Sigma_z(X)$ is a hypersurface of degree at least 2 in $\Pi_{x,y}$ since $z \notin X$ (it it were a linear space, then $SX = X$ and X would be linear). Then $\Sigma_z(X)$ is a quadric hypersurface by the trisecant lemma and the conclusion follows by arguing as in the previous lemma. □

We restrict ourselves for a moment to the case of surfaces and prove that, if X is not a cone, two general fibers of $\pi_x : X \dashrightarrow C_x \subset \mathbb{P}^2$ are linearly equivalent Cartier divisors intersecting transversally at x; and more precisely that every fiber of π_x

is a smooth conic so that the closure of two arbitrary fibers are linear equivalent Cartier divisors, which are smooth conics in the fixed embedding.

1.4.5. Lemma. (Bertini, [**Be**]) *Let $X \subset \mathbb{P}^5$ be a non-degenerate irreducible projective surface, not a cone, such that* $\dim(SX) = 4$*. Then $(T_xX \cap X)_{\text{red}} = x$, the closure of every fiber of $\pi_x : X \dashrightarrow C_x \subset \mathbb{P}^2$ is a smooth conic and two fibers of π_x are linearly equivalent Cartier divisors on X intersecting transversally at x.*

Proof. Suppose that for a general point $x \in X$, there exists $p_x \in T_xX \cap X$, $p_x \neq x$. Fix a general x and take a general point $y \in X$. By lemma 1.4.3, if $\pi_x : X \dashrightarrow C_x$ is the tangential projection, then C_x is a smooth conic. Take the line $< y, p_y >$. Thus π_x is defined at y, since $\mathrm{Vert}(X) = \emptyset$ and since x, y are general points. It cannot be $\pi_x(y) \neq \pi_x(p_y)$, because otherwise the line $< y, p_y >$ would not cut T_xX so that it would project onto $T_{\pi_x(y)}C_x$ and this line would cut C_x at least in 3 points counted with multiplicity, contrary to the fact that C_x is a conic. If $\pi_x(y) = \pi_x(p_y)$, then the line $< y, p_y >$ cuts T_xX necessarily at $p_{x,y} = T_xX \cap T_yX$ and the line $< y, p_y >=< y, p_{x,y} >= T_yF_y$ would cut the smooth conic F_y in at least 3 points counted with multiplicity, which is impossible.

Therefore two fibers of π_x can intersect only at x and they are linearly equivalent divisors by definition. The closure of each fiber is then a Cartier divisor which is a conic on X passing necessarily through $x = (T_xX \cap X)_{\text{red}}$. Since $(T_xX \cap X)_{\text{red}} = x$, there is no line through X and the closure of every fiber is a smooth conic.

If two general fibers meet along a fixed tangent direction $l \subset T_xX$ at x, then the tangent spaces at two general points of these fibers, let us say $y \in X$, respectively $z \in X$, will cut the fixed line in different points since X is not a cone (otherwise $p_{x,y} = p_{x,z} \in \mathrm{Vert}(X)$ and X would be a cone). Then $S(l, SX) = SX$ by Terracini lemma since $\dim(< l, < T_zX, T_yX >>) = \dim(< T_zX, T_yX >) = \dim(SX)$. This forces $x \in l \subseteq \mathrm{Vert}(SX)$, which by the generality of $x \in X$, yields $X \subseteq \mathrm{Vert}(SX) = \mathbb{P}^l$, $l \leq 2$ (recall that SX is not linear and has dimension 4), i.e. $X = \mathbb{P}^2$. □

We can easily prove theorem 1.4.1.

Proof. (1^{st} proof of theorem 1.4.1). By Scorza Lemma and by lemma 1.4.5 the fibers of the tangential projections at x and at y, $x, y \in X$ general points, are linearly equivalent Cartier divisors of selfintersection 1. Moreover, since there exists a conic through x and y which is a fiber of both projections, we constructed a base point free two dimensional linear system of Cartier divisors on X of autointerction 1. The associated morphism $\phi : X \to \mathbb{P}^2$ is birational.

Let $\psi : \mathbb{P}^2 \dashrightarrow X \subset \mathbb{P}^5$ be the composition of $\phi^{-1} : \mathbb{P}^2 \dashrightarrow X$ with the inclusion $i : X \hookrightarrow \mathbb{P}^5$. Since lines in $\mathbb{P}^2$ are mapped into the two dimensional linear system of divisors constructed before, which are conics in the fixed embedding, the map ψ is given by a linear system of conics of dimension 5, i.e. by the complete linear system of conics, so that $\psi : \mathbb{P}^2 \to X$ is an isomorphism and X is projectively equivalent to $\nu_2(\mathbb{P}^2) \subset \mathbb{P}^5$. □

Proof. (2^{nd} proof of theorem 1.4.1). Fix a general point $x \in X$ and consider the tangential projection $\pi_x : X \dashrightarrow C_x \subset \mathbb{P}^2$. This rational map resolves to a morphism $\widetilde{\pi_x} : Bl_xX \to C_x \simeq \mathbb{P}^1$ such that every fiber is isomorphic to $\mathbb{P}^1$, i.e. it is a $\mathbb{P}^1$-bundle over $\mathbb{P}^1$ and in particular Bl_xX and hence X are smooth surfaces. Since Bl_xX contains the (-1)-curve E as a section of $\widetilde{\pi_x}$, then $Bl_xX \to C_x$ is isomorphic as a $\mathbb{P}^1$-bundle to $\pi : \mathbb{F}_1 \to \mathbb{P}^1$. By contracting E we get $X \simeq \mathbb{P}^2$. Since it contains conics in the fixed embedding and $N = 5$, it is necessarily the complete 2-Veronese embedding of $\mathbb{P}^2$. □

The reason for which we included the second proof, apparently more complicated, is for the analogy with the argument used by Mori to prove Hartshorne's conjecture that $\mathbb{P}^N$ is the only smooth projective variety of dimension N having ample tangent bundle, see [**Mo1**]. There one shows that for a general point $x \in X$ $\mathbb{P}(\mathcal{T}_x X) \simeq \mathbb{P}^{N-1}$ and then by blowing-up x, it proves that $Bl_x X \to \mathbb{P}^{N-1}$ is a $\mathbb{P}^1$-bundle, see *loc. cit.*

Now we can prove theorem 1.4.2.

Proof. (of theorem 1.4.2) Suppose X is not a cone over a curve. By lemma 1.4.3 we get $N = n+3$ and that through x there passes a line. In fact $T_x F_x \cap F_x \subset X$ is a quadric in $T_x F_x = \mathbb{P}^{n-1}$ and since $n - 1 \geq 2$, through the point x there passes at least a line $l_x \subset T_x X \cap F_x \subset X$. Then X is a cone by lemma 1.2.7 and since it is not a cone over a curve, its linear section with a general $\mathbb{P}^5 \subset \mathbb{P}^{n+3}$ is an irreducible non-degenerate surface $Y \subset \mathbb{P}^5$, which is not a cone, and such $\dim(SY) \leq 4$. By theorem 1.1.6 $Y \subset \mathbb{P}^5$ is a Veronese surface and the conclusion follows. □

It is worth of note also the following geometrical characterization of the Veronese surface given by Ran: it is the unique smooth surface in $\mathbb{P}^5$ which is not contained in an irreducible 3-fold non-singular along the surface, see [**Ra**].

1.5. Dual varieties and contact loci of general tangent linear spaces

Let $X \subset \mathbb{P}^N$ be a projective, irreducible non-degenerate variety of dimension n; let $\mathrm{Sm}(X) := X \setminus \mathrm{Sing}(X)$ be the locus of non-singular points of X. By definition $\mathrm{Sm}(X) = \{x \in X \,:\, \dim(T_x X) = n\}$.

If we take an hyperplane section of X, $Y = X \cap H$, where $H = \mathbb{P}^{N-1}$ is an arbitrary hyperplane, then for every $y \in Y$ we get

$$T_y Y = T_y X \cap H. \tag{1.5.1}$$

Since Y is a pure dimensional scheme of dimension $n-1$, we see that $\mathrm{Sing}(Y) \setminus (\mathrm{Sing}(X) \cap H) = \{y \in Y \setminus \mathrm{Sing}(X) \cap Y \,:\, T_y X \subseteq H\}$, which is an open subset in the locus of points of X at which H is tangent. In particular to show that an hyperplane section has non-singular points, we have to exhibit an hyperplane H which is not tangent at all the points in which it intersects X. It naturally arises the need of patching together all the "bad" hyperplanes and eventually show that there always exists an hyperplane section of X, non-singular at least outside $\mathrm{Sing}(X)$. Since hyperplane can be naturally thought as points in the dual projective space ($\mathbb{P}^{N*}$, we can define a subvariety of $\mathbb{P}^{N*}$ parametrizing hyperplane sections which are singular also outside $\mathrm{Sing}(X)$. This locus is the so-called *dual variety*.

1.5.1. Definition. (**Dual variety**) Let $X \subset \mathbb{P}^N$ be as above and let

$$\mathcal{P}_X := \overline{\{((x, H) \,:\, x \in \mathrm{Sm}(X),\ T_x X \subseteq H\}} \subset X \times \mathbb{P}^{N*},$$

the so called *conormal variety of X*.

