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A two-stage variational inequality for medical supply in emergency management

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Background

Equilibrium is a central concept in numerous disciplines including economics, management science/operations research, and engineering.

Methodologies that have been applied to the formulation, qualitative analysis, and computation of equilibria have included:

- systems of equations;
- optimization theory.

Variational inequality theory is a powerful unifying methodology for the study of equilibrium problems.

Background

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- The breakthrough in finite-dimensional theory occurred in 1980 when *Dafermos* recognized that the traffic network equilibrium conditions as stated by *Smith* (1979) had a structure of a variational inequality, with a focus on **transportation**.

Background

- Variational inequality theory was introduced by *Hartman and Stampacchia* (1966) as a tool for the study of partial differential equations with applications principally drawn from mechanics.
- The breakthrough in finite-dimensional theory occurred in 1980 when *Dafermos* recognized that the traffic network equilibrium conditions as stated by *Smith* (1979) had a structure of a variational inequality, with a focus on **transportation**.
- Observe that many of the applications explored to-date that have been formulated, studied, and solved as variational inequality problems are, in fact, network problems.

  In addition, as we shall see, many of the advances in variational inequality theory have been spurred by needs in practice!

Variational inequality theory

Variational inequality theory provides us with a tool for:

- formulating a variety of equilibrium problems;
- qualitatively analyzing the problems in terms of existence and uniqueness of solutions, stability and sensitivity analysis, and
- providing us with algorithms with accompanying convergence analysis for computational purposes.

Variational Inequality

Definition

The finite-dimensional variational inequality problem, $VI(F, K)$, is to determine a vector $x^* \in \mathbb{R}$, such that $F(x^*)^T(x - x^*) \geq 0$, $\forall x \in K$

where F is a given continuous function from K to $\mathbb{R}^n$, K is a given closed convex set.

Optimization Problem

An optimization problem is characterized by its specific objective function that is to be maximized or minimized, depending upon the problem and, in the case of a constrained problem, a given set of constraints.

Proposition

Let x^* be a solution to the optimization problem:

$$\text{Minimize } f(x) \tag{1}$$

$$\text{subject to } : x \in K, \tag{2}$$

where f is continuously differentiable and K is closed and convex. Then x^* is a solution of the variational inequality problem:
$\nabla f(x^*)^T(x - x^*) \geq 0, \forall x \in K$

Existence and uniqueness of the solution

Theorem (1)

Let $K \subseteq \mathbb{R}^n$ a closed, convex non empty and bounded set.
Let $F : K \to \mathbb{R}^n$ be a continuous function, then $\exists x^* \in K$ s.t.
$F(x^*)^T(x - x^*) \geq 0 \; \forall x \in K$

Theorem (2)

Let $K \subseteq \mathbb{R}^n$ a closed, convex non empty and bounded set.
Let $F : K \to \mathbb{R}^n$ be a continuous and strictly monotonous function, then $\exists! x^* \in K$ s.t. $F(x^*)^T(x - x^*) \geq 0 \; \forall x \in K$

Supply Chain Model

Related Literature

Our research is inspired by the following articles:

In (13), the authors consider a game theory model for multiple humanitarian organizations engaged in disaster relief using a two stage stochastic model.

In (15), the authors develop a Generalized Nash Equilibrium model with stochastic demands to model competition among organizations at demand points for medical supplies.

In (19), the authors construct a stochastic Generalized Nash Equilibrium model for the study of competition among countries for limited supplies of medical items during Covid-19 pandemic.

Medical Supply Model

We present a medical supply model, where different medicine items has to be shipped from warehouses to hospitals, and propose an optimization formulation for **minimizing** transportation cost, transportation time and purchasing cost.

