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The problem of complementability for some spaces of vector measures of bounded variation with values in Banach spaces containing copies of c_0

by

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Abstract. Let (S,Σ,m) be any atomless finite measure space, and X any Banach space containing a copy of c_0. Then the Bochner space $L^1(m;X)$ is uncomplemented in $ccabv(\Sigma,m;X)$, the Banach space of all m-continuous vector measures that are of bounded variation and have a relatively compact range; and $ccabv(\Sigma,m;X)$ is uncomplemented in $cabv(\Sigma,m;X)$. It is conjectured that this should generalize to all Banach spaces X without the Radon–Nikodym property.

1. Introduction. We start by explaining some basic notation used in this paper. (In general, our Banach space and vector measure terminology and notation follow [3], [4] and [13].)

Throughout, (S,Σ,m) is an atomless probability measure space, and X is a Banach space. Several Banach spaces of (countably additive) vector measures $\mu:\Sigma\to X$ will be encountered below. For convenience, we first mention the space $ca(\Sigma,X)$ of all such measures μ, equipped with the supnorm $\|\mu\|=\sup_{E\in\Sigma}\|\mu(E)\|$, and its closed subspaces

$$\begin{aligned} cca(\Sigma,X)&=\{\mu\in ca(\Sigma,X):\mu(\Sigma)\text{ is relatively compact}\},\\ ca(\Sigma,m;X)&=\{\mu\in ca(\Sigma,X):\mu\ll m\},\\ cca(\Sigma,m;X)&=cca(\Sigma,X)\cap ca(\Sigma,m;X).\end{aligned}$$

However, our primary interest here is rather in $cabv(\Sigma,X)$, the space of all measures $\mu:\Sigma\to X$ of bounded variation, considered with the variation

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norm $\|\mu\|_1 = |\mu|(S)$, and its closed subspaces

$$ccabv(\Sigma, X) = cca(\Sigma, X) \cap cabv(\Sigma, X),$$
$$cabv(\Sigma, m; X) = ca(\Sigma, m; X) \cap cabv(\Sigma, X),$$
$$ccabv(\Sigma, m; X) = cca(\Sigma, m; X) \cap cabv(\Sigma, X).$$

In addition, $L^1(m; X) = L^1(S, \Sigma, m; X)$, the Banach space of all Bochner m-integrable functions $f : S \to X$ under the norm $\|f\|_1 = \int_S \|f(\cdot)\| \, dm$, can (and will) be identified via the linear isometric embedding $f \mapsto m_f(\cdot) = \int_{(\cdot)} f \, dm$ with a subspace of $ccabv(\Sigma, m; X)$ (cf. [4; II.3.9]). Using this convention we can therefore write

$$(*) \qquad L^1(m; X) \subset ccabv(\Sigma, m; X) \subset cabv(\Sigma, m; X).$$

The present paper originated from an attempt to prove the following conjecture:

(C) Whenever a proper inclusion occurs between some two spaces in the chain $(*)$, then the smaller space is an uncomplemented subspace of the bigger.

At this point let us recall that, as follows from the results of Chatterji [2] and Bourgain [1], respectively, each of the equalities $L^1(m; X) = cabv(\Sigma, m; X)$ and $L^1(m; X) = ccabv(\Sigma, m; X)$ is necessary and sufficient for the Banach space X to have the Radon–Nikodym property. (We thank Z. Lipecki and K. Musiał for calling our attention to the results of [1].) In view of this, the most essential part of our conjecture seems to be that if X does not have the Radon–Nikodym property, then $L_1(m; X)$ is not complemented in $cabv(\Sigma, m; X)$ and $ccabv(\Sigma, m; X)$ ([1]).

So far we have been able to verify (C) only for those Banach spaces X which contain an isomorphic copy of c_0 (Theorems 3.1 and 3.3). (For such spaces X it is relatively easy to see that both inclusions in $(*)$ are proper.) In achieving this goal, we heavily depend on some special isomorphic embeddings of l_∞ into the spaces of measures involved in (C), which we construct in Section 2.

We refer the reader to [5], [7], [10], [11] and [12] (a highly incomplete list of references!), where problems analogous to (C) were treated for some spaces of continuous functions, some other spaces of vector measures, and some spaces of operators.

([1]) This was disproved by F. Freniche and L. Rodríguez-Piazza (University of Sevilla, Spain) in November 1991. They showed that, for Lebesgue measure m on $[0,1]$ and $X = L^1(m)$, $L^1(m; X)$ is complemented in $cabv(\Sigma, m; X)$. (Note added November 1992.)

2. Special isomorphic embeddings of l_∞ into $cabv(\Sigma, m; X)$ and $ccabv(\Sigma, m; X)$. In what follows, the sequence of unit vectors in c_0 and l_∞ is denoted by (e_n), and a basic sequence in a Banach space which is equivalent to the basis (e_n) of c_0 is briefly called a *c_0-sequence*. Given a Banach space Z, let us denote by $c_0(Z, w)$ the Banach space of all weakly null sequences (z_n) in Z equipped with the supnorm $\|(z_n)\| = \sup_n \|z_n\|$. In the lemma below we collect some elementary (and fairly well known) facts about c_0-valued measures; most of these facts appear in some of the examples in [4].

