SPACES OF BOCHNER INTEGRABLE FUNCTIONS AND SPACES OF REPRESENTABLE OPERATORS AS $\mathfrak{U}$-IDEALS

By G. EMMANUELE *and* T. S. S. R. K. RAO[1]

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Introduction

IN this paper we study the geometry of the space of Bochner integrable functions as a subspace of the space of vector valued countably additive measures of finite variation and that of the space of representable operators as a subspace of the space of bounded linear operators.

Let E be a Banach space and $(\Omega, \mathcal{A}, \mu)$ a finite measure space; let $cabv(\mu, E)$ denote the space of countably additive E-valued measures of finite variation that are absolutely continuous with respect to μ. Drewnowski and Emmanuele [6] have proved recently that if E has a copy of c_0 then $L^1(\mu, E)$, the space of Bochner integrable functions, is not complemented in $cabv(\mu, E)$. However, if one is interested in weaker geometric conditions like being locally 1-complemented (recall from [12], that a closed subspace Y of a Banach space X is said to be locally 1-complemented if $Y^{\perp}$ is the kernel of a norm one projection in X^*; such a subspace was called an "ideal" in [9]) then $L^1(\mu, E)$ is always locally 1-complemented in $cabv(\mu, E)$. Similarly $\mathcal{R}(L^1(\mu), E)$ the space of representable operators from $L^1(\mu)$ to E is a locally 1-complemented subspace of $\mathcal{L}(L^1(\mu)E)$, the space of bounded linear operators (see [14]). Here again, if E has a copy of c_0 then $\mathcal{R}(L^1(\mu), E)$ is not complemented in $\mathcal{L}(L^1(\mu), E)$ (see [8]).

These results suggest that better and more reasonable geometric properties to study in this context are the notions of $\mathcal{U}$-ideal and $\mathcal{U}$-summand recently introduced by Godefroy, Kalton and Saphar in [9]. Let us recall from [9], that a subspace Y of X is said to be a $\mathcal{U}$-summand if Y is the range of a (unique) projection P in X satisfying $\|I - 2P\| = 1$ and Y is said to be a $\mathcal{U}$-ideal if $Y^{\perp}$ is a $\mathcal{U}$-summand in X^*. Since the condition $\|I - 2P\| = 1$ implies $\|P\| = 1$ (and $\|I - P\| = 1$) any $\mathcal{U}$-ideal is clearly an ideal. Following the approach of Godefroy et al, who use the notion of unconditional compact approximation property ($UKAP$) to study similar questions in the context of the space of compact operators,

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in the first part of this paper we introduce the notion of a Radon-Nikodym approximation property and show that if E has such a property then $L^1(\mu, E)$ is a $\mathscr{U}$-ideal in $cabv(\mu, E)$. After exhibiting a class of Banach spaces that satisfy this property, we show that taking quotient by suitable subspaces leads to more natural examples of this phenomenon.

Let us recall from [4] that, given an operator T: $L^1(\mu) \to E$, there is a $G \in cabv(\mu, E)$ such that

