PELCZYNSKI'S PROPERTY (V) AND WEAK* BASIC SEQUENCES

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Abstract. In this note we study the property (V) of Pelczynski, in a Banach space X, in relation with the presence, in the dual Banach space X^*, of suitable *weak** basic sequences. We answer negatively to a question posed by John and we prove that, if X is a Banach space with the Property (V) of Pelczynski and the Gelfand Phillips property, then X is reflexive if and only if every quotient with a basis is reflexive. Moreover, we prove that, if X is a Banach space with the property (V) of Pelczynski, then either X is a Grothendieck space or, $W(X,Y)$ is uncomplemented in $L(X,Y)$ provided that Y is a Banach space such that $W(X,Y) \neq L(X,Y)$.

In [8] the authors gave some existence theorems for *weak** basic sequences. In particular they proved that the following assertions hold:
1) every weak* null normalized sequence $(x_n^*)_n$ in the dual of a separable Banach space admits a weak* basic subsequence
2) if $(x_n^*)_n$ is a weak* basic sequence, then the quotient space $X/((x_n^*)_n)^\perp$ has a basis and $\widetilde{span}((x_n^*)_n)$ (see below for the notations) can be identified with the dual space $\left(X/((x_n^*)_n)^\perp\right)^*$.

In this note we prove that the property (V) of Pelczynski in a (not necessarily separable) Banach space X, that is not a Grothendieck space, implies the existence in X^* of a suitable *weak** basic sequence $(x_n^*)_n$, equivalent to the unit vector basis in l_1, such that the weak* closure of its linear span coincides with its norm closure and is complemented in X^*. The same result holds if X has the weak property (V) and, in this case, our Theorem gives more properties of the sequence $(x_n^*)_n$ than Proposition 2.1 (implication 1) $\Rightarrow$ 2)) in [2].

As a Corollary, we give a characterization of the Grothendieck spaces with the property (V) of Pelczynski.

In [7] the author proved that a separable Banach space X is reflexive if and only if each of its quotients with a basis is reflexive and he asked if this equivalence holds even without the hypothesis of separability. We observe that, in general, the answer is negative, and we prove (see Proposition 12) that the equivalence remains true for the class of all Banach spaces with the Gelfand Phillips property and the property (V) of Pelczynski. The equivalence in John's Theorem is also true if X is a Banach lattice with the Gelfand Phillips property (even without the property (V) of Pelczynski), as we shall prove in Proposition 14 below.

Moreover, using the main Theorem of the paper, Proposition 15 relates the presence of the property (V) of Pelczynski in a Banach space X with the largely explored

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problem of the uncomplementability of the space $W(X,Y)$ into the space $L(X,Y)$ provided that Y is a Banach space such that $W(X,Y) \neq L(X,Y)$.

Finally, we give a sufficient condition, in relation to the presence of suitable sequences in the dual space X^*, in order that the space X could have the property (V) of Pelczynski.

Throughout X,Y will denote Banach spaces. If X is a Banach space, then X^* denotes its dual Banach space. The symbols $L(X,Y)$, $W(X,Y)$ and $UC(X,Y)$ will denote respectively the space of all linear continuous operators from X into Y, the subspace of all weakly compact operators and the subspace of all unconditionally converging operators. If $T \in L(X,Y)$, then $T^* \in L(Y^*, X^*)$ will be its adjoint operator.

A sequence (x_n, x_n^*) with $(x_n)_n \subseteq X$ and $(x_n^*)_n \subseteq X^*$ is called biorthogonal provided that $x_m^*(x_n) = \delta_{nm}$ for all ever $n, m \in \mathbb{N}$.

A sequence (x_n) in X is a basis for X provided that for each $x \in X$ there is a unique sequence $(a_n^x)_n$ of scalars for which $\lim_n \sum_{i=1}^{n} a_i^x x_i = x$. If, for each $n \in \mathbb{N}$, x_n^* is the linear continuous functional defined by the law

$$x_n^*(x) = a_n^x \quad \forall x \in X,$$

then, as one can easily see, (x_n, x_n^*) is biorthogonal and $(x_n^*)_n$ forms a basis for its norm closed linear span.