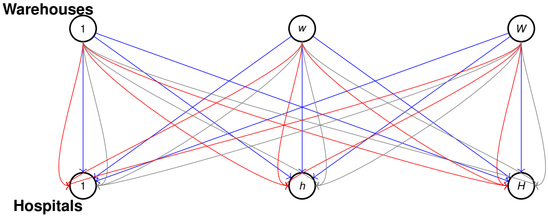

Figure: The Network representation

Notation for the Stochastic Model

- $\mathcal{W}$ set of **warehouses**, with typical warehouse denoted by w, $card(\mathcal{W}) = W$
- $\mathcal{H}$ set of **hospitals**, with typical hospital denoted by h, $card(\mathcal{H}) = H$
- $\mathcal{K}$ set of **different medical items**, with typical item denoted by k, $card(\mathcal{K}) = K$
- $\mathcal{M}$ set of **transportation modes**, with typical mode denoted by m, $card(\mathcal{M}) = M$

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- $\mathcal{M}$ set of **transportation modes**, with typical mode denoted by m, $card(\mathcal{M}) = M$
- d_h^k demand of medical item k of hospital h in stage one
- $d_h^k(\omega)$ demand of medical item k of hospital h in stage two under scenario ω
- Q_w^k the amount of the medical item type k in warehouse w
- $Q_w = \sum_{k \in \mathcal{K}} Q_w^k$ the total amount of the medical items in warehouse w
- e_k maximum amount available of medical item k

Notation for the Stochastic Model

- x^k_{wh} amount of medical item k from warehouse w to hospital h in stage one
- x_{wh} amount of medical items delivered from warehouse w to hospital h in stage one
- x amount of total medical items from all warehouses to all hospitals in stage one
- $y_{wh}(\omega)$ amount of medical items to be delivered from w to h in stage two under ω
- $z^k_h(\omega)$ amount of unfulfilled demand at hospital h of medical supply item k under scenario ω

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- x amount of total medical items from all warehouses to all hospitals in stage one
- $y_{wh}(\omega)$ amount of medical items to be delivered from w to h in stage two under ω
- $z_h^k(\omega)$ amount of unfulfilled demand at hospital h of medical supply item k under scenario ω
- ρ_w^k unitary price of medical item k at warehouse w
- $t_{wh}^m(x_{wh})$ transportation time from warehouse w to hospital h with mode m
- $c_{wh}^m(y_{wh}(\omega), \omega)$ transportation cost from warehouse w to hospital h with mode m under scenario ω
- $\pi_h^k(z_h^k(\omega), \omega)$ penalty for unfulfilled demand at hospital h of medical supply item k under scenario ω

First Stage

For each hospital h, the first-stage problem is given by

$$\min \sum_{w \in \mathcal{W}} \left(\sum_{k \in \mathcal{K}} \rho_w^k x_{wh}^k + \sum_{m \in \mathcal{M}} t_{wh}^m(x_{wh}) \right) + \mathbb{E}_\xi(\Phi_h(x, \xi(\omega))) \qquad (3)$$

subject to

$$\sum_{h \in \mathcal{H}} \sum_{k \in \mathcal{K}} x_{wh}^k \leq Q_w, \ \forall w \in \mathcal{W}, \qquad (4)$$

$$\sum_{w \in \mathcal{W}} x_{wh}^k = d_h^k, \ \forall k \in \mathcal{K}, \qquad (5)$$

$$\sum_{w \in \mathcal{W}} x_{wh}^k \leq e_k, \ \forall k \in \mathcal{K}, \qquad (6)$$

$$x_{wh}^k \geq 0, \ \forall w \in \mathcal{W}, \ \forall k \in \mathcal{K}. \qquad (7)$$

Second Stage

For a given realization $\omega \in \Omega$, $\Phi_h(x, \xi(\omega))$ is the optimal value of the second-stage problem (8)-(11) of hospital h, where the constraints hold almost surely (P-a.s.).

$$\Phi_h(x, \xi(\omega)) = \min \sum_{w \in \mathcal{W}} \sum_{m \in \mathcal{M}} c_{wh}^m(y_{wh}(\omega), \omega) + \sum_{k \in \mathcal{K}} \pi_h^k(z_h^k(\omega), \omega) \quad (8)$$

subject to

$$\sum_{h \in \mathcal{H}} \sum_{k \in K} y_{wh}^k(\omega) \leq Q_w(\omega) - \sum_{h \in \mathcal{H}} \sum_{k \in \mathcal{K}} x_{wh}^k, \; \forall w \in \mathcal{W}, P\text{-}a.s., \quad (9)$$