$$\|G(A)\| \leq \mu(A) \|T\| \qquad \forall A \in \mathscr{A},$$

and conversely any such G corresponds to a $T \in \mathscr{L}(L^1(\mu), E)$. Further such a T is said to be representable if there exists a $g \in L^\infty(\mu, E)$ such that $T(f) = \int fg \, d\mu$ for all $f \in L^1(\mu)$ and $T \to g$ is an onto-isometry between $\mathscr{R}(L^1(\mu), E)$ and $L^\infty(\mu, E)$. Hence by analogy with the situation described above for the space $L^1(\mu, E)$, we study in the second half of this paper the $\mathscr{U}$-ideal question for $\mathscr{R}(L^1(\mu), E)$ as a subspace of $\mathscr{L}((L^1(\mu), E)$. Motivated by a recent observation of the first author (see [8]) that when $L^1(\mu, E)$ is an L-summand in $cabv(\mu, E)$ then $\mathscr{R}(L^1(\mu), E)$ is complemented in $\mathscr{L}(L^1(\mu), E)$ by a projection of norm one, we show that if $L^1(\mu, E)$ is a $\mathscr{U}$-summand in $cabv(\mu, E)$ and the corresponding projection commutes with characteristic projections then $\mathscr{R}(L^1(\mu), E)$ is a $\mathscr{U}$-summand in $\mathscr{L}(L^1(\mu), E)$. For any Banach space Y such that Y^* is separable and $X = \mathscr{K}(Y, L^1[0, 1]) = \mathscr{L}(Y, L^1[0, 1])$ we show also that $\mathscr{R}(L^1(\mu), X)$ is a $\mathscr{U}$-summand in $\mathscr{L}(L^1(\mu), X)$.

All the Banach spaces considered in this paper are over the real scalar field. We refer to the monograph [4] for the terminology and results related to vector measures and to [10] for concepts from L and M-structure theory

Section 1

In this section we introduce the notion of a Radon-Nikodym approximating sequence and after considering an example we show that when E has such an approximating sequence of operators then $L^1(\mu, E)$ is a $\mathscr{U}$-ideal in $cabv(\mu, E)$. We also consider this question for certain quotient spaces.

Definition. Let E be a Banach space and $T_n \in \mathscr{L}(E)$ be a sequence of operators each one factoring through a Banach space having the Radon-Nikodym property. We say that $\{T_n\}$ is a Radon-Nikodym approximating sequence if $\|T_n(x) - x\| \to 0$ for all $x \in X$. Such a sequence is said to be unconditional if $\limsup_n \|I - 2T_n\| \leq 1$.

Clearly any unconditional compact or weakly compact approximating sequence satisfies the above property.

We are now ready to prove the main result of this section.

THEOREM 1. *Let E be a Banach space admitting a unconditional Radon-Nikodym approximating sequence. Let $(\Omega, \mathscr{A}, \mu)$ be a finite measure space. Then $L^1(\mu, E)$ is a $\mathscr{U}$-ideal in $cabv(\mu, E)$.*

Proof. Let $\{R_n\}$ be such a sequence of operators. We define an operator P: $cabv(\mu, E)^* \to cabv(\mu, E)^*$ in the following way. Let $\wedge \in cabv(\mu, E)^*$, $v \in cabv(\mu, E)$. Since R_n factors through a Banach space having the Radon-Nikodym property, clearly $R_n v$ has a Bochner density w.r.t. μ, i.e. $R_n v \in L^1(\mu, E)$; it also is a bounded sequence.

Put $P(\wedge)(v) = L(\{(\wedge(R_n v)\})$ where L is a Banach limit of norm one. Clearly P is a well defined linear map. We first note that if $v \in L^1(\mu, E)$, then $R_n v \to v$ weakly. Thus $\wedge(R_n v) \to \wedge(v)$ in this case and $P(\wedge)(v) = \wedge(v)$.

To see this observe that since $\|R_n x - x\| \to 0$, $\|R_n v(s)\| \to 0$ a.e. Recall from [5] that elements of $L^1(\mu, E)^*$ are represented by g^*: $\Omega \to E^*$, a w^*-measurable function, such that $s \to \|g^*(s)\|$ is in $L^\infty(\mu)$. Thus

$$|g^*(s)(R_n v(s) - v(s))| \to 0$$

almost everywhere and

$$|g^*(s)(R_n v(s) - v(s))| \leq h \; \|v(s)\| \; \|g^*(S)\|$$

where $h = 1 + \sup_n \|R_n\|$. Therefore by the Lebesgue Dominated convergence theorem we get that $R_n v \to v$ weakly. From this it is clear that $\operatorname{Ker} P = L^1(\mu, E)^\perp$. Also