Let us consider the projections of $\mathcal{P}_X$ onto the factors X and $\mathbb{P}^{N*}$,

$$\begin{tikzcd} & \mathcal{P}_X \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ X & & \mathbb{P}^{N*}. \end{tikzcd}$$

The dual variety to X, X^*, is the scheme-theoretic image of $\mathcal{P}_X$ in $\mathbb{P}^{N*}$, i.e. the algebraic variety

$$X^* := p_2(\mathcal{P}_X) \subseteq \mathbb{P}^{N*}.$$

The set $\mathcal{P}_X$ is easily seen to be a closed subset. For $x \in \mathrm{Sm}(X)$, we have $p_1^{-1}(x) \simeq (T_xX)^* = \mathbb{P}^{N-n-1} \subset \mathbb{P}^{N*}$. Then the set $\mathcal{P}_X$ is irreducible since $p_1^{-1}(\mathrm{Sm}(X)) \to \mathrm{Sm}(X)$ is a $\mathbb{P}^{N-n-1}$-bundle and clearly $\dim(\mathcal{P}_X) = N-1$. Then $\dim(X^*) \leq N-1$ and the *dual defect of* X, $\mathrm{def}(X)$, is defined as

$$\mathrm{def}(X) = N - 1 - \dim(X^*) \geq 0.$$

A variety is said to be *reflexive* if the natural isomorphism between $\mathbb{P}^N$ and $\mathbb{P}^{N**}$ induces an isomorphism between $\mathcal{P}_X$ and $\mathcal{P}_{X^*}$. This clearly implies that the natural identification between $\mathbb{P}^N$ and $\mathbb{P}^{N**}$ induces an isomorphism $X \simeq X^{**} = (X^*)^*$.

Let us take $H \in X^*$. By definition

$$C_H := C_H(X) = p_2^{-1}(H) = \overline{\{x \in \mathrm{Sm}(X) \,:\, T_xX \subset H\}}$$

is exactly the closure of non-singular points of X where H is tangent to X, it is not empty so that $H \cap X$ is singular outside $\mathrm{Sing}(X)$. On the contrary if $H \notin X^*$, the hyperplane section $H \cap X$ can be singular only along $\mathrm{Sing}(X)$. This is the classical "Bertini theorem".

In particular we proved the following result.

1.5.2. Theorem. *Let $X \subset \mathbb{P}^N$ be a projective, irreducible non-degenerate variety of dimension $n =$. Then for every $H \in (\mathbb{P}^N)^* \setminus X^*$ the divisor $H \cap X$ is non-singular outside* $\mathrm{Sing}(X)$.

In particular if X has at most a finite number of singular points $p_1, \ldots, p_m$, then for every $H \notin X^* \cup (p_1)^* \cup \ldots \cup (p_m)^*$, the hyperplane section $H \cap X$ is a non-singular subscheme of pure codimension 1.

Later we shall see that if $n \geq 2$, then every hyperplane section is connected. For non-singular varieties, the hyperplane sections with hyperplanes $H \notin X^*$, being connected and non-singular are also irreducible so that are irreducible non-singular algebraic varieties.

To justify the name of conormal variety for $\mathcal{P}_X$ and to get some practice with the definitions, one could solve the following exercise. It is also a training for the language of locally free sheaves and their projectivizations.

1.5.3. Exercise. Prove the following facts.

(1) Let $X \subsetneq \mathbb{P}^M \subsetneq \mathbb{P}^N$ be a degenerate variety. Prove that $X^* \subset \mathbb{P}^{N*}$ is a cone of vertex $\mathbb{P}^{M*} = \mathbb{P}^{N-M-1} \subset \mathbb{P}^{N*}$ over the dual variety of X in $\mathbb{P}^M$. Suppose $X = S(L, X')$ is a cone of vertex $L = \mathbb{P}^l$, $l \geq 0$, over a variety $X' \subset M = \mathbb{P}^{N-l-1}$, $M \cap L = \emptyset$. Then $X^* \subset (\mathbb{P}^l)^* = \mathbb{P}^{N-l-1} \subset (\mathbb{P}^N)^*$ is degenerated. Is there any relation between X^* and the dual of X' in M?

Suppose $X \subset \mathbb{P}^N$ is a cone. Prove that $X^* \subset \mathbb{P}^{N*}$ is degenerated. Conclude that $X \subset \mathbb{P}^N$ is degenerated if and only if $X^* \subset \mathbb{P}^{N*}$ is a cone; and, dually, that $X \subset \mathbb{P}^N$ is a cone if and only if $X^* \subset \mathbb{P}^{N*}$ is degenerated.

(2) Let $C \subset \mathbb{P}^N$ be an irreducible non-degenerate projective curve. Then $p_2 : \mathcal{P}_C \to C^* \subset (\mathbb{P}^N)^*$ is a finite morphism so that $\operatorname{def}(C) = 0$.

(3) Let $X \subset \mathbb{P}^N$ be a non-singular variety, then $\mathcal{P}_X \simeq \mathbb{P}(\mathcal{N}^*_{X/\mathbb{P}^N}(1))$ (Grothendieck's notation), where $\mathcal{N}^*_{X/\mathbb{P}^N}(1)$ is the the twist of the conormal bundle of X in $\mathbb{P}^N$ by $\mathcal{O}_{\mathbb{P}^N}(1)$. Show that $p_2 : \mathcal{P}_X \to X^* \subset \mathbb{P}^{N*}$ is given by a sub-linear system of $|\mathcal{O}_{\mathcal{N}^*_{X/\mathbb{P}^N}(1)}(1)|$. (Hint: restrict Euler sequence to X and use the standard conormal sequence; interpret these sequences in terms of the associated projective bundles and of the incidence correspondence defining $\mathcal{P}_X$).

(4) Let $X \subset \mathbb{P}^N$ be a smooth complete intersection. Deduce by the previous exercise that $p_2 : \mathcal{P}_X \to X^* \subset \mathbb{P}^{N*}$ is a finite morphism so that $\dim(X^*) = N - 1$, i.e. $\operatorname{def}(X) = 0$ (Hint: show that $\mathcal{N}^*_{X/\mathbb{P}^N}(1)$ is a sum of very ample line bundles; deduce that $\mathcal{O}_{\mathcal{N}^*_{X/\mathbb{P}^N}(1)}(1)$ is very ample and finally that $p_2 : \mathcal{P}_X \to X^* \subset \mathbb{P}^{N*}$ is a finite morphism).

(5) Suppose char(K)=0 and let $C \subset \mathbb{P}^2$ be an irreducible curve, not a line. Show that C^* is an irreducible curve of degree at least 2. Take a tangent line at a point $x \in C$. Show that if T_xC is tangent at another point $y \in C$, $y \neq x$, then the point $(T_xC)^* \in C^*$ is a singular point of C^*. Deduce that if char(K)=0, then a general tangent line is tangent to C only at one point. Deduce that the same is true for an irreducible curve $C \subset \mathbb{P}^N$, $N \geq 3$.

(6) Let $X = \mathbb{P}^1 \times \mathbb{P}^n \subset \mathbb{P}^{2n+1}$, $n \geq 1$, be the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^n$. Identify $\mathbb{P}^{2n+1}$ with the projectivization of the vector space of $2 \times n + 1$ matrices and show that, due to the fact that there are only two orbits for the action of $GL(2)$ on $\mathbb{P}^N$ and on $(\mathbb{P}^N)^*$, $(\mathbb{P}^1 \times \mathbb{P}^n)^* \simeq \mathbb{P}^1 \times \mathbb{P}^n$ so that $\operatorname{def}((\mathbb{P}^1 \times \mathbb{P}^n)) = n - 1$. Interpret this result geometrically and reverse the construction for $n = 2$ to show directly that $X = X^*$.

(7) Use the same argument as above to show that if $X = \nu_2(\mathbb{P}^2) \subset \mathbb{P}^5$, or if $X = \mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^8$, then $X^* \simeq SX$ and $SX^* \simeq X$.

As we have seen the dual varieties encode informations about the tangency of hyperplanes. Terracini Lemma says that linear spaces containing tangent spaces to higher secant varieties are tangent along $(\Sigma_z^k)_{\mathrm{red}} \setminus \operatorname{Sing}(X)$, see corollary 1.3.6. Hence if the maximal dimension of the fibers of $p_2 : \mathcal{P}_X \to X^* \subset \mathbb{P}^{N*}$ is an upper bound for $\delta_k(X)$ as soon as $S^kX \subsetneq \mathbb{P}^N$, as we shall immediately see. More refined versions with the higher Gauss maps γ_m, see below, can be formulated but in those cases the condition expressed by the numbers $\varepsilon_m(X)$, which can be defined as below, is harder to control.

1.5.4. Theorem. (**Dual variety and higher secant varieties**) *Let $X \subset \mathbb{P}^N$ be an irreducible non-degenerate projective variety. Let $p_2 : \mathcal{P}_X \to X^* \subset \mathbb{P}^{N*}$ be as above and let $\varepsilon(X) = \max\{\dim(p_2^{-1}(H)),\, H \in X^*\}$. If $S^kX \subsetneq \mathbb{P}^N$, then $\delta_k(X) \leq \varepsilon(X)$. In particular if $p_2 : \mathcal{P}_X \to X^*$ is a finite morphism, then* $\dim(S^kX) = \min\{(k+1)n + k, N\}$.