If A is a subset of the Banach space X, then $\overline{A}$ denotes the norm closure of A in X, while, for a subset A of X^*, the symbol $\widetilde{A}$ denotes the *weak** closure of A in X^*. If A is a subset of a dual space X^*, then $A^\perp$ is the annihilator of A in X, i.e., $A^\perp = \{x \in X : x^*(x) = 0 \quad \forall x^* \in A\}$.

A Banach space X is called a Grothendieck space if every weak* convergent sequence in the dual space X^* is also weak convergent (see [3]).

Definition 1. [8] [12] *A sequence $(x_n^*)_n$ in X^* is called weak* basic if there is a sequence $(x_n)_n$ in X so that $\{(x_n, x_n^*) : n \in N\}$ is biorthogonal and, for each $x^* \in \widetilde{span}(x_n^*)$, one has*

$$x^* = \lim_n \sum_{i=1}^{n} x^*(x_i) x_i^*$$

in the weak topology of X^*.*
Note that the sequence $(x_n)_n$ is not uniquely determined by $(x_n^*)_n$, but, if $(x_n)_n$ and $(x'_n)_n$ are two sequences such that (x_n, x^*) and (x'_n, x_n^*) are biorthogonal, then

$$x^*(x_n) = x^*(x'_n) \quad \forall n \in N, \quad \forall x^* \in \widetilde{span((x_n^*)_n)}.$$

Definition 2. [10] *A Banach space X has the property (V) of Pelczynski if every unconditionally converging operator defined on it with values in a Banach space Y is weakly compact.*

It is well known, (see[10]), that, for every compact space K, the space $C(K)$ has the property (V) of Pelczynski. All Banach spaces not containing a copy of l_1 with

property (u) (see [11] for this definition) have property (V) of Pelczynski. Moreover, if X is a Banach space not containing a copy of l_1, with property (u), then, for every compact Hausdorff space K, the space $C(K,X)$ has the property (V) of Pelczynski too.
In [10] the author gave the following

Definition 3. *A bounded subset K in X^* is a V-set if*

$$\lim_n \sup_{x^* \in K} \mid x^*(x_n) \mid = 0$$

for every weakly unconditionally Cauchy series $\sum_{n=1}^{\infty} x_n$ *in X.*

Proposition 4. [10] *A Banach space X has the property (V) of Pelczynski if and only if every V-set in the dual space X^* is weakly relatively compact.*

In order to obtain our main result we need the following

Proposition 5. *Let X be a Banach space. A bounded subset K in X^* is a V-set if and only if, for every linear and continuous operator $T : c_0 \to X$, it follows that $T^*(K)$ is a relatively compact set.*

Proof. Let K be a V-set in X^* and let $T \in L(c_0, X)$. Then the set $T^*(K)$ is also a V-set in l_1. Since c_0 has the property (V) [10], by Proposition 4, $T^*(K)$ is a relatively weakly compact subset of l_1 and then, by the Schur property of l_1, it is a relatively compact set. Conversely, let $\sum_{n=1}^{\infty} x_n$ be a weakly unconditionally Cauchy series in X and let $T : c_0 \to X$ be the linear continuous operator defined by the law

$$T(e_n) = x_n \quad \forall n \in \mathbb{N}.$$

Easily,

$$T^*(x^*) = (x^*(x_n))_n \quad \forall x^* \in X^*.$$

By the hypothesis, $T^*(K)$ is relatively compact, so, using the characterization of the compact sets in l_1, we have

$$\lim_n \sup_{x^* \in K} \mid x^*(x_n) \mid = 0$$

that is our claim. □

Lemma 6. *Let (x_n^*) be a weak* null, not relatively compact sequence in l_1. Then there are a subsequence $(x_{k_n}^*)_n$ of $(x_n^*)_n$ and a sequence $(x_n)_n$ in c_0, equivalent to the unit vector basis in c_0, such that $(x_n, x_{k_n}^*)$ is biorthogonal.*