$$\begin{aligned} P(P(\wedge))(v) &= L(\{P(\wedge)(R_n v)\})(\{P(\wedge)(R_n v)\}) \\ &= L(\{\wedge(R_n(v)\}) = P(\wedge)(v) \end{aligned}$$

for any $\wedge \in cabv(v, E)^*$ and $v \in cabv(\mu, E)$. Therefore P is a projection. Finally, fix $\wedge \in cabv(\mu, E)^*$, $\|\wedge\| = 1$ and $v \in cabv(\mu, E)$, $\|v\| = 1$. Using standard properties of Banach limits we have:

$$\begin{aligned} |\wedge(v) - 2L(\{\wedge(R_n v)\})| &= |L(\{\wedge(v) - 2\wedge(R_n v)\})| \\ &= |L(\{\wedge(v - 2R_n v)\})| \\ &\leq \limsup_n \|I - 2R_n\| \end{aligned}$$

Therefore $\|I - 2P\| \leq 1$. Thus $L^1(\mu, E)$ is a $\mathscr{U}$-ideal in $cabv(\mu, E)$.

The next proposition gives a procedure for constructing spaces that satisfy the hypothesis of the above theorem, among spaces of operators.

A Banach space F is said to have the unconditional metric approximation property (UMAP) if there is a sequence $\{K_n\}$ of finite rank operators with $\|K_n(x) - x\| \to 0$ for all $x \in F$ and $\lim_n \|I - 2K_n\| = 1$ (see [**1**]).

PROPOSITION 1. *Let E be a Banach space such that E^* has the Radon-Nikodym property and let F be a Banach space having the UMAP. Then $K(E, F)$ admits an unconditional Radon-Nikodym approximating sequence.*

Proof. Let $\{K_n\}$ be a sequence of finite-rank operators in F such that $\|K_n(x) - x\| \to 0$ for all $x \in F$ and $\lim_n \|I - 2K_n\| = 1$. Define

$$\mathbf{K}_n: \mathcal{K}(E, F) \to \mathcal{K}(E, F) \qquad \text{by } \mathbf{K}_n(T) = K_n \circ T$$

Since K_n is a finite-rank operator, $\mathbf{K}_n$ takes values into some $\mathcal{K}(E, F')$ where F' is a finite dimensional subspace of F. Since E^* has the RNP, $\mathcal{K}(E, F')$ has the RNP (see [3]). Also since T is a compact operator

$$\|K_n \circ T - T\| \to 0$$

Therefore $\{\mathbf{K}_n\}$ is a Radon-Nikodym approximating sequence. A routine verification shows that the sequence $\{\mathbf{K}_n\}$ is unconditional.

Remark 1. Note that when E is infinite dimensional, it follows from a result of Vala [**17**], that $\mathbf{K}_n$ is not a compact operator. Also if E is reflexive and F has the UKAP, then it follows from a result of Saksman-Tylli [**16**] that the $\mathbf{K}_n$ constructed above is a weakly compact operator. To ensure non-triviality $K(E, F)$ should of course fail the RNP (take E reflexive and $F = \ell^p$, $1 < p < \infty$, see [**3**]).

More examples of this nature can be constructed by starting from a E that has the unconditional Radon-Nikodym approximation by a sequence of operators say R_n and a subspace F of E such that $R_n(F) \subset F$ and restricting the $R_n's$ to F. The following example illustrates this.

EXAMPLE. Suppose E and F are such that $K(E, F)$ has an unconditional Radon-Nikodym approximating sequence $\{R_n\}$ constructed as above. Let $G \subset E$ be a closed subspace. Note that $K(E/G, F)$ is isometric to a subspace of $K(E, F)$ via the map $T \to T \circ \pi$. Clearly $R_n(K(E/G, F)) \subset K(E/G, F)$. Therefore the restriction of $R_n's$ to $K(E/G, F)$ works.

Remark 2. Note that for an infinite dimensional separable nonreflexive E such that $K(E)$ is an M-ideal in $L(E)$ then E contains an isomorphic copy of c_0 (see [**10**]) and by a result of Kalton [**10**, page 299] we have that E has a compact unconditional approximating sequence.