Proof. Let $z \in S^kX$ be a general point. There exists $x \in \Sigma_z^k(X) \cap \mathrm{Sm}(X)$ and moreover T_zS^kX is contained in a hyperplane H. Then $p_1(p_2^{-1}(H)) \supseteq \mathrm{Sing}(X \cap H) \setminus (\mathrm{Sing}(X) \cap H)$ (and more precisely $\mathrm{Sing}(X \cap H) \setminus (\mathrm{Sing}(X) \cap H)$) contains the irreducible component of $\Sigma_z^k(X) \setminus (\mathrm{Sing}(X) \cap \Sigma_z^k(X))$ passing through x by corollary 1.3.6. Then $p_1(p_2^{-1}(H))$ has dimension at least $\delta_k(X) = \dim(\Sigma_z^k(X))$ and the conclusion follows. □

1.5.5. Corollary. (cfr. corollaries 1.2.3 and 1.3.6).
Let $X \subset \mathbb{P}^N$ be either an irreducible non-degenerate curve or a smooth non-degenerate complete intersection. Then

$$\dim(S^kX) = \min\{(k+1)n + k, N\}.$$

Proof. By exercise 1.5.3, we know that in both cases $p_2 : \mathcal{P}_X \to X^*$ is a finite morphism. □

More generally one would study the locus of points at which a general hyperplane is tangent, the so called *contact locus*. For reflexive varieties it is a linear space of dimension $\mathrm{def}(X)$. This is an interpretation of the isomorphism $X \simeq (X^*)^*$. One should be careful in the interpretation of the result: it does not mean that the hyperplane remains tangent along the whole "contact locus", see remark 1.3.7 and adapt it to the more general situation of a ruling of a cone. This is true only for non-singular varieties. In particulat reflexive varieties of positive dual defect contain positive dimensional families of linear spaces.

1.5.6. Proposition. *Let $X \subset \mathbb{P}^N$ be a reflexive variety. Then for $H \in \mathrm{Sm}(X^*)$,*

$$p_2^{-1}(H) = \overline{\{x \in \mathrm{Sm}(X) \, : \, T_xX \subset H\}} = (T_HX^*)^* = \mathbb{P}^{\mathrm{def}(X)}.$$

The following result will not be proved here but the reader can consult [**Ha**], pg. 208 for an elementary and direct proof. It is considered as a classical theorem, know at least to C. Segre.

1.5.7. Theorem. (**Reflexivity Theorem**) *Let $X \subset \mathbb{P}^N$ be an irreducible variety. Suppose* char(K)*=0. Then X is reflexive.*

Another natural and similar problem is to know if a general tangent space to a variety X is tangent at more than one point. During the discussion we will always suppose char(K)=0 to avoid artificial problems, since the natural ones are enough interesting.

We have seen in exercise 1.5.3 that for irreducible curves a general tangent space is tangent only at one point. On the other hand if X is a cone over a curve, we know that a a general tangent space is tangent exactly along the ruling passing through the point. The unique common feature of irreducible algebraic varieties from this point of view seems to be the linearity of the locus of points at which a general linear space is tangent.

1.5.8. Definition. (**Gauss maps**) Let $X \subset \mathbb{P}^N$ be an irreducible projective variety and let $m \geq n$. Let

$$\mathcal{P}_X^m := \overline{\{((x, L) \, : \, x \in \mathrm{Sm}(X), \; T_xX \subseteq L\}} \subset X \times \mathbb{G}(m, N).$$

Let us consider the projections of $\mathcal{P}^m_X$ onto the factors X and $(\mathbb{P}^N)^*$,

$$\begin{tikzcd} & S_X \arrow[dl, "p_1"'] \arrow[dr, "\gamma_m"] & \\ X & & \mathbb{G}(m,N). \end{tikzcd}$$

The variety of m-dimensional tangent subspaces to X, X^*_m, is the scheme-theoretic image of $\mathcal{P}^m_X$ in $\mathbb{G}(m,N)$, i.e. the algebraic variety

$$X^*_m := \gamma_m(\mathcal{P}^m_X) \subset \mathbb{G}(m,N).$$

For $m = N-1$, we recover the dual variety and its definition, while for $m = n$, we get the usual Gauss map $\mathcal{G}_X : X \dashrightarrow \mathbb{G}(n,N)$ which associates to a point $x \in \mathrm{Sm}(X)$ its tangent space T_xX. For such $x \in \mathrm{Sm}(X)$ $\mathcal{G}_X(x) := \gamma_n(x) = T_xX$.

If $X \subset \mathbb{P}^N$ is an hypersurface, then $n = N-1$ and clearly the Gauss map $\mathcal{G}_X : X \dashrightarrow \mathbb{P}^{N*}$ associates to a smooth point p of X its tangent hyperplane, so that in coordinates is given by

$$\mathcal{G}_X(p) = (\frac{\partial f}{\partial X_0}(p) : \ldots : \frac{\partial f}{\partial X_N}(p)).$$

The following theorem is once again a consequence of reflexivity and it is a generalization of proposition 1.5.6 and of the properties of cones. One can consult [**Z2**], pg. 21, for a proof.

1.5.9. THEOREM. (**Linearity of general contact loci**) *Let $X \subset \mathbb{P}^N$ be an irreducible projective non-degenerate variety. Assume* char(K)=0*. The general fiber of the morphism $\gamma_m : \mathcal{P}^m_X \to X^*_m$ is a linear space of dimension* $\dim(\mathcal{P}^m_X) - \dim(X^*_m)$*. In particular the closure of a general fiber of $\mathcal{G}_X : X \dashrightarrow X^*_n \subset \mathbb{G}(n,N)$ is a linear space of dimension $n - \dim(\mathcal{G}_X(X))$ so that a general linear tangent space is tangent along an open subset of a linear space of dimension $n - \dim(\mathcal{G}_X(X))$.*

To conclude the section and the chapter, we prove via Terracini Lemma a relation between X^* and $(S^kX)^*$, $k < k_0(X)$, assuming char(K)=0.

1.5.10. PROPOSITION. *Let $X \subset \mathbb{P}^N$ be an irreducible non-degenerate projective variety. Assume* char(K)=0 *and $SX \subsetneq \mathbb{P}^N$. Then $(SX)^* \subseteq \mathrm{Sing}(X^*) \subsetneq X^*$, i.e. a general bitangent hyperplane represents a singular point of X^*. More generally for a given $k \geq 2$ such that $k < k_0(X)$, we have $(S^kX)^* \subseteq \mathrm{Sing}((S^{k-1}X)^*) \subsetneq (S^{k-1}X)^*$, i.e. a general $(k+1)$-tangent hyperplane represents a singular point of $(S^{k-1}X)^*$.*

PROOF. Take $H \in (SX)^*$ general point. Then $H \supseteq T_zSX$, with $z \in SX$ general point. By corollary 1.3.6, H is tangent to X along $\Sigma_z(X) \setminus (\Sigma_z(X) \cap \mathrm{Sing}(X))$ so that $H \in X^*$. Since X is non-degenerate, then $z \notin X$ implies that the contact locus of H is not linear, yielding $H \in \mathrm{Sing}(X^*)$ by proposition 1.5.6.

Take more generally $H \in (S^kX)^*$ general and write $S^kX = S(X, S^{k-1}X)$. Then $H \subseteq T_zS^kX$, with $z \in S^kX$ general point. Then there exists $y \in \mathrm{Sm}(S^{k-1}X)$ with $y \in \Sigma^k_z(X)$ and such that $z \in < x, y >$, $x \in X$, $x \neq y$. By Terracini Lemma $T_zS^kX \supseteq T_yS^{k-1}X$ so that $H \in (S^{k-1}X)^*$. Since $x \in X$, $x \in \mathrm{Sing}(H \cap S^{k-1}X)$, so that $p_2^{-1}(H) \subseteq S^{k-1}X$ is not linear since once again $z \in S^kX \setminus S^{k-1}X$ by the non-linearity of S^kX. $\square$

Recall that to a non-degenerate irreducible closed subvariety $X \subset \mathbb{P}^N$ we associated an ascending filtration of irreducible projective varieties, see equation 1.2.5,

$$X = S^0X \subsetneq SX \subsetneq S^2X \subsetneq \ldots \subsetneq S^{k_0}X = \mathbb{P}^N.$$

The above proposition says that at least over a filed of characteristic zero, there exists also a strictly descending dual filtration:

$$X^* \supsetneq \mathrm{Sing}(X^*) \supseteq (SX)^* \supsetneq \ldots \supseteq (S^{k_0-2}X)^* \supsetneq \mathrm{Sing}((S^{k_0-2}X)^*) \supseteq (S^{k_0-1}X)^*.$$

CHAPTER 2

Fulton-Hansen connectedness theorem and some applications to projective geometry

2.1. Connectedness principle of Enriques-Zariski-Grothendieck-Fulton-Hansen and some classical theorems in algebraic geometry

In the first chapter we introduced the main definitions of classical projective geometry and furnished rigorous proofs of many classical results. Many theorems in classical projective geometry deal with "general" objects. For example the classical Bertini theorem on hyperplane sections, see theorem 1.5.2. A more refined version of this theorem says that if $f : X \to \mathbb{P}^N$ is morphism, with X proper and such that $\dim(f(X)) \geq 2$, and if $H = \mathbb{P}^{N-1} \subset \mathbb{P}^N$ is a general hyperplane, then $f^{-1}(H)$ is irreducible, see [**Ju**] theorem 6.10 for a modern reference. The "Enriques-Zariski principle" says that "limits of connected varieties remain connected" and it is for example illustrated in the previous example because for an arbitrary $H = \mathbb{P}^{N-1} \subset \mathbb{P}^N$, $f^{-1}(H)$ is connected as we shall prove below.

This result is particularly interesting because, as shown by Deligne and Jouanolou, a small generalization of it proved by Grothendieck, [**Gr**] XIII 2.3, yields a simplified proof of a beautiful and interesting connectedness theorem of Fulton and Hansen in [**FH**], whose applications are deep and appear in different areas of algebraic geometry and topology. Moreover, Deligne's proof generalizes to deeper statements involving higher homotopy groups when studying complex varieties, see [**D1**], [**D2**], [**Fu**], [**FL**].