Proof. By Rosenthal's Theorem $(x_n^*)_n$ admits a subsequence $(x_{k_n}^*)_n$ that is equivalent to the unit vector basis $(e_n^*)_n$ of l_1. Let J be the isomorphism from $\overline{span((x_{k_n}^*)_n)}$ onto $\overline{span((e_n^*)_n)}$ such that $J(x_{k_n}^*) = e_n^*$ for every $n \in \mathbb{N}$. If we denote with $(e_n)_n$ the unit vector basis in c_0, we have

$$\delta_{nm} =< e_n^*, e_m >=< J(x_{k_n}^*), e_m >=< x_{k_n}^*, J^* e_m > \quad \forall n, m \in \mathbb{N}.$$

Therefore, if we put
$$x_m = J^* e_m \quad \forall m \in \mathbb{N},$$
it follows
$$\delta_{nm} =< x^*_{k_n}, x_m > \quad \forall n, m \in \mathbb{N}.$$
It is easy to see that the sequence $(x_m)_m$ is equivalent to the canonical basis in c_0 and so we are done. □

Theorem 7. *Let X be a Banach space with the property (V) of Pelczynski. Then, for every weak* null sequence $(x^*_n)_n$, that is not a weakly relatively compact set in X^*, there is a weak* basic subsequence $(x^*_{k_n})_n$ such that*
1) $\widetilde{span}((x^*_{k_n})_n) = \overline{span((x^*_{k_n})_n)}$;
2) the quotient space $X/((x^*_{k_n})_n)^\perp$ is isomorphic to c_0.
Moreover there is a linear continuous operator $P : X \to X$ such that P^* is a projection from X^* into $\widetilde{span}((x^*_{k_n})_n)$.

Proof. By Proposition 4, since X has the property (V) of Pelczynski and the bounded set $(x^*_n)_n$ is not weakly relatively compact, it follows that $(x^*_n)_n$ is not a V-set. Therefore, by Proposition 5, there is $T \in L(c_0, X)$ such that $(T^*(x^*_n))_n$ is not relatively (weakly) compact in l_1. Since l_1 is weakly sequentially complete, it does not exist a weak Cauchy subsequence of $(T^*(x^*_n))_n$ and then $(x^*_n)_n$ does not admit a weak Cauchy subsequence. By Rosenthal's Theorem, we can suppose (by passing, if necessary, to a subsequence) that $(x^*_n)_n$ is equivalent to the unit vector basis in l_1. Since the sequence $(T^*(x^*_n))_n$ is *weak** null, the hypotheses of Lemma 6 are verified. So, there are a subsequence $(x^*_{k_n})_n$ of $(x^*_n)_n$, that is still equivalent to the unit basis of l_1 and a sequence $(f_n)_n$ in c_0, equivalent to the unit vector basis of c_0, such that $(T^*(x^*_{k_n}), f_n)$ is a biorthogonal sequence.
Since $(f_n)_n$ is equivalent to the unit vector basis of c_0, the series
$$\sum_{n=1}^{\infty} |x^*(T(f_n))| = \sum_{n=1}^{\infty} |T^* x^*(f_n)|$$
converges for every $x^* \in X^*$. Since $(T^*(x^*_{k_n}), f_n)$ is a biorthogonal sequence, the sequence $(\|T(f_n)\|)_n$ does not converge to zero. It follows that the series $\sum_{n=1}^{\infty} T(f_n)$ is weakly unconditionally Cauchy, but it is not unconditionally convergent. By [4, Theorem 8, pag 45], X contains a copy of c_0 and, more precisely, we can find a subsequence of $(T(f_n))_n$ that is equivalent to the unit vector basis of c_0. We can suppose that $(T(f_n))_n$ is itself equivalent to the unit vector basis of c_0.
Now let
$$x_n = T(f_n) \quad \forall n \in \mathbb{N}.$$
Let $Y = \overline{span((x_n)_n)}$ and let $Q : X \to Y$ be defined by the law
$$Q(x) = \sum_{n=1}^{\infty} x^*_{k_n}(x) x_n \quad \forall x \in X.$$