Remark 3. If $L^1(\mu, E)$ is a $\mathcal{U}$-ideal in $cabv(\mu, E)$ and E has no isomorphic copy of c_0, then it follows from a result of Kwapien [**11**] that $L^1(\mu, E)$ has no copy of c_0. Hence by applying Theorem 3.5 of [**9**] we get that $L^1(\mu, E)$ is a $\mathcal{U}$-summand in $cabv(\mu, E)$.

The next proposition will be used to show that $L^1(\mu, E)$ is a $\mathcal{U}$-ideal in another naturally occuring superspace.

PROPOSITION 2. *Let $E \subset F \subset G$. If E is a $\mathcal{U}$-ideal in F and F is an L-summand in G then E is a $\mathcal{U}$-ideal in G.*

Proof. Let P: $G \to G$ be the L-projection i.e., $\|Pg\| + \|g - Pg\| = \|g\|$ for all $g \in G$ whose range is F. To show that E is a $\mathcal{U}$-ideal in G, it is enough to verify the "local characterization" given by Proposition 3.6 in [**9**]. Accordingly, let $\mathcal{F}$ be a finite dimensional subspace of G and let $\varepsilon > 0$. Since E is a $\mathcal{U}$-ideal in F, for this $\varepsilon > 0$ and for $P(\mathcal{F})$ there exists an operator L: $P(\mathcal{F}) \to E$ such that $L(x) = x$ for $x \in P(\mathcal{F}) \cap E$ and

$$\|x - 2L(x)\| \leq (1 + \varepsilon)\,\|x\| \qquad \forall x \in F.$$

Now if $L' = L \circ P$: $\mathcal{F} \to E$ then $L' = L$ on $E \cap \mathcal{F}$.
Also for $x \in G$

$$\begin{aligned}\|x - 2L'(x)\| &= \|x - 2L(P(x))\| \\ &\leq \|x - P(x)\| + \|P(x) - 2L(P(x))\| \\ &\leq \|x - P(x)\| + (1 + \varepsilon)\,\|P(x)\| \\ &\leq (1 + \varepsilon)(\|x - Px\| + \|P(x)\| \\ &= (1 + \varepsilon)\,\|x\|.\end{aligned}$$

Therefore E is a $\mathcal{U}$-ideal in G.

COROLLARY 1. *For a compact Hausdorff space Ω and a finite regular Borel measure μ on the Borel σ-field, if $L^1(\mu, E)$ is a $\mathcal{U}$-ideal in $rcabv(\mu, E)$ (regular measures) then it is a $\mathcal{U}$-ideal in $rcabv(E)$.*

Proof. It is well-known that the Lebesgue decomposition is a L-projection from $rcabv(E)$ onto $rcabv(\mu, E)$. Hence the conclusion follows from the above proposition.

If $L^1(\mu, E)$ is a $\mathcal{U}$-ideal in $cabv(\mu, E)$ in some special situations, for a subspace $F \subset E$, $L^1(\mu, E/F)$ will be an $\mathcal{U}$-ideal in $cabv(\mu, E/F)$. $L^1(\mu, E/F)$ can be identified with the quotient space $L^1(\mu, E)/L^1(\mu, F)$ in the canonical way. However the same is not in general true of $cabv(\mu, E/F)$ and the quotient space $cabv(\mu, E)/cabv(\mu, F)$.

PROPOSITION 3. *Suppose E admits an unconditional Radon-Nikodym approximating sequence, say R_n. If $F \subset E$ is an M-ideal such that $R_n(F) \subset F$ for all n, then for any separable measure space $L^1(\mu, E/F)$ is an $\mathcal{U}$-ideal in $cabv(\mu, E/F)$.*

Proof. We first show that Q: $cabv(\mu, E) \to cabv(\mu, E/F)$ defined by $Q(v) = \pi \circ v$ is a quotient map. Let $v \in cabv(\mu, E/F)$. Then there is a bounded linear map T: $L^1(|v|) \to E/F$ such that $T(\chi_A) = v(A)$.