To illustrate this circle of ideas and the "connectedness principle", we describe how the theorem of Fulton-Hansen includes some classical theorems in algebraic geometry and generalizes them. In our treatment we strictly follow the surveys [**Fu**] and [**FL**]. Another interesting source, where the ideas of Grothendieck behind this theorem and their generalizations to d-connectedness and to weighted projective spaces are explained in great detail, are the notes of a course of Bădescu, [**B1**], and his book [**B2**].

Now we recall four classical theorem with emphasis on the connectedness results in the idea of looking for a common thread. When dealing with homotopy groups π_i, we are assuming $K = \mathbb{C}$ and referring to the classical topology.

2.1.1. **Four classical theorems**. Let us list the following more or less known theorems.

(1) (Bézout) Let X and Y be closed subvarieties of $\mathbb{P}^N$. If $\dim(X)+\dim(Y) \geq N$, then $X \cap Y \neq \emptyset$. If $\dim(X) + \dim(Y) > N$, then $X \cap Y$ is connected and more precisely $(\dim(X) + \dim(Y) - N)$-connected.

(2) (Bertini) Let $f : X \to \mathbb{P}^N$ be a morphism, with X proper variety, and let $L = \mathbb{P}^{N-l} \subset \mathbb{P}^N$ be a linear space. If $l \leq \dim(f(X))$, then $f^{-1}(L) \neq \emptyset$. If $l < \dim(f((X))$, then $f^{-1}(L)$ is connected.

(3) (Lefschetz) If $X \subset \mathbb{P}^N$ is a closed irreducible subvariety of dimension n and if $L = \mathbb{P}^{N-l} \subset \mathbb{P}^N$ is a linear space containing $\operatorname{Sing}(X)$, then

$$\pi_i(X, X \cap L) = 0 \quad \text{for } i \leq n - l.$$

Equivalently the morphism

$$\pi_i(X \cap L) \to \pi_i(X)$$

is an isomorphisms if $i \leq n - l$ and surjective if $i = n - l$.

(4) (Barth-Larsen) If $X \subset \mathbb{P}^N$ is a closed irreducible non-singular subvariety of dimension n, then

$$\pi_i(\mathbb{P}^N, X) = 0 \quad \text{for } i \leq 2n - N + 1.$$

(Recall that $\pi_i(\mathbb{P}^N) = \mathbb{Z}$ for $i = 0, 2$ and $\pi_i(\mathbb{P}^N) = 0$ for $i = 1, 3, 4, \ldots, 2N$).

As we said at the beginning usually the names of the classical theorem refers to properties of general linear sections, for which a better property can be expected, as in the case of Bertini theorem for example, or as in the case of Bézout theorem (when the intersection is transversal one usually computes $\#(X \cap Y)$). In the classical Lefschetz theorem the variety was non-singular and L was general.

Let us remark that the two parts of theorem 1) can be reformulated by mean of homotopy groups. The first part is equivalent to

$$\pi_0(X \cap Y) \to \pi_0(X \times Y)$$

is surjective, the second one to the fact that the above morphism is an isomorphism. Similarly theorem 2) can be reformulated as

$$\pi_0(f^{-1}(L)) \to \pi_0(X)$$

is an isomorphism.

A common look at the above theorems comes from the following observation of Hansen, [**FL**],[**FH**]. All the above theorems are statement about the not emptiness, respectively connectedness, of the inverse image of $\Delta_{\mathbb{P}^N} \subset \mathbb{P}^N \times \mathbb{P}^N$ under a proper morphism $f : W \to \mathbb{P}^N \times \mathbb{P}^N$ such that $\dim(f(W)) \geq N$, respectively $\dim(f(W)) > N$.

Suppose this is true and take $W = X \times Y$ for theorem 1) or $W = X \times L$ in theorem 2) and 3) at least to deduce the connectedness parts. Theorem 4) can be deduced by taking $W = X \times X$, see [**FL**] and [**Fu**].

These results can be explained from other points of view as consequences of the ampleness of the normal bundle of a smooth subvarieties, or of complete intersections in $\mathbb{P}^N$. On the other hand the same positivity holds for $\Delta_{\mathbb{P}^N} \subset \mathbb{P}^N \times \mathbb{P}^N$ since $N_{\Delta_{\mathbb{P}^N}/\mathbb{P}^N \times \mathbb{P}^N} \simeq T_{\mathbb{P}^N}$ and the tangent bundle to $\mathbb{P}^N$, $T_{\mathbb{P}^N}$, is ample by Euler sequence.

The above discussion and further generalizations by Faltings, Goldstein and Hansen revealed a connectedness principle, which we now state and later justify why one should expect its validity.

2.1.2. **Connectedness Principle**, [**Fu**], pg. 18. Let P be a smooth projective variety.

Given a "suitable positive" embedding $Y \hookrightarrow P$ of codimension l and a proper morphism $f : W \to P$, $n = \dim(W)$,

$$\begin{tikzcd} f^{-1}(Y) \arrow[r, hook] \arrow[d] & W \arrow[d, "f"] \\ Y \arrow[r, hook] & P, \end{tikzcd}$$

we should have

$$\pi_i(W, f^{-1}(Y)) \xrightarrow{\simeq} \pi_i(P, Y) \qquad \text{for } i \leq n - l - \text{"defect"}.$$

This defect should be measured by (a) lack of positivity of Y in P; (b) singularities of W; (c) dimensions of the fibers of f. Usually $\pi_i(P, Y) = 0$ for small i, so the conclusion is that, as regards connectivity, $f^{-1}(Y)$ must look like W. If the defect is zero we deduce that

$$f^{-1}(Y) \neq \emptyset \qquad \text{if } n \geq l,$$

$$f^{-1}(Y) \quad \text{is connected and } \pi_1(f^{-1}(Y)) \to \pi_1(W) \quad \text{is surjective if } n > l.$$

The most basic case is with $P = \mathbb{P}^N$ and $Y = \mathbb{P}^{N-l}$ a linear subspace. In this case the principle furnishes the theorems of Bertini and Lefschetz by taking $W = X$. As we explained before the case which allows one to include all the classical theorems is $P = \mathbb{P}^N \times \mathbb{P}^N$, and $Y = \Delta_{\mathbb{P}^N}$ diagonally embedded in P. Indeed $W = X \times Y$ gives Bézout theorem, while theorems 2) and 3) are recovered by setting $W = X \times L$. Theorem 4) can be obtained with $W = X \times X$.

When $\mathbb{P}^N$ is replaced by other homogeneous spaces, one could measure the defect of positivity of its tangent bundle and one expects the principle to hold with this defect, see [**Fa**], [**Go**], [**BS**].

Why should one expect this connectedness principle to be valid? In some cases one can define a Morse function which measures distance from Y. Positivity should imply that all the Morse indices of this function are at least $n-l-1$ (perhaps minus a defect). Then one constructs W from $f^{-1}(Y)$ by adding only cells of dimension at least $n - l - 1$, which yields the required vanishing of relative homotopy groups, see [**Fu**] for a proof giving theorems 3) and 4) above.

Before ending this long introduction to the connectedness theorem we recall for completeness the following statements for later reference. They are particular forms or consequences of results of Barth and Barth and Larsen. Chronologically part 2) has been stated before the Barth-Larsen Theorem involving higher homotopy groups and recalled above.

2.1.3. THEOREM. *Let $X \subset \mathbb{P}^N$ be a smooth, irreducible projective variety and let $H \subset X$ be a hyperplane section.*

(1) *If $n \geq \frac{N+1}{2}$, then $\pi_1(X) = 1$* (Barth-Larsen).
(2) *If $n \geq \frac{N+i}{2}$, then the restriction map*

$$H^i(\mathbb{P}^N, \mathbb{Z}) \to H^i(X, \mathbb{Z})$$

is an isomorphism (Barth).

(3) *If* $n \geq \frac{N+2}{2}$, *then*

$$\mathrm{Pic}(X) \simeq \mathbb{Z} < H > \text{(Barth)}.$$

We come back to the algebraic setting and to the proof of the theorem of Fulton-Hansen and hence of the non-emptiness and connectedness parts ($i = 0$) of theorems 1), 2), 3) (and 4)). The theorem appears as a consequence of the connectedness of preimages of linear spaces under proper morphisms, a result due to Grothendieck and which follows from the "classical" Bertini theorem we quoted at the beginning. We start with the connectedness theorem and later prove some interesting results having their own interest and leading to its proof. In [**B1**], Lucian Bădescu extends the connectedness theorem to weighted projective spaces using the original ideas of Grothendieck, so that many geometrical consequences of the result are valid also for this class of homogeneous varieties.

2.1.4. Theorem. (**Fulton-Hansen Connectedness Theorem**, [**FH**]) *Let* X *be an irreducible variety, proper over an algebraically closed filed* K. *Let* $f : X \to \mathbb{P}^N \times \mathbb{P}^N$ *be a morphism and let* $\Delta = \Delta_{\mathbb{P}^N} \subset \mathbb{P}^N \times \mathbb{P}^N$ *be the diagonal.*

(1) *If* $\dim(f(X)) \geq N$, *then* $f^{-1}(\Delta) \neq \emptyset$.
(2) *If* $\dim(f(X)) > N$, *then* $f^{-1}(\Delta)$ *is connected.*

We begin by recalling the following "classical" Bertini theorem in a more general form. For a proof we refer to [**Ju**], theorem 6.10, where the hypothesis $K = \overline{K}$ is relaxed.