$$\sum_{w \in \mathcal{W}} y_{wh}^k(\omega) + z_h^k(\omega) = d_h^k(\omega), \; \forall k \in \mathcal{K}, P\text{-}a.s., \quad (10)$$

$$\sum_{w \in \mathcal{W}} y_{wh}^k(\omega) \leq e_k(\omega), \; \forall k \in \mathcal{K}, P\text{-}a.s., \quad (11)$$

$$y_{wh}^k(\omega) \geq 0, z_h^k(\omega) \geq 0 \; \forall w \in \mathcal{W}, \; \forall k \in \mathcal{K}, P\text{-}a.s. \quad (12)$$

Assumptions

We assume that:

a) $t^m_{wh}(\cdot)$, is continuously differentiable and convex for all w, h, k;

b) $c^m_{wh}(\cdot, \omega)$, $\pi^k_h(\cdot, \omega)$, a.e. in Ω, are continuously differentiable and convex for all w, h, k, m;

c) for each $u \in \mathbb{R}^{WH}$, $c^m_{wh}(u, \cdot)$ is measurable with respect to the random parameter in Ω for all w, h, m;

d) for each $v \in \mathbb{R}^{HK}$, $\pi^k_h(v, \cdot)$ is measurable with respect to the random parameter in Ω for all h, k;

e) $y^k_{wh} : \Omega \to \mathbb{R}$ and $z^k_h : \Omega \to \mathbb{R}$ are measurable mappings for all w, h, k;

f) $d^k_h : \Omega \to \mathbb{R}$ is a measurable mapping for all h and all k.

Unique large scale problem

If the random parameter $\omega \in \Omega$ follows a discrete distribution with finite support $\Omega = \{\omega_1, \ldots, \omega_r\}$ and probabilities $p(\omega_r)$ associated with each realization ω_r, $r \in \mathcal{R} = \{1, \ldots, R\}$, then the two-stage problem of hospital h can be formulated as the unique large scale problem:

$$\min \sum_{w \in \mathcal{W}} \left(\sum_{k \in \mathcal{K}} \rho^k_{wh} x^k_{wh} + \sum_{m \in \mathcal{M}} t^m_{wh}(x_{wh}) \right)$$

$$+ \sum_{r \in \mathcal{R}} p(\omega_r) \left(\sum_{w \in \mathcal{W}} \sum_{m \in \mathcal{M}} c^m_{wh}(y_{wh}(\omega_r), \omega_r) + \sum_{k \in \mathcal{K}} \pi^k_h(z^k_h(\omega_r), \omega_r) \right) \quad (13)$$

subject to

$$\sum_{h \in \mathcal{H}} \sum_{k \in \mathcal{K}} x^k_{wh} \leq Q_w, \ \forall w \in \mathcal{W}, \quad (14)$$

$$\sum_{w \in \mathcal{W}} x^k_{wh} = d^k_h, \ \forall k \in \mathcal{K}, \quad (15)$$

Unique large scale problem

$$\sum_{w \in \mathcal{W}} x_{wh}^k \leq e_k, \; \forall k \in \mathcal{K}, \tag{16}$$

$$\sum_{h \in \mathcal{H}} \sum_{k \in \mathcal{K}} y_{wh}^k(\omega_r) \leq Q_w(\omega_r) - \sum_{h \in \mathcal{H}} \sum_{k \in \mathcal{K}} x_{wh}^k, \; \forall w \in \mathcal{W}, \forall r \in \mathcal{R}, \tag{17}$$

$$\sum_{w \in \mathcal{W}} y_{wh}^k(\omega_r) + z_h^k(\omega_r) = d_h^k(\omega_r), \; \forall k \in \mathcal{K}, \forall r \in \mathcal{R}, \tag{18}$$

$$\sum_{w \in \mathcal{W}} y_{wh}^k(\omega_r) \leq e_k(\omega_r), \; \forall k \in \mathcal{K}, \forall r \in \mathcal{R}, \tag{19}$$

$$x_{wh}^k \geq 0, \; \forall w \in \mathcal{W}, \; \forall k \in \mathcal{K}, \tag{20}$$

$$y_{wh}^k(\omega_r) \geq 0, \; \forall w \in \mathcal{W}, \; \forall k \in \mathcal{K}, \forall r \in \mathcal{R}, \tag{21}$$

$$z_h^k(\omega_r) \geq 0, \; \forall k \in \mathcal{K}, \forall r \in \mathcal{R}. \tag{22}$$

Stochastic Generalized Nash equilibrium

Competition for medical supplies among hospitals can be studied as a game.