Since $(x^*_{k_n})_n$ is *weak** null, Q is well defined [4, Theorem 6, pag 44]. Moreover, Q is a linear and continuous operator and, using the fact that $(x_n, x^*_{k_n})$ is a biorthogonal sequence, it is easy to verify that Q is a projection. For every index n, let z^*_n be the restriction of $x^*_{k_n}$ on Y. It follows that $(z^*_n)_n$ is a basis for Y^*. Indeed, if $y^* \in Y^*$, we have

$$\begin{aligned} \langle y^*, y\rangle &= \left\langle y^*, \sum_{m=1}^{\infty} x^*_{k_m}(y)x_m \right\rangle \\ &= \left\langle \sum_{m=1}^{\infty} y^*(x_m)z^*_m,\ y \right\rangle \quad \forall y \in Y \end{aligned}$$

Further, since (x_n, z^*_n) is a biorthogonal sequence, we have

$$\begin{aligned} Q^*(z^*_n)(x) &= z^*_n(Q(x)) \\ &= z^*_n\left(\sum_{n=1}^{\infty} x^*_{k_n}(x)x_n\right) \\ &= x^*_{k_n}(x) \quad \forall x \in X. \end{aligned}$$

We also observe that the sequences $(x^*_{k_n})_n$ and $(z^*_n)_n$ are equivalent to the unit vector basis of l_1 and so, for every sequence $(a_n)_n \in l_1$, the series $\sum\limits_{n=1}^{\infty} a_n x^*_{k_n}$ converges if and only if the series $\sum_{n=1}^{\infty} a_n z^*_n$ converges. It is easy to see that

$$Q^*(Y^*) = \overline{span((x^*_{k_n})_n)}$$

and that Q^* is an isomorphism onto the set $\overline{span((x^*_{k_n})_n)}$. Moreover, since $Q^*(Y^*)$ is *weak** closed, it follows that

$$\overline{span((x^*_{k_n})_n)} = span\widetilde{((x^*_{k_n})}_n)$$

and then one can see that

$$x^* = \lim_n \sum_{i=1}^{n} x^*(x_i)x^*_{k_i} \quad \forall x^* \in span\widetilde{((x^*_{k_n})}_n)$$

in the *weak** topology. So $(x^*_{k_n})_n$ is a *weak** basic sequence.

Now, applying [8, Proposition 2.2], it follows that the set $X/((x^*_{k_n})_n)^{\perp}$ is isomorphic to c_0. Let

$$J : Y \to X$$

be the natural embedding. Look at

$$Q^* \circ J^* : X^* \to X^*$$

and its restriction on $\overline{span((x^*_{k_n})_n)}$.

It follows that

$$
\begin{aligned}
Q^*(J^*(x^*)) &= Q^*J^*\left(\sum_{n=1}^{\infty} a_n x^*_{k_n}\right)\\
&= \sum_{n=1}^{\infty} a_n Q^*(J^*(x^*_{k_n}))\\
&= \sum_{n=1}^{\infty} a_n Q^*(z^*_n)\\
&= \sum_{n=1}^{\infty} a_n x^*_{k_n}\\
&= x^* \quad \forall x^* = \sum_{n=1}^{\infty} a_n x^*_{k_n} \in \overline{span((x^*_{k_n})_n)}
\end{aligned}
$$

where we have used the fact that

$$J^*(x^*_{k_n)})(y) = x^*_{k_n}(J(y)) = x^*_{k_n}(y) = z^*_n(y) \quad \forall y \in Y.$$

It follows that $Q^* \circ J^*$ is a projection onto $\overline{span((x^*_{k_n})_n)}$. Defining

$$P := J \circ Q : X \to X,$$

our claim is proved. □

In [13] the authors introduced the following

Definition 8. *A Banach space X has the weak property (V) if and only if every V-set in the dual space X^* is weakly conditionally compact.*

In [2, Proposition 2.1, implication 1) → 2)] the authors, proved that, if a Banach space X has the weak property (V), then, for each sequence in X^*, that is equivalent to the unit basis of l_1, there are $\epsilon > 0$, a subsequence $(x^*_{n_k})_k$ and a sequence $(x_k)_k \subset X$ such that the series $\sum x_n$ is unconditionally weak Cauchy and $< x^*_{n_k}, x_k >\geq \epsilon$ for every $k \in \mathbb{N}$.