2.1.5. Theorem. (**Bertini Theorem**, see [**Ju**]) *Let* X *be an irreducible variety and let* $f : X \to \mathbb{P}^N$ *be a morphism. For a fixed integer* $l \geq 1$, *let* $\mathbb{G}(N-l, N)$ *be the Grassmann variety of linear subspaces of* $\mathbb{P}^N$ *of codimension* l. *Then*

(1) *if* $l \leq \dim(\overline{f(X)})$, *then there is a non-empty open subset* $U \subseteq \mathbb{G}(N-l, N)$ *such that for every* $L \in U$,

$$f^{-1}(L) \neq \emptyset;$$

(2) *if* $l < dim(\overline{f(X)})$, *then there is a non-empty open subset* $U \subseteq \mathbb{G}(N-l, N)$ *such that for every* $L \in U$,

$$f^{-1}(L) \quad \textit{is irreducible.}$$

We now show that the Enriques-Zariski principle is valid in this setting by proving the next result, which is the key point towards theorem 2.1.4. We pass from general linear sections to arbitrary ones and for simplicity we suppose $K = \overline{K}$ as always.

2.1.6. Theorem. ([**Gr**], [**FH**], [**Ju**], theorem 7.1) *Let* X *be an irreducible variety and let* $f : X \to \mathbb{P}^N$ *be a morphism. Let* $L = \mathbb{P}^{N-l} \subset \mathbb{P}^N$ *be an arbitrary linear space of codimension* l.

(1) *If* $l \leq \dim(\overline{f(X)})$ *and if* X *is proper over* K, *then*

$$f^{-1}(L) \neq \emptyset.$$

(2) *If* $l < dim(\overline{f(X)})$ *and if* X *is proper over* K, *then*

$$f^{-1}(L) \quad \textit{is connected.}$$

More generally for an arbitrary irreducible variety X, if $f : X \to \mathbb{P}^N$ is proper over some open subset $V \subseteq \mathbb{P}^N$, and if $L \subseteq V$, then, when the hypothesis on the dimensions are satisfied, the same conclusions hold for $f^{-1}(L)$.

Proof. (According to [**Ju**]). We prove the second part of the theorem from which the statements in 1) and 2) follow.

Let $W \subseteq \mathbb{G}(N-l, N)$ be the open subset consisting of linear spaces contained in V and let

$$Z = \{(x, L') \in X \times W \, : \, f(x) \in L'\} \subset \{(x, L') \in X \times \mathbb{G}(N-l, N) \, : \, f(x) \in L'\} = \mathcal{I}.$$

The scheme Z is irreducible since it is an open subset of the Grassmann bundle $p_1 : \mathcal{I} \to X$. Since f is proper over V, the second projection $p_2 : Z \to W$ is a proper morphism. Consider its Stein factorization:

$$\begin{tikzcd} X \arrow[d, "q"'] \arrow[dr, "p_2"] & \\ W' \arrow[r, "r"'] & W; \end{tikzcd}$$

the morphism q is proper with connected fibers and surjective, while r is finite. By theorem 2.1.5 r is dominant and hence surjective if $l \leq \dim(\overline{f(X)})$, respectively generically one-to-one and surjective if $l < \dim(\overline{f(X)})$. In the first case $p_2 : Z \to W$ is surjective so that $f^{-1}(L) \neq \emptyset$ for every $L \in W$. In the second case, since W is smooth, it follows that r is one to one everywhere so that $f^{-1}(L) = q^{-1}(r^{-1}(L))$ is connected for every $L \in W$. $\square$

2.1.7. Remark. The original proof of Grothendieck used an analogous local theorem proved via local cohomology. His method has been used and extended by Hartshorne, Ogus, Speiser and Faltings. Faltings proved with similar techniques a connectedness theorem for other homogeneous spaces, see [**Fa**], at least in characteristic zero. A different proof of a special case of the above theorem was also given by Barth in 1969.

Now we are in position to prove the connectedness theorem.

Proof. (of theorem 2.1.4, according to Deligne, [**D1**]). The idea is to pass from the diagonal embedding $\Delta \subset \mathbb{P}^N \times \mathbb{P}^N$ to a linear embedding $L = \mathbb{P}^N \subset \mathbb{P}^{2N+1}$, a well known classical trick.

In $\mathbb{P}^{2N+1}$ separate the $2N+2$ coordinates into $[X_0 : \ldots : X_N]$ and $[Y_0 : \ldots : Y_N]$ and think these two sets as coordinates on each factor of $\mathbb{P}^N \times \mathbb{P}^N$. The two N dimensional linear subspaces $H_1 : \; X_0 = \ldots = X_N = 0$ and $H_2 : \; Y_0 = \ldots = Y_N = 0$ of $\mathbb{P}^{2N+1}$ are disjoint. If $V = \mathbb{P}^{2N+1} \setminus (H_1 \cup H_2)$ since there is a unique secant line to $H_1 \cup H_2$ passing through each $p \in V$, there is a morphism

$$\phi : V \to H_1 \times H_2 = \mathbb{P}^N \times \mathbb{P}^N,$$

which to p associates the points $(p_1, p_2) = (< H_2, p > \cap H_1, < H_1, p > \cap H_2)$. In coordinates, $\phi([X_0 : \ldots : X_N : Y_0 : \ldots : Y_N] = ([X_0 : \ldots : X_N], [Y_0 : \ldots : Y_N])$. Then $\phi^{-1}(\phi(p)) =< p_1, p_2 > \setminus \{p_1, p_2\} \simeq \mathbb{A}^1_K \setminus 0$. Let $L = \mathbb{P}^N \subset V$ be the linear subspace of $\mathbb{P}^{2N+1}$ defined by $X_i = Y_i$, $i = 0, \ldots, N$. Then

$$\phi_{|L} : L \xrightarrow{\simeq} \Delta$$

is an isomorphism. Given $f : X \to \mathbb{P}^N \times \mathbb{P}^N$ we construct the following Cartesian diagram

$$\begin{tikzcd} X' \arrow[r, "\phi'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ V \arrow[r, "\phi"] & \mathbb{P}^N \times \mathbb{P}^N, \end{tikzcd}$$

where

$$X' = V \times_{\mathbb{P}^N \times \mathbb{P}^N} X.$$

Clearly ϕ' induces an isomorphism between $f'^{-1}(L)$ and $f^{-1}(\Delta)$. To prove the theorem it is sufficient to verify the corresponding assertion for $f'^{-1}(L)$. To this aim we apply theorem 2.1.6. Let us verify the hypothesis.

Since $\phi'^{-1}(x) \simeq \phi^{-1}(f(x)) = \mathbb{A}^1_K \setminus 0$ for every $x \in X$, the scheme X' is irreducible and of dimension $\dim(X) + 1$. The morphism f is proper, so that also $f' : X' \to V$ is proper and moreover $\dim(f(X')) = \dim(f(X)) + 1$. If $\dim(f(X)) \geq N$, then $\dim(f(X')) \geq N + 1 = \operatorname{codim}(L, \mathbb{P}^{2N+1})$. If $\dim(f(X')) > N$, then $\dim(f(X')) > N + 1 = \operatorname{codim}(L, \mathbb{P}^{2N+1})$. $\square$

2.2. Zak's applications to Projective Geometry

In this section we come back to projective geometry and apply Fulton-Hansen theorem to prove some interesting and non-classical results in projective geometry. Most of the ideas and the results are due to Fyodor L. Zak, see [**Z2**], [**FL**], [**LV**], and they will be significant improvements of the classical material presented in the first chapter. Other applications to new results in algebraic geometry can be found in [**FH**], [**FL**], [**Fu**].

We begin with the following key result, which refines a result of Johnson, [**Jo**].

2.2.1. Theorem. ([**FH**], [**Z2**]) *Let $Y \subseteq X \subset \mathbb{P}^N$ be a closed subvariety of dimension $r = \dim(Y) \leq \dim(X) = n$, with X irreducible and projective. Then either*

(1) $\dim(T^*(Y, X)) = r + n$ *and* $\dim(S(Y, X)) = r + n + 1$*, or*
(2) $T^*(Y, X) = S(Y, X)$.

Proof. We can suppose Y irreducible and then apply the same argument to each irreducible component of Y. We know that $T^*(Y, X) \subseteq S(Y, X)$ and that $\dim(T^*(Y, X)) \leq r + n$ by construction. Suppose that $\dim(T^*(Y, X)) = r + n$. Since $S(Y, X)$ is irreducible and $\dim(S(Y, X)) \leq r + n + 1$, the conclusion holds.

Suppose now $\dim(T^*(Y, X)) = t < r + n$. We prove that $\dim(S(Y, X)) \leq t$ so that $T^*(Y, X) = S(Y, X)$ follows from the irreducibility of $S(Y, X)$. There exists $L = \mathbb{P}^{N-t-1}$ such that $L \cap T^*(Y, X) = \emptyset$. The projection $\pi_L : \mathbb{P}^N \setminus L \to \mathbb{P}^t$ restricts to a finite morphism on X and on Y, since $L \cap X = \emptyset$, see definition 1.2.4. Then $(\pi_L \times \pi_L)(X \times Y) \subset \mathbb{P}^t \times \mathbb{P}^t$ has dimension $r + n > t$ by hypothesis. By theorem 2.1.4, the closed set

$$\widetilde{\Delta} = (\pi_L \times \pi_L)^{-1}(\Delta_{\mathbb{P}^t}) \subset Y \times X$$

is connected and contains the closed set $\Delta_Y \subset Y \times X$ so that Δ_Y is closed in $\widetilde{\Delta}$.