The underlying equilibrium concept is then that of a stochastic generalized Nash equilibrium (**SGNE**), namely a Nash equilibrium when the cost functions are expected value functions and the players are subject to shared constraints.

We define the sets:

$$S_h = \left\{ x_h = (x_{wh}^k)_{w,k} \in \mathbb{R}^{WK} : (5) - (7) \text{ hold} \right\},$$

$$X = \{ x = (x_h)_h \in \mathbb{R}^H : x \text{ satisfies } (4) \},$$

$$T_h = \left\{ (y_h(\omega), z_h(\omega)) = \left((y_{wh}^k(\omega))_{w,k}, (z_h^k(\omega))_k \right) \in \mathbb{R}^{WK+K} : (10) - (12) \text{ hold}, \right.$$

$$V = \{ (y(\omega), z(\omega)) = (y_h(\omega), z_h(\omega))_h \in \mathbb{R}^{2H} : (9) \text{ holds}, \ P\text{-}a.s. \}.$$

We also define $S = \prod_h S_h$ and $T = \prod_h T_h$.

Stochastic Generalized Nash equilibrium

We refer to the objective function (3) for $h \in \mathcal{H}$ as the function

$$\mathbb{J}_h(x_h, x_{-h}) = \sum_{w \in \mathcal{W}} \left(\sum_{k \in \mathcal{K}} \rho^k_{wh} x^k_{wh} + \sum_{m \in \mathcal{M}} t^m_{wh}(x_{wh}) \right) + \mathbb{E}_\xi(\Phi_h(x_h, x_{-h}, \xi(\omega))),$$

where x_{-h} denotes the amount of medical items required by all hospitals except for h.

Definition

A vector of medical items $x^* = (x^*_h, x^*_{-h}) \in S \cap X$ is a stochastic generalized Nash equilibrium of the first stage if for each $h \in \mathcal{H}$

$$\mathbb{J}_h(x^*_h, x^*_{-h}) \leq \mathbb{J}_h(x_h, x^*_{-h}), \quad \forall x_h \in S_h, \forall x \in X.$$

Analogously, we can define the **SGNE** for the second stage.
A solution of such a problem can be found solving a quasi-variational inequality.

Finite-dimensional VI problem

Theorem

The vector $(x^*, y^*(\omega_r), z^*(\omega_r))$, $\forall \omega_r, r \in \mathcal{R}$, is an optimal solution of the medical item procurement planning if and only if:

1. *the vector $x^* = (x^*_h, x^*_{-h}) \in S \cap X$ is a solution of the variational inequality*

$$\sum_{w\in\mathcal{W}}\sum_{h\in\mathcal{H}}\sum_{k\in\mathcal{K}}\left(\rho^k_{wh} + \sum_{m\in\mathcal{M}}\frac{\partial t^m_{wh}(x^*_{wh})}{\partial x^k_{wh}} + \sum_{r\in\mathcal{R}} p(\omega_r)\frac{\partial \Phi_h(x^*, \xi(\omega_r))}{\partial x^k_{wh}}\right) \times (x^k_{wh} - x^{*k}_{wh}) \geq 0,$$
$$\forall x \in S \cap X; \qquad (23)$$