Since $L^1(|v|)$ is separable space with the MAP and F is an M-ideal in E by Theorem 2.1 in [10], we get a lifting $\tilde{T}$: $L^1(|v|) \to E$. Now define $\tilde{v}$: $\mathcal{A} \to E$ by

$$\tilde{v}(A) = \tilde{T}(\chi_A)$$

Then $\tilde{v} \in cabv(\mu, E)$ and $Q(\tilde{v}) = v$. By hypothesis we have that $L^1(\mu, E)$ is a $\mathcal{U}$-ideal in $cabv(\mu, E)$. Let P: $cabv(\mu, E)^* \to cabv(\mu, E)^*$ be the $\mathcal{U}$-projection with $\operatorname{Ker} P = L^1(\mu, E)^\perp$.

Since $cabv(\mu, F/F) = cabv(\mu, E)/cabv(\mu, F)$, $cabv(\mu, E/F)^* = cabv(\mu, F)^\perp \subset cabv(\mu, E)^*$.

Therefore P: $cabv(\mu, E/F)^* \to cabv(\mu, E/F)^*$ is a $\mathcal{U}$-projection with $\operatorname{Ker} P = L^1(\mu, E/F)^\perp$ (here is where the definition of P and the fact that $R_n(F) \subset F$ is being used). Hence $L^1(\mu, E/F)$ is a $\mathcal{U}$-ideal in $cabv(\mu, E/F)$.

Section 2

In this section we consider the $\mathcal{U}$-ideal question for the space of representable operators $\mathcal{R}(L^1(\mu), E)$ as a subspace of $\mathcal{L}(L^1(\mu), E)$. Note that unlike the $L^1(\mu, E)$ situation, $\mathcal{R}(L^1(\mu), E)$ always has an isometric copy of c_0 as this space is isometric to $L^\infty(\mu, E)$.

For any $A \in \mathcal{A}$ and $v \in cabv(\mu, E)$, by v/A we denote the measure on $\mathcal{A}$ defined by $v/A(B) = v(A \cap B)$. The projection $v \to v/A$ in $cabv(\mu, E)$ is called a characteristic projection.

THEOREM 1. *Let P: $cabv(\mu, E) \to cabv(\mu, E)$ be a $\mathcal{U}$-projection (i.e. $\|I - 2P\| = 1$) with Range $P = L^1(\mu, E)$. Assume further that P commutes with the projection $v \to v/A$ for every $A \in \mathcal{A}$. Then $R(L^1(\mu), E)$ is a $\mathcal{U}$-summand in $\mathcal{L}(L^1(\mu), E)$.*

Proof. Let $T \in L(L^1(\mu), E)$. As we mentioned in the 'Introduction', there exists a unique $\nu \in cabv(\mu, E)$, given by

$$\nu(A) = T(\chi_A)$$

verifying the relation

$$\|T\| = \sup\left\{\frac{\|\nu(A)\|}{\lambda(A)}: A \in \mathscr{A}, \lambda(A) > 0\right\};$$

and conversely given a $\nu \in cabv(\mu, E)$ with the supremum on the right hand side finite, there corresponds a $T \in L(L^1(\mu), E)$. We first claim that under the given hypothesis $P(\nu)$ gives rise to a representable operator in $L(L^1(\mu), E)$. Let $A \in \mathscr{A}$; since P commutes with the projection $\nu \to \nu/A$, we have $P(\nu)/A = P(\nu/A)$.

Therefore

$$|P(\nu)|(A) = |P(\nu)/A| = |P(\nu/A)| \leq |\nu/A| \leq \|T\| \lambda(A).$$

(Here $|\ |$ denotes the total variation and then the total variation norm in $cabv(\mu, E)$.)