We claim that

$$\Delta_Y = \widetilde{\Delta}.$$

This yields $L \cap S(Y, X) = \emptyset$ and hence $\dim(S(Y, X)) \leq N - 1 - \dim(L) = t$.

Suppose $\overline{\widetilde{\Delta} \setminus \Delta_Y} \neq \emptyset$. We find $y' \in Y$ such that $\emptyset \neq T^*_{y'}(Y,X) \cap L \subseteq T^*(Y,X) \cap L$ contrary to the assumption. If $\overline{\widetilde{\Delta} \setminus \Delta_Y} \neq \emptyset$, the connectedness of $\widetilde{\Delta}$ implies the existence of $(y', y') \in \overline{\widetilde{\Delta} \setminus \Delta_Y} \cap \Delta_Y$. Let notation be as in definition 1.2.1. i. e. $p_2(p_1^{-1}(y,x)) =< x, y >$ if $x \neq y$ and $p_2(p_1^{-1}(y,x)) = T^*_y(Y,X)$ if $x = y \in Y$. Since for every $(y,x) \in \widetilde{\Delta} \setminus \Delta_Y$ we have $< y, x > \cap L \neq \emptyset$ by definition of π_L ($\pi_L(y) = \pi_L(x)$, $y \neq x$, if and only if $< y, x > \cap L \neq \emptyset$), the same holds for (y', y') so that $p_2(p_1^{-1}(y,x)) \cap L \neq \emptyset$ forces $p_2(p_1^{-1}(y', y')) \cap L \neq \emptyset$. $\square$

2.2.2. Corollary. *Let $X \subset \mathbb{P}^N$ be an irreducible projective variety of dimension n. Then either*

(1) $\dim(T^*X) = 2n$ *and* $\dim(SX) = 2n + 1$, *or*
(2) $T^*X = SX$.

The following theorem well illustrates the passage from general to arbitrary linear spaces, as regards to tangency.

2.2.3. Theorem. (**Zak's Theorem on Tangencies**) *Let $X \subset \mathbb{P}^N$ be an irreducible projective non-degenerate variety of dimension n. Let $L = \mathbb{P}^m \subset \mathbb{P}^N$ be a linear subspace, $n \leq m \leq N - 1$, which is J-tangent along the closed set $Y \subseteq X$. Then* $\dim(Y) \leq m - n$.

Proof. Without loss of generality we can suppose that Y is irreducible and then apply the conclusion to each irreducible component. By hypothesis and by definition we get $T^*(Y,X) \subseteq L$. Since $X \subseteq S(Y,X)$ and since X is non-degenerate, $S(Y,X)$ is not contained in L so that $T^*(Y,X) \neq S(Y,X)$. By theorem 2.2.1 we have $\dim(Y) + n = \dim(T^*(Y,X)) \leq \dim(L) = m$. $\square$

We now come back to the problem of tangency and to contact loci of smooth varieties to furnish two beautiful applications of the theorem on Tangencies. We begin with the finiteness of the Gauss map of a smooth variety.

2.2.4. Corollary. (**Gauss map is finite for smooth varieties**, Zak) *Let $X \subsetneq \mathbb{P}^N$ be a smooth irreducible non-degenerate projective variety of dimension n. Then the Gauss map $\mathcal{G}_X : X \to \mathbb{G}(n,N)$ is finite. If moreover* char(K)=0, *the $\mathcal{G}_X$ is birational onto the image, i.e. X is a normalization of $\mathcal{G}_X(X)$.*

Proof. As always it is sufficient to prove that $\mathcal{G}_X$ has finite fibers. For every $x \in X$, $\mathcal{G}_X^{-1}(\mathcal{G}_X(x))$ is the locus of points at which the tangent space T_xX is tangent. By theorem 2.2.3 it has dimension less or equal than $\dim(T_xX) - n = 0$.

If char(K)=0, then every fiber $\mathcal{G}_X^{-1}(\mathcal{G}_X(x))$ is linear by theorem 1.5.9 and of dimension zero by the first part, so that it reduces to a point. $\square$

The next result reveals a special feature of non-singular varieties, since the result is clearly false for cones, see exercise 1.5.3.

2.2.5. Corollary. (Zak) *Let $X \subset \mathbb{P}^N$ be a smooth projective non-degenerate variety. Let $X^* \subset \mathbb{P}^{N*}$ be its dual variety. Then* $\dim(X^*) \geq \dim(X)$. *In particular, if also X^* is smooth, then* $\dim(X^*) = \dim(X)$.

Proof. By the theorem of the dimension of the fiber, letting notation as in definition 1.5.1, $\dim(X^*) = N - 1 - \dim(p_2^{-1}(H))$, $H \in X^*$ general point. By theorem 2.2.3, $\dim(p_2^{-1}(H)) \leq N - 1 - \dim(X)$ and the conclusion follows. $\square$

2.2.6. Remark. In exercise 1.5.3, we saw that $(\mathbb{P}^1 \times \mathbb{P}^n)^* \simeq \mathbb{P}^1 \times \mathbb{P}^n$ for every $n \geq 1$. In [**Ei**], L. Ein shows that if $N \geq 2/3 \dim(X)$, if X is smooth, if char(K)=0 and if $\dim(X) = \dim(X^*)$, then $X \subset \mathbb{P}^N$ is either a hypersurface, or $\mathbb{P}^1 \times \mathbb{P}^n \subset \mathbb{P}^{2n+1}$ Segre embedded, or $\mathbb{G}(1,4) \subset \mathbb{P}^9$ Plücker embedded, or the 10-dimensional spinor variety $S^{10} \subset \mathbb{P}^{15}$. In the last three cases $X \simeq X^*$.

We apply the theorem on Tangencies to deduce some strong properties of the hyperplane sections of varieties of small codimension. By the theorem of Bertini proved in the previous section we know that arbitrary hyperplane sections of varieties of dimension at least 2 are connected. When the codimension of the variety is small with respect to the dimension, some further restrictions for the scheme structure hold.

If $X \subset \mathbb{P}^N$ is a non-singular irreducible nondegenerate variety, we recall that for every $H \in X^*$

$$\mathrm{Sing}(H \cap X) = \{x \in X \,:\, T_xX \subset H\},$$

i.e. it is the locus of points at which H is tangent. By theorem 2.2.3 we get

$$\dim(\mathrm{Sing}(X \cap X) \leq N - 1 - \dim(X),$$

i.e.

$$\mathrm{codim}(\mathrm{Sing}(X \cap H), X \cap H) \geq 2\dim(X) - N.$$

Recall that $H \cap X$ is a Cohen-Macaulay scheme of dimension $\dim(X) - 1$ and that such a scheme is reduced as soon as it is generically reduced ($R_0 + S_1 \Leftrightarrow R_1$).

If $N \leq 2\dim(X) - 1$, then $H \cap X$ is a reduced scheme being non-singular in codimension zero and in particular generically reduced. The condition forces $\dim(X) \geq 2$, so that it is also connected by Bertini theorem.

If $N \leq 2\dim(X) - 2$, which forces $\dim(X) \geq 3$, then $H \cap X$ is also non-singular in codimension 1, so that it is normal being Cohen-Macaulay. Since it is connected and integral, it is also irreducible. The case of the Segre 3-fold $\mathbb{P}^1 \times \mathbb{P}^2 \subset \mathbb{P}^5$ shows that this last result cannot be improved, since an hyperplane containing a $\mathbb{P}^2$ of the ruling yields a reducible, reduced, hyperplane section. Clearly in the same way, if $N \leq 2\dim(X) - k - 1$, $k \geq 0$, then $X \cap H$ is connected, Cohen-Macaulay and non-singular in codimension k. We summarize these result in the following corollary to the theorem on Tangencies.

2.2.7. Corollary. (Zak) *Let $X \subset \mathbb{P}^N$ be a smooth non-degenerate projective variety of dimension n. Then*

(1) *if $N \leq 2n - 1$, then every hyperplane section is connected and reduced;*
(2) *if $N \leq 2n - 2$, then every hyperplane section is irreducible and normal;*
(3) *let $k \geq 2$. If $N \leq 2n - k - 1$, then every hyperplane section is irreducible, normal and non-singular in codimension k.*

CHAPTER 3

Hartshorne's conjectures and Severi varieties

3.1. Hartshorne's conjectures and Zak's theorem on linear normality

After the period in which new and solid foundations to the principles of algebraic geometry were rebuilt especially by Zariski, Grothendieck and their schools, at the beginning of the '70 a new trend began. There was a renewed interest in solving concrete problems and in finding applications of the new methods and ideas. One can consult the beautiful book of Robin Hartshorne, [**H1**], to have a picture of that situation. In [**H1**] many outstanding questions, such as the set-theoretic complete intersection of curves in $\mathbb{P}^3$ (still open), the characterization of $\mathbb{P}^N$ among the smooth varieties with ample tangent bundle (solved by Mori in [**Mo1**] and which cleared the path to the foundation of Mori theory, [**Mo2**]) were discusses, or stated and a lot of other problems solved. In related fields we only mention Deligne proof of the Weil conjectures or later Faltings proof of the Mordell conjecture, which used the new machinery.