2. *the vector $(y^*(\omega_r), z^*(\omega_r)) \in T \cap V$, $\forall \omega_r, r \in \mathcal{R}$, is a solution of the variational inequality*

$$\sum_{r\in\mathcal{R}} p(\omega_r)\sum_{w\in\mathcal{W}}\sum_{h\in\mathcal{H}}\sum_{k\in\mathcal{K}}\left(\sum_{m\in\mathcal{M}}\frac{\partial c^m_{wh}(y^*_{wh}(\omega_r), \omega_r)}{\partial y^k_{wh}(\omega_r)}\right) \times (y^k_{wh}(\omega_r) - y^{*k}_{wh}(\omega_r))$$
$$+ \sum_{r\in\mathcal{R}} p(\omega_r)\sum_{h\in\mathcal{H}}\sum_{k\in\mathcal{K}}\frac{\partial \pi^k_h(z^{*k}_h(\omega_r), \omega_r)}{\partial z^k_h(\omega_r)} \times (z^k_h(\omega_r) - z^{*k}_h(\omega_r)) \geq 0,$$
$$\forall (y(\omega_r), z(\omega_r)) \in V \cap T. \qquad (24)$$

Lagrangian relaxation approach

Alternative Method

1. *Find $x^* = (x^*_h, x^*_{-h}) \in S$ such that*

$$\sum_{w\in\mathcal{W}}\sum_{h\in\mathcal{H}}\sum_{k\in\mathcal{K}}\left(\rho^k_{wh} + \sum_{m\in\mathcal{M}}\frac{\partial t^m_{wh}(x^*_{wh})}{\partial x^k_{wh}} + \sum_{r\in\mathcal{R}} p(\omega_r)\frac{\partial \Phi_h(x^*, \xi(\omega_r))}{\partial x^k_{wh}} + \lambda_w\right) \times (x^k_{wh} - x^{*k}_{wh})$$
$$+ \sum_{w\in\mathcal{W}}\left(Q_w - \sum_{h\in\mathcal{H}}\sum_{k\in\mathcal{K}} x^{*k}_{wh}\right) \times (\lambda_w - \lambda^*_w) \geq 0, \forall x \in S, \lambda \in \mathbb{R}^W_+. \quad (25)$$

2. *Find $(y^*(\omega_r), z^*(\omega_r)) \in T$, $\forall \omega_r, r \in \mathcal{R}$, such that*

$$\sum_{r\in\mathcal{R}} p(\omega_r)\sum_{w\in\mathcal{W}}\sum_{h\in\mathcal{H}}\sum_{k\in\mathcal{K}}\sum_{m\in\mathcal{M}}\left(\frac{\partial c^m_{wh}(y^*_{wh}(\omega_r), \omega_r)}{\partial y^k_{wh}(\omega_r)} + \mu_w(\omega_r)\right) \times (y^k_{wh}(\omega_r) - y^{*k}_{wh}(\omega_r))$$
$$+ \sum_{r\in\mathcal{R}} p(\omega_r)\sum_{h\in\mathcal{H}}\sum_{k\in\mathcal{K}}\frac{\partial \pi^k_h(z^{*k}_h(\omega_r), \omega_r)}{\partial z^k_h(\omega_r)} \times (z^k_h(\omega_r) - z^{*k}_h(\omega_r))$$
$$+ \sum_{r\in\mathcal{R}} p(\omega_r)\sum_{w\in\mathcal{W}}\left(Q_w(\omega_r) - \sum_{h\in\mathcal{H}}\sum_{k\in\mathcal{K}}(x^k_{wh} + y^{*k}_{wh}(\omega_r))\right) \times (\mu_w(\omega_r) - \mu^*_w(\omega_r)) \geq 0$$
$$\forall (y(\omega_r), z(\omega_r)) \in T, \forall \mu(\omega_r) \in \mathbb{R}^W_+. \quad (26)$$

Conclusions

Innovative Aspects

- *We studied a two-stage procurement planning model in a random environment.*
- *We obtained a plan of medical item procurement/distribution for each demand location in the first stage by the evaluation of adaptive plans in the second stage under different disaster scenarios.*
- *For each hospital, we minimized the purchasing costs and the transportation time of the first stage with the expected overall costs.*
- *We provided a variational inequality formulation of the problem.*

Possible Extensions

The results presented are just a small part of a bigger study that is still under work.

- *Use a continuous distribution of probability instead of discrete one.*
- *Analyse the two-stage using a quasi-variational inequality approach.*

THANK YOU FOR YOUR ATTENTION

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