Theorem 7 holds even if X has the weak property (V) and there is a weak* null sequence $(x^*_n) \subset X^*$ that is not weakly conditionally compact, and, under these hypotheses, it gives more conditions than the mentioned result in [2].

Corollary 9. *Let X be a Banach space with the property (V) of Pelczynski. The following assertions are equivalent*
1) X is not a Grothendieck space
2) there is a weak $\widetilde{\text{basic}}$ subsequence $(x^*_n)_n \subset X^*$ such that it is equivalent to the basis of l_1, and $\widetilde{span}((x^*_n)_n) = \overline{span((x^*_n)_n)}$ is complemented in X^*. Moreover there is a quotient of X, having a basis, that is isomorphic to c_0*
3) there is a quotient of X, having a basis, that is not reflexive.

Proof. 1) ⇒ 2) By the hypothesis, there is $T \in L(X, c_0)$ that is not weakly compact. Therefore there is a bounded sequence $(y^*_n)_n$ in l_1 that is weak star null

but none of the subsequences of $(T^*(y_n^*))_n$ is weakly convergent. Then it is enough to apply Theorem 7.
2) $\Rightarrow$ 3) obvious
3) $\Rightarrow$ 1) is well known (see, for example, [1]) $\square$

Remark 10. *It is well known that X is a Grothendieck space if and only if X^* is weakly sequentially complete and there is a quotient of X that is isomorphic to c_0 (see* [6] *and references there). Therefore the equivalence* $1 \Leftrightarrow 3$ *in Corollary 9 is true even if the hypothesis of the presence of the property (V) of Pelczynski is replaced by the weaker hypothesis of the weak sequential completeness of X^*.*

In [7] the author proved that a separable Banach space X is reflexive if and only if each of its quotients with a basis is reflexive and he asked if this equivalence holds even without the hypothesis of separability. It seems that the answer is negative. Indeed, l_∞ has property (V) of Pelczynski, and, since it is a Grothendieck space, by Corollary 9, every its quotient with a basis is reflexive. For the existence of a quotient with a basis of the space l_∞ we refer the reader to [9, Theorem 4.1].

More generally, every not reflexive Grothendieck space that admits quotients with basis works to answer negatively to John's question.

In the following Proposition we prove that the result in [7] is true even for (not separable) Banach spaces with the Gelfand Phillips property and the property (V) of Pelczynski. We recall the following

Definition 11. *A bounded subset K in a Banach space X is limited if*

$$\lim_n \sup_{x \in K} \mid x_n^*(x) \mid = 0$$

for every weak null sequence $(x_n^*)_n$ in X^*.*

A Banach space has the Gelfand Phillips property if every its limited set is relatively compact.

Banach spaces having the Gelfand Phillips property are, among the others, separable Banach spaces, Schur spaces, separably complemented spaces, reflexive Banach spaces, spaces with weak* sequentially compact dual balls and spaces C(K), where K is both compact and sequentially compact. It is easy to see that the Gelfand Phillips property is inherited by closed subspaces. We refer the reader to [14] for more about this property.

Proposition 12. *Let X be a Banach space with the Gelfand Phillips property and the property (V) of Pelczynski. Then the following assertions are equivalent*
1) X is reflexive
2) every quotient of X having a basis is reflexive.