The interplay between topology and algebraic geometry returned to flourish. Lefschetz theorem and Barth-Larsen theorem, see subsection 2.1.1 and theorem 2.1.3, also suggested that smooth varieties, whose codimension is small with respect to their dimension, should have very strong restrictions both topological, both geometrical. To have a feeling we remark that a codimension 2 smooth complex subvariety of $\mathbb{P}^N$, $N \geq 5$, has to be simply connected for example. If $N \geq 6$, there are no known examples of codimension 2 smooth varieties with the exception of the trivial ones, the complete intersection of two hypersurfaces, i.e. the transversal intersection of two hypersurfaces, smooth along the subvariety. In fact, at least for the moment, one is able to construct only these kinds of varieties whose codimension is sufficiently small with respect to dimension. Let us recall the following definition and some notable properties of complete intersections analogous to varieties whose codimension is small with respect to dimension.

3.1.1. Definition. (**Complete intersection**) A variety $X \subset \mathbb{P}^N$ of dimension n is a *complete intersection* if there exist $N-n$ homogeneous polynomials $f_i \in K[X_0, \ldots, X_N]$ of degree $d_i \geq 1$, generating the homogeneous ideal $I(X) \subset K[X_0, \ldots, X_N]$, i.e. $I(X) =< f_1, \ldots, f_{N-n} >$.

Let us recall that since $f_1, \ldots, f_{N-n}$ form a regular sequence in $K[X_0, \ldots, X_N]$, the homogeneous coordinate ring $S(X) = K[X_0, \ldots, X_N]/I(X)$ has depth $n+1$, i.e. $X \subset \mathbb{P}^N$ is an arithmetically Cohen-Macaulay variety. Thus a complete intersection $X \subset \mathbb{P}^N$ is projectively normal, i.e. the restriction morphisms

$$H^0(\mathcal{O}_{\mathbb{P}^N}(m)) \to H^0(\mathcal{O}_X(m))$$

are surjective for every $m \geq 0$, so that X is connected, and $H^i(\mathcal{O}_X(m)) = 0$ for every i such that $0 < i < n$ and for every $m \in \mathbb{Z}$. Moreover, by Grothendieck theorem on complete intersections, $\mathrm{Pic}(X) \simeq \mathbb{Z} < \mathcal{O}_X(1) >$, as soon as $n \geq 3$, see [**H1**]. By Lefschetz theorem complete intersections defined over $K = \mathbb{C}$ are simply connected, as soon as $n \geq 2$ and have the same cohomology $H^i(X, \mathbb{Z})$ of the projective spaces containing them for $i < n$.

Based on some empirical observations, inspired by the theorem of Barth and Larsen and, according to Fulton and Lazarsfeld, "on the basis of few examples", Hartshorne was led to formulate the following conjectures.

3.1.2. CONJECTURE. (1^{st} **Conjecture of Hartshorne, or Complete Intersection Conjecture**, [**H2**]) *Let $X \subset \mathbb{P}^N$ be a smooth irreducible non-degenerate projective variety.*

If $N < \dfrac{3}{2}\dim(X)$, *i.e. if* $\operatorname{codim}(X) < \dfrac{1}{2}\dim(X)$, *then X is a complete intersection.*

Let us quote Hartshorne: *While I am not convicted of the truth of this statement, I think it is useful to crystallize one's idea, and to have a particular problem in mind* ([**H2**]).

Hartshorne immediately remarks that the conjecture is sharp, due to the examples of the Grassmann variety of lines in $\mathbb{P}^4$, $\mathbb{G}(1,4) \subset \mathbb{P}^9$, Plücker embedded, and of the spinorial variety of dimension 10, $S^{10} \subset \mathbb{P}^{15}$; moreover, the examples of cones over curves in $\mathbb{P}^3$, not complete intersection, reveals the necessity of the non-singularity assumption. Varieties for which $N = \frac{3}{2}\dim(X)$ and which are not complete intersection are usually called *Hartshorne varieties*. No other example of Hartshorne variety is known till today. It is not a case that these varieties are homogeneous since a technique for constructing varieties of not too high codimension is exactly via algebraic groups, see for example [**Z2**], chapter 3, or the appendix to [**LV**].

One of the main difficulties of the problem is a good translation in geometrical terms of the algebraic condition of being a complete intersection and in general of dealing with the equations defining a variety.

It is not here the place to remark how many important results originated and still today arise from this open problem in the areas of vector bundles on projective space, of the study of defining equations of a variety and k-normality and so on. The list of these achievements is too long that we preferred to avoid citations, being confident that everyone has met sometimes a problem or a result related to it.

Let us recall the following definition.

3.1.3. DEFINITION. (**Linear normality**) A non-degenerate irreducible variety $X \subset \mathbb{P}^N$ is said to be *linearly normal* if the linear of hyperplane sections is complete, i.e. if the injective, due to non-degenerateness, restriction morphism

$$H^0(\mathcal{O}_{\mathbb{P}^N}(1)) \xrightarrow{r} H^0(\mathcal{O}_X(1))$$

is surjective and hence an isomorphism.

If a variety $X \subset \mathbb{P}^N$ is not linearly normal, then the complete linear system $|\mathcal{O}_X(1)|$ is of dimension greater than N and embeds X as a variety $X' \subset \mathbb{P}^M$, $M > N$. Moreover, there exists a linear space $L = \mathbb{P}^{M-N-1}$ such that $L \cap X' = \emptyset$ and such that $\pi_L : X' \to X \subset \mathbb{P}^N$ is an isomorphism. Indeed, if $V = r(H^0(\mathcal{O}_{\mathbb{P}^N}(1))) \subsetneq H^0(\mathcal{O}_X(1))$ and if $U \subset H^0(\mathcal{O}_{\mathbb{P}^N}(1))$ is a complementary subspace of V in $H^0(\mathcal{O}_{\mathbb{P}^N}(1))$, the one can take $\mathbb{P}^M = \mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^N}(1)))$, $L = \mathbb{P}(U)$ and the claim follows from the fact that $\pi_L : X' \simeq X \to X \subset \mathbb{P}^N = \mathbb{P}(V)$ is given by the very ample linear system $|V|$. On the contrary, if X is an isomorphic linear projection of a variety $X' \subset \mathbb{P}^M$, $M > N$, then X is not linearly normal.

In the same survey paper Hartshorne posed another conjecture, based on the fact that complete intersections are linearly normal and on some examples in low dimension.

3.1.4. CONJECTURE. (2^{nd} **Conjecture of Hartshorne, or Linear Normality Conjecture**, [**H2**]) *Let $X \subset \mathbb{P}^N$ be a smooth irreducible non-degenerate projective variety.*

If $N < \frac{3}{2}\dim(X)+1$, i.e. if $\operatorname{codim}(X) < \frac{1}{2}\dim(X)+1$, then X is linearly normal.

Recalling proposition 1.2.8 and the above discussion, we can equivalently reformulate it by means of secant varieties putting "$N = N+1$".

$$\textit{If } N < \frac{3}{2}\dim(X)+2, \ \textit{then } SX = \mathbb{P}^N.$$

Let us quote once again Hartshorne point of view on this second problem: *Of course in settling this conjecture, it would be nice also to classify all nonlinearly normal varieties with $N = \frac{3n}{2}+1$, so as to have a satisfactory generalization of Severi's theorem. As noted above, a complete intersection is always linearly normal, so this conjecture would be a consequence of our original conjecture, except for the case $N = \frac{3n}{2}$. My feeling is that this conjecture should be easier to establish than the original one* ([**H2**]). Once again the bound is sharp taking into account the example of the projected Veronese surface in $\mathbb{P}^4$.

The conjecture on linear normality was proved by Zak at the beginning of the '80's and till now it is the major evidence for the possible truth of the complete intersection conjecture. As we shall see conjecture 3.1.4 is now an immediate consequence of Terracini Lemma and of theorem 2.2.1. Later we will furnish another proof of this theorem, cfr. theorem **??**.

3.1.5. THEOREM. (**Zak Theorem on Linear Normality**) *Let $X \subset \mathbb{P}^N$ be a smooth non-degenerate projective variety of dimension n. If $N < \frac{3}{2}n+2$, then $SX = \mathbb{P}^N$. Or equivalently if $SX \subsetneq \mathbb{P}^N$, then $\dim(SX) \geq \frac{3}{2}n+1$ and hence $N \geq \frac{3}{2}n+2$.*

PROOF. Suppose that $SX \subsetneq \mathbb{P}^N$, then there exists a hyperplane H containing the general tangent space to SX, let us say T_zSX. Then by corollary 1.3.6, the hyperplane H is tangent to X along $\Sigma_z(X)$, which by the generality of z has pure dimension $\delta(X) = 2n+1-\dim(SX)$. Since $T(\Sigma_z(X), X) \subseteq H$, the non-degenerate variety $S(\Sigma_z(X), X) \supseteq X$ is not contained in H, yielding $T(\Sigma_z(X), X) \neq S(\Sigma_z(X), X)$. By theorem 2.2.1 we get

$$2n+1-\dim(SX)+n+1 = \dim(S(\Sigma_z(X), X)) \leq \dim(SX),$$

i.e.

$$3n+2 \leq 2\dim(SX)$$

implying

$$N-1 \geq \dim(SX) \geq \frac{3}{2}n+1.$$

□

3.2. Severi varieties

Theorem 3.1.5 opens the problem of investigating examples for which the result is sharp, i.e. to try to classify smooth varieties of dimension n, $X \subset \mathbb{P}^{\frac{3}{2}n+2}$ such that $SX \subsetneq \mathbb{P}^{\frac{3}{2}n+2}$, or equivalently smooth not linearly normal varieties of dimension n, $\widetilde{X} \subset \mathbb{P}^{\frac{3}{2}n+1}$. Clearly n is even so that the first case to be considered is $n = 2$ and so one would like to classify smooth surfaces in $\mathbb{P}^5$ such that $SX \subsetneq \mathbb{P}^5$. The answer is thus contained in the classical and well known theorem of Severi, [**Sev1**], which is theorem 1.4.1 here, saying that X is projectively equivalent to the Veronese surface $\nu_2(\mathbb{P}^2) \subset \mathbb{P}^5$. This justifies the name given by Zak to such varieties.