Thus we can define a map

$$\tilde{P}: \mathscr{L}(L^1(\mu), E) \to \mathscr{L}(L^1(\mu), E)$$

by

$$\tilde{P}(T) = P(\nu).$$

It is easy to verify that $\tilde{P}$ is a well defined linear projection whose range is in $\mathscr{R}(L^1(\mu), E)$. To verify that $\|I - 2\tilde{P}\| = 1$, fix $T \in \mathscr{L}(L^1(\mu, E))$, $\|T\| = 1$. Let ν correspond to T. By the defining property, we need to show that $\|\nu(A) - 2P(\nu)(A)\| \leq \lambda(A)$ for any $A \in \mathscr{A}$ with $\lambda(A) > 0$.
But

$$\|\nu(A) - 2P(\nu)(A)\| \leq |\nu - 2P\nu|(A) \leq \lambda(A)$$

since

$$\|I - 2P\| = 1.$$

Therefore $R(L^1(\mu), E)$ is a $\mathscr{U}$-summand in $\mathscr{L}(L^1(\mu), E)$.

Remark 1. There are several natural situations where the hypothesis of the above Theorem is satisfied. For instance if P is an L-projection then since $\nu \to \nu/A$ is an L-projection and since any two L-projections in a space commute (see **[10]** Theorem I 1.10) the hypothesis is satisfied. Also, when E is a Banach lattice not containing c_0 it is known that the projection P is a band projection (**[2]**) as well as $\nu \to \nu/A$ is, for any

$A \in \mathcal{A}$; and so they commute. In this last case we have a u-projection (see [**9**]), that, in general, is not a L-projection.

In the next lemma we give another way of obtaining a commuting projection. Let $\pi: E \to E/F$ be the quotient map. Define $Q: cabv(\mu, E) \to cabv(\mu, E/F)$ by

$$Q(v) = \pi \circ v.$$

Assume that

1) Q is a quotient map and
2) $P(cabv(\mu, F)) \subset L^1(\mu, F)$.

Then $\tilde{P}$: $cabv(\mu, E/F) \to L^1(\mu, E/F)$ defined as $\tilde{P}(Q(v)) = Q(P(v))$ is a well defined map and is an onto projection (see [**7**]).

LEMMA 1. *Let P: $cabv(\mu, E) \to L^1(\mu, E)$ be a projection that commutes with characteristic projections. Let $F \subset E$ be a closed subspace and suppose that the two conditions mentioned above are satisfied. Then $\tilde{P}$ again commutes with characteristic projections.*

Proof. Fix $A \in \mathcal{A}$. For any $B \in \mathcal{A}$

$$\begin{aligned}(Q(v)/A)(B) &= \pi(v(A \cap B)) \\ &= \pi(v/A(B)) \\ &= Q(v/A)(B).\end{aligned}$$

Therefore

$$Q(v)/A = Q(v/A).$$

Since P commutes with characteristic projections, it is clear now that $\tilde{P}$ also commutes with characteristic projections.

We now give an example to illustrate this situation using concepts and results from the work of Godefroy, see [**10**], Chapter IV. We also keep to the notation of that monograph. We first need a lemma.

LEMMA 2. *If $E \subset F \subset G^*$ and E is w^*-closed in G^*, then Q: $cabv(\mu, F) \to cabv(\mu, F/E)$ defined by $Q(v) = \pi \circ v$ is a quotient map.*

Proof. Let $v \in cabv(\mu, F/E)$. Clearly $v \in cabv(\mu, G^*/E)$. It follows from the proof of Corollary 7 in [**7**], that there exists a $\tilde{v} \in cabv(\mu, G^*)$ such that

$$\pi(\tilde{v}(A)) = v(A) \qquad \forall A \in \mathcal{A}$$

Fix $A \in \mathcal{A}$ and let $\pi(x) = v(A) = \pi(\tilde{v}(A))$ for some $x \in F$.