Proof. Obviously we have to prove only that 2) $\Rightarrow$ 1). By Corollary 9, X is a Grothendieck space. So it does not contain a complemented copy of c_0. On the other hand, if a Banach space with the Gelfand Phillips property contains a copy of c_0, it must contain a complemented copy of c_0 (see [14]), so X cannot contain c_0. Since X has property (V) of Pelczynski, it must be reflexive. $\square$

Remark 13. *The proof of Proposition 12 allows us to characterize the class of all reflexive spaces as the class of all Banach spaces with Grothendieck property, Gelfand Phillips property and (V) property.*

Now we reformulate Proposition 12 for Banach lattices.

Proposition 14. *Let X be a Banach lattice with the Gelfand Phillips property. Then the following assertions are equivalent*
1) X is reflexive
2) every quotient of X having a basis is reflexive.

Proof. Again we have to prove that 2) $\Rightarrow$ 1). As before, by Corollary 9, the hypothesis implies that X is a Grothendieck space. Therefore, every operator T from X into a Gelfand Phillips space is weakly compact. In particular $Id : X \to X$ is weakly compact so we are done. $\square$

Observe that all Grothendieck Banach space X such that B_{X^*} is weakly* sequentially compact are reflexive. We do not know if there is a Grothendieck Banach space (not Banach lattice) not reflexive with the Gelfand Phillips property. If there is such a space, then it cannot contain c_0, so it does not have property (V) of Pelczynski.

Proposition 15. *Let X be a Banach space with the property (V) of Pelczynski. Then either X is a Grothendieck space or, for every Banach space Y such that $W(X,Y) \neq L(X,Y)$, it follows that $W(X,Y)$ is uncomplemented in $L(X,Y)$.*

Proof. Suppose that X is not a Grothendieck space and that $W(X,Y) \neq L(X,Y)$. Then c_0 embeds into Y. Indeed if not, $UC(X,Y) = L(X,Y)$, on the other hand, by property (V), $W(X,Y) = UC(X,Y)$, so we have a contradiction. Now, since X is not a Grothendieck space, following the proof of Theorem 7, X contains a complemented coy of c_0 and then $W(X,Y)$ is uncomplemented in $L(X,Y)$ by [5, Corollary 3.4]. $\square$

We end this note presenting a converse, in a certain sense, of Theorem 7, that is a sufficient condition in order that a Banach space could have the property (V) or the weak property (V). A similar condition has been given in [2].

Theorem 16. *Let X be a Banach space with the following property: for every sequence $(x_n^*)_n$ that is not a weakly conditionally (respectively relatively) compact set, there is a subsequence $(x_{k_n}^*)_n$ such that*
1) $X/\underbrace{((x_{k_n}^*)_n)}^{\perp}$ is isomorphic to c_0 and
2) $\widetilde{span}((x_n^*)_n)$ is complemented in X^* by a projection that is an adjoint operator. Then X has the weak property (V) (respectively the property (V)) of Pelczynski.

Proof. Suppose that there exists a bounded V-set $K \subset X^*$ that is not weakly conditionally compact. Then there is a bounded sequence $(x_n^*)_n$ in K without a weak Cauchy subsequence. By the hypothesis, there exist a subsequence $(x_{k_n}^*)_n$ such that $X/((x_{k_n}^*)_n)^\perp$ is isomorphic to c_0 and an operator $P : X/((x_{k_n}^*)_n)^\perp \to X$ such that $P^* : X^* \to \widetilde{span}((x_{k_n}^*)_n) \cong \left(X/((x_{k_n}^*)_n)^\perp\right)^*$ is a projection. Let $J : c_0 \to X/((x_{k_n}^*)_n)^\perp$ be such an isomorphism. Now look at $P \circ J : c_0 \to X$. By lemma 5, the set $(P \circ J)^*(K)$ is relatively compact in l_1. Since P^* is a projection, it follows that

$$(P \circ J)^*(x^*_{k_n}) = J^*(P^*(x^*_{k_n})) = J^*(x^*_{k_n}) \quad \forall n \in \mathbb{N}.$$

So $(J^*(x^*_{k_n}))_n$ is relatively compact in l_1. Since J is an isomorphism, $(x^*_{k_n})_n$ is a relatively compact in X^*, and it is in contrast with the choice of $(x^*_{k_n})_n$. $\square$

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