3.2.1. Definition. (**Severi variety**) A smooth irreducible non-degenerate variety of dimension n, $X \subset \mathbb{P}^{\frac{3}{2}n+2}$, is said to be a *Severi variety* if $SX \subsetneq \mathbb{P}^{\frac{3}{2}n+2}$.

By theorem 3.1.5, it follows that $SX \subset \mathbb{P}^{\frac{3}{2}n+2}$ is necessarily an hypersurface, i.e. $\dim(SX) = \frac{3}{2}n + 1$.

In exercise 1.1.6 we showed that the Segre variety $X = \mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^8$ is an example of Severi variety of dimension 4. Indeed $N = 8 = \frac{3}{2} \cdot 4 + 2$ and SX is a cubic hyersurface, see *loc. cit.*. By the classical work of Scorza, last page of [**S1**], it turns out that $\mathbb{P}^2 \times \mathbb{P}^2$ is the only Severi variety of dimension 4. We shall furnish a short, geometrical and elementary proof of this fact below, see theorem **??**.

The realization of the Grassmann variety of lines in $\mathbb{P}^5$ Plücker embedded, $X = \mathbb{G}(1, 5) \subset \mathbb{P}^{14}$, as the variety given by the pfaffians of the general antisymmetric 6×6 matrix, yields that $\mathbb{G}(1, 5)$ is a Severi variety of dimension 8 such that its secant variety is a degree 3 hypersurface, see for example [**Ha**] pg. 112 and 145, for the last assertion.

A less trivial examples is a variety studied by Elie Cartan and also by Room. It is a homogeneous complex variety of dimension 16, $X \subset \mathbb{P}^{26}$, associated to the representation of E_6 and for this reason called E_6*-variety*, or *Cartan variety* by Zak. It has been shown by Lazarsfeld and Zak that its secant variety is a degree 3 hypersurface, see for example [**LV**] and [**Z2**], chapter 3.

There is a unitary way to look at these 4 examples, by realizing them as "Veronese surfaces over the composition algebras over K", $K = \overline{K}$ and char(K)=0, [**Z2**] chapter 3. Let $\mathcal{U}_0 = K$, $\mathcal{U}_1 = K[t]/(t^2 + 1)$, $\mathcal{U}_2 =$ quaternion algebra over K, $\mathcal{U}_3 =$ Cayley algebra over K. For $K = \mathbb{C}$, we get $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and the octonions numbers $\mathbb{O}$. Let $\mathcal{I}_i$, $i = 0, \ldots, 3$, denote the Jordan algebra of Hermitian (3×3)-matrices over $\mathcal{U}_i$, $i = 0, \ldots, 3$. A matrix $A \in \mathcal{I}_i$ is called *Hermitian* if $\overline{A}^t = A$, where the bar denotes the involution in $\mathcal{U}_i$. Let

$$X_i = \{[A] \in \mathbb{P}(\mathcal{I}_i) \,:\, \mathrm{rk}(A) = 1\,\} \subset \mathbb{P}(\mathcal{I}_i).$$

Then

$$N_i = \dim(\mathbb{P}(\mathcal{I})) = 3 \cdot 2^i + 2, \;\; n_i = \dim(X_i) = 2^{i+1} = 2\dim_K(\mathcal{U}_i),$$

and

$$SX = \{[A] \in \mathbb{P}(\mathcal{I}_i) \,:\, \mathrm{rk}(A) \leq 2\,\} = V(\det(A)) \subset \mathbb{P}(\mathcal{I}_i)$$

is a degree 3 hypersurface. By definition $X_i \subset \mathbb{P}(\mathcal{I}_i)$ is a Severi variety of dimension 2^{i+1}, which is seen to be one of the above examples.

A theorem of Jacobson states that over a fixed algebraically closed field K there are only four Jordan algebras, the algebras $\mathcal{U}_i$'s, and hence these are the only examples which can be constructed in this way.

The highly non-trivial and very beautiful result, which is essentially equivalent to Jacobson classification theorem, is the following classification theorem of Severi varieties proved by Zak.

3.2.2. THEOREM. (**Zak classification of Severi varieties**, [**Z1**], [**Z2**], [**LV**], [**La**], [**Ru6**]) *Let $X \subset \mathbb{P}^{\frac{3}{2}n+2}$ be a Severi variety of dimension n, defined over an algebraically closed field K of characteristic 0. Then X is projectively equivalent to one of the following:*

(1) *the Veronese surface $\nu_2(\mathbb{P}^2) \subset \mathbb{P}^5$;*
(2) *the Segre 4-fold $\mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^8$;*
(3) *the Grassmann variety $\mathbb{G}(1,5) \subset \mathbb{P}^{14}$;*
(4) *the E_6-variety $X \subset \mathbb{P}^{26}$.*

A complete proof of this theorem for $n > 8$ is beyond the scope of these notes and of the lectures and it can be found in the above cited references. We prefer to sketch the basic ideas leading to the restriction $n = 2, 4, 8, 16$ for the dimension using some results from the theory of *quadric entry locus varieties*, [**Ru6**], and to study the analogies with the theorem we proved for $n = 2$, classifying Severi varieties in dimension 2,4 (and 8) and explaining why there exists only one case more. A proof of the classification can be found in above cited references. From now on we will suppose char(K)=0, or equivalently $K = \mathbb{C}$.

3.2.3. THEOREM. (**Dimension of a Severi variety, [Z2], theorem 3.10, pg. 84,** [**Ru6**]) *Let $X \subset \mathbb{P}^{\frac{3n}{2}+2}$ be a Severi variety. Then $n = 2$, 4, 8 or 16. In particular, $\delta(X) = 1$, 2, 4 or 8.*

PROOF. In [**Ru6**] it is shown that a Severi variety $X \subset \mathbb{P}^{\frac{3n}{2}+2}$ is a quadric variety of type $\delta = \frac{n}{2}$. We can also suppose $n \geq 6$, i.e. $\delta \geq 3$.

For a smooth quadric variety of type $\delta \geq 3$, $X \subset \mathbb{P}^N$, one defines

$$r_X = \sup\{r \in \mathbb{N} : \delta \geq 2r+1\}.$$

By the main result of [**Ru6**], 2^{r_X} divides $n - \delta = \frac{n}{2} = \delta$ so that 2^{r_X+1} divides n. Hence $\delta = \frac{n}{2}$ is even and, by definition of r_X, $\frac{n}{2} = 2r_X + 2$. Thus, for some integer $m \geq 1$,

$$m2^{r_X+1} = n = 4(r_X+1).$$

Therefore either $r_X = 1$, i.e. $n = 8$, or $r_X = 3$, i.e. $n = 16$. $\square$

The classification theorem 3.2.2 is now easy to deduce. Indeed, for $n = 8$ the variety $X \subset \mathbb{P}^{14}$ is a Mukai variety, being a Fano variety of index 6 with $b_2(X) = 1$. Indeed, by Barth theorem 2.1.3, $\mathrm{Pic}(X) = \mathbb{Z} < H >$ and $X \subset \mathbb{P}^{14}$ contains moving lines, so that it is Fano. In [**Ru6**], it is shown that for a quadric variety of type $\delta \geq 3$ and for a line $l \subset X$, $-K_X \cdot l = \frac{n+\delta}{2}$, so that $i(X) = 6$. It is classically well known that $X \simeq \mathbb{G}(1,5) \subset \mathbb{P}^{14}$ Plücker embedded. A uniform approach connecting the original ideas of Zak and a careful study of lines on X in dimension 4, 8 and 16 and leading to a quick classification of Severi varieties in dimension 4, 8 and 16 is described in [**Ru6**]. This approach does not depend on any previous classification result, with the exceptions of the rational representations on $\mathbb{P}^n$ of the varieties appearing, as done by Zak.

We would like to comment briefly the main differences between our approach, if correct, to the classification of Severi varieties and the ones present in the literature. Zak's approach, see also [**LV**], is based on some preliminary analysis of the geometry of SX and of the fact that all the entry loci are smooth quadric of dimension $\frac{n}{2}$. The central point in Zak's classification is a careful study of the linear spaces on the quadrics of the family $\widetilde{Q}_x$, i.e. of the entry loci of the variety to obtain $n \leq 16$ and $n \equiv 0$ (mod. 4). This is very geometric but full of details and verifications. The first part of Zak's analysis was used by Chaput, [**Ch**], to prove a priori that X is homogeneous and then one deduces the classification from the known description of homogeneous varieties. There is a different proof of the classification by Landsberg, [**La**], via local differential geometry and second fundamental form. Landsberg derives some restrictions on the linear system of quadrics describing the second fundamental form, deduces the bound of the dimension from the classification of Clifford modules and then reconstruct the variety via moving frames.

In [**Ru6**] we concentrate on conics and lines contained in a Severi variety. The classification of Severi varieties, or better of the possible dimension of such a variety, becomes a particular case of the study of conics and lines on varieties defined by quadratic equations, generalizing the case of the quadric hypersurface in $\mathbb{P}^N$, [**Ru6**]. The reconstruction of the Severi varieties in dimension 4, 8 and 16 in [**Ru6**] is analogous to Zak's one but follows the opposite direction, showing a priori, in the possible dimensions, the description of the cones $T_xX \cap X$.

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