Since $x - \tilde{v}(A) \in E \subset F$, we get that

$$\tilde{v}(A) \in F.$$

Therefore $\tilde{v} \in cabv(\mu, F)$ and $Q(\tilde{v}) = v$.

EXAMPLE. Let T be the unit circle and let $\wedge$ be a Riesz subset of $\mathbb{Z}$ that is not nicely placed. Then $L^1_\wedge$ is a w^*-closed subspace of $C(T)^*$ having the Radon-Nikodym property. Therefore Q: $cabv(\mu, L^1) \to cabv(\mu, L^1/L^1_\wedge)$ is a quotient map. Further since $L^1_\wedge$ has the RNP we have $P(cabv(\mu, L^1_\wedge)) = P(L^1(\mu, L^1_\wedge))$, where P: $cabv(\mu, L^1) \to L^1(\mu, L^1)$ is an L-projection (hence commutes with characteristic projections; see [**8**], [**13**] for the existence of this projection).

Thus the projection:

$$\tilde{P}\colon cabv(\mu, L^1/L^1_\wedge) \to L^1(\mu, L^1/L^1_\wedge)$$

commutes with characteristic projections.

Observe that we are assuming the $\wedge$ is not a nicely placed set to ensure that this example does not obviously follow from the lifting properties enjoyed by quotients of L-embedded spaces by L-embedded subspaces.

Out next result gives another class of Banach spaces for which $R(L^1(\mu), E)$ is a $\mathcal{U}$-summand in $\mathcal{L}(L^1(\mu), E)$. In it we shall assume that F is a Banach space such that F^* is separable and $\mathcal{K}(F, L^1[0, 1]) = \mathcal{L}(F, L^1[0, 1])$. (This for example is satisfied if F^* has the Schur property, in particular when $F = c_0$. Also for any $p > 2$, $\mathcal{K}(L^p, L^1[0, 1]) = \mathcal{L}(L^p, L^1[0, 1])$.)

THEOREM 2. *Let F be a Banach space such that $E = \mathcal{K}(F, L^1[0, 1]) = \mathcal{L}(F, L^1[0, 1])$ and F^* separable. Then $R(L^1(\mu), E)$ is a $\mathcal{U}$-summand in the space $\mathcal{L}(L^1(\mu), E)$.*

Proof. From Theorem 1 and the subsequent remarks we have that there exists a $\mathcal{U}$-projection P: $\mathcal{L}(L^1(\mu), L^1[0, 1]) \to \mathcal{L}(L^1(\mu), L^1[0, 1])$ with range $P = \mathcal{R}(L^1(\mu), L^1[0, 1])$. Fix $T \in \mathcal{L}(L^1(\mu), E)$. Let v be the E-valued measure associated with T. For $x \in F$, define v_x: $\mathfrak{A} \to L^1[0, 1]$ by $v_x(A) = v(A)(x)$. Clearly v_x is a countably additive vector measure.

Also

$$\|v_x(A)\| \le \|x\| \, \|v(A)\| \le \|x\| \, \|T\| \, \mu(E),$$

so that the operator corresponding to v_x (also denoted by v_x) is a member of the space $\mathcal{L}(L^1(\mu), L^1[0, 1])$. Now $P(v_x) \in L^\infty(\mu, L^1[0, 1])$ and $\|P(v_x)\|_\infty \le \|T\| \, \|x\|$. Thus outside a null set N_x we have $\|P(v_x)(w)\| \le \|x\| \, \|T\|$. Since F is separable, choose a dense sequence $\{x_n\}$ in the unit ball of F. For any n-tuple of rationals $(r_1, \cdots, r_n)$ by repeating the above process we get an element $P(v_{\sum_{i=1}^n r_i x_i})$ of

$L^\infty(\mu, L^1[0,1])$ and a null set $N_{(r_1,\dots,r_n)}$ such that outside this null set

$$\|P(v_{\sum_{i=1}^n r_i x_i})(w)\| \leq \left\|\sum_{i=1}^n r_i x_i\right\| \|T\|.$$

Put $N = \bigcup_n \bigcup_{(r_n,\dots,r_n)} N_{(r_1,\dots,r_n)}$. Now $\mu(N) = 0$ and for $w \notin N$

$$\|P(v_{\sum_{i=1}^n r_i x_i})(w)\| \leq \left\|\sum_{i=1}^n r_i x_i\right\| \|T\|.$$

We also have for any $A \in \mathscr{A}$ and any n-tuple of rationals $(r_1, \cdots, r_n)$

$$\begin{aligned}\int_A \sum_{i=1}^n r_i P(v_{x_i})(w)\, d\mu &= \sum_{i=1}^n r_i \int_A P(v_{x_i})(w)\, d\mu \\ &= \sum_{i=1}^n r_i v(A)(x_i) = v(A)\Big(\sum_{i=1}^n r_i x_i\Big) \\ &= \int_A P(v_{\sum_{i=1}^n r_i x_i})\, d\mu.\end{aligned}$$

Therefore by the uniqueness of the representing function we get $P(v_{\sum_{i=1}^n r_i x_i}) = \sum_{i=1}^n r_i P(v_{x_i})$ a.e.

Hence, outside a suitable null set N

$$\left\|\sum_{i=1}^n r_i p(v_{x_i})(w)\right\| \leq \|T\| \left\|\sum_{i=1}^n r_i x_i\right\|$$

Thus by a standard procedure we can define g: $\Omega \to E$ such that for $x \in F$ we have $g(w)(x) = P(v_x)(w)$ a.e. (this part of the proof follows the lines for the proof of the main theorem in [**2**]).

Since F^* is separable and $\mathscr{K}(F, L^1[0,1]) = \mathscr{L}(F, L^1[0,1])$ we have that E is a separable space. Therefore from the definition we have that g is strongly measurable. Also $\|g\|_\infty \leq \|T\|$. Therefore $\tilde{P}$: $\mathscr{L}(L^1(\mu), E) \to \mathscr{L}(L^1(\mu), E)$ defined by $\tilde{P}(T) = g$ is a well defined projection whose range is $\mathscr{R}(L^1(\mu), E)$. It is also easy to verify that for $f \in L^1(\mu)$, $\tilde{P}(T)(f) = P(\hat{x} \circ T)(f)$ a.e. $x \in F$ (here $\hat{x} \circ T$: $L^1(\mu) \to L^1[0,1]$ is defined by $(\hat{x} \circ T)(g) = T(g)(x)$). Now to show that $\|I - 2\tilde{P}\| \leq 1$, fix T, f, x vectors of norm one in $\mathscr{L}(L^1(\mu), E)$, $L^1(\mu)$ and F respectively.

$$\begin{aligned}&\|T(f)(x) - 2\tilde{P}(T)(f)(x)\| \\ &\quad = \|T(f)(x) - 2P(\hat{x} \circ T)(f)\| \\ &\quad = \|(I - 2P)(\hat{x} \circ T)(f)\| \\ &\quad \leq 1.\end{aligned}$$

Hence $\|I-2\hat{P}\|=1$ so that $\mathcal{R}(L^1(\mu),E)$ is a $\mathcal{U}$-summand in $\mathcal{L}(L^1(\mu),E)$.

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Note added: The first author has recently shown that if X is such that $L^1(\mu, X)$ is complemented in $cabv(\mu, X)$ by a projection that commutes with characteristic projections and F a Banach space with F^* separable and $K(F,X)=L(F,X)$ then again the conclusion of Theorem 2 (section 2) holds for $E=K(F,X)$.

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Department of Mathematics
University of Catania
Viale A. Doria, 6
95125 Catania, Italy
E-mail: emmanuele@dipmat.unict.it.

Statistical and Mathematics Unit
Indian Statistical Institute
R.V. College Post
Bangalore 560 059, India
E-mail: TSS@isibang.ernet.in.