2.1. LEMMA. *There is an isomorphism between the spaces $c_0(L^1(m), w)$ and $ca(\Sigma, m; c_0)$ under which the measure $\phi : \Sigma \to c_0$ assigned to a weakly null sequence (f_n) in $L^1(m)$ is given by the formula*

$$\phi(E) = \sum_{n=1}^{\infty} \int_E f_n \, dm \cdot e_n .$$

Moreover, if ϕ is given in the above form, then

(a) $\phi \in cca(\Sigma, m; c_0) \Leftrightarrow \|f_n\|_1 \to 0$.

(b) $\phi \in cabv(\Sigma, m; c_0) \Leftrightarrow (f_n)$ *is order bounded in $L^1(m)$, i.e., $\sup_n |f_n| \in L^1(m)$; in this case*

$$|\phi|(E) = \int_E \sup_n |f_n| \, dm \qquad \textit{for all } E \in \Sigma .$$

(c) $\phi \in ccabv(\Sigma, m; c_0) \Leftrightarrow \sup_n |f_n| \in L^1(m)$ *and* $\|f_n\|_1 \to 0$.

(d) $\phi \in L^1(m; c_0) \Leftrightarrow \sup_n |f_n| \in L^1(m)$ *and* $f_n \to 0$ *m-a.e.*

Proof. If $\phi \in ca(\Sigma, m; c_0)$ then, using the Radon–Nikodym theorem coordinatewise, ϕ can be uniquely represented in the above form with $\int_E f_n \, dm \to 0$ for all $E \in \Sigma$ or, equivalently, $f_n \to 0$ weakly in $L^1(m)$. Conversely, if (f_n) is weakly null in $L^1(m)$, then the formula for ϕ makes sense and $\phi \in ca(\Sigma, m; c_0)$ by the Nikodym and Vitali–Hahn–Saks theorems. Finally, since the standard norm $\|f\|_1 = \int_S |f| \, dm$ and the norm $\|f\| = \|m_f\| = \sup_{E \in \Sigma} |\int_E f \, dm|$ are equivalent in $L^1(m)$, so are the norms $\|\phi\| = \sup_{E \in \Sigma} \sup_n |\int_E f_n \, dm|$ and $\|\phi\|' = \sup_n \int_S |f| \, dm$ in $ca(\Sigma, m; c_0)$. In other words, the mapping $(f_n) \mapsto \phi$ is an isomorphism.

(a) By a well known compactness criterion in Banach spaces with Schauder bases, $\phi(\Sigma)$ is relatively compact in c_0 if and only if

$$\sup_{E \in \Sigma} \sup_{n \geq N} \left| \int_E f_n \, dm \right| \to 0 \qquad \text{as } N \to \infty ,$$

or equivalently, $\sup_{n \geq N} \|f_n\|_1 \to 0$, i.e., $\|f_n\|_1 \to 0$.

(b) ϕ is m-continuous and of bounded variation if and only if there is a finite positive measure $v \ll m$ such that for every $E \in \Sigma$, $\|\phi(E)\| \le v(E)$, or $|\int_E f_n\,dm| \le \int_E F\,dm$ $(n = 1, 2, \dots)$, where $F = dv/dm$. This in turn is equivalent to the set of inequalities $|f_n| \le F$ m-a.e. $(n = 1, 2, \dots)$. Clearly, the smallest such v (i.e., $|\phi|$) is obtained by taking $F = \sup_n |f_n|$.

(c) follows from (a) and (b), and (d) is obvious. ∎

2.2. THEOREM. *Suppose the Banach space X contains a subspace X_0 isomorphic to c_0. Then there exists an isomorphic embedding*

$$J : l_\infty \to cabv(\Sigma, m; X_0) \subset cabv(\Sigma, m; X)$$

such that

(i) $J(c_0) \subset L^1(m; X_0)$;
(ii) $J(c_0) = J(l_\infty) \cap ccabv(\Sigma, m; X)$, *and*
(iii) $J(c_0)$ *is complemented in* $ccabv(\Sigma, m; X)$.

Note that assertions (i) and (iii) give an improvement of the result from [9] that $L^1(m; X)$ contains a complemented copy of c_0 provided $X \supset c_0$.

The above theorem follows immediately from the following more precise result (see also Remark 2.4); some parts of its proof combine the arguments already employed in [6], [7] and [9].

2.3. PROPOSITION. *Let (x_n) be a c_0-sequence in X, and let (f_n) be a sequence in $L^1(m)$ satisfying the following conditions*:

(1) $\|f_n\|_1 = 1$ *for all n (or, more generally,* $\inf_n \|f_n\|_1 > 0$);
(2) $f_n \to 0$ *weakly in* $L^1(m)$, *and*
(3) $\sup_n |f_n| =: F \in L^1(m)$.

Then the formula

$$(Ja)(E) = \sum_{n=1}^{\infty} a_n \int_E f_n\,dm \cdot x_n\,, \qquad a = (a_n) \in l_\infty,\ E \in \Sigma\,,$$

defines an isomorphic embedding $J : l_\infty \to cabv(\Sigma, m; X)$ satisfying conditions (i) *and* (ii) *of Theorem* 2.2.

Moreover, if there exists a weak null sequence (h_n) in $L^\infty(m)$ with*

$$\int_S h_n f_n\,dm = 1 \qquad \textit{for all } n \in \mathbb{N},$$

and if (x_n^*) is a bounded sequence in X^* obtained by applying the Hahn–Banach theorem to the coefficient functionals of (x_n), then the formula

$$(P\mu)(E) = \sum_{n=1}^{\infty} a_n(\mu)\Big(\int_E f_n\,dm\Big) \cdot x_n,$$

where

$$a_n(\mu) = \int_S h_n \, d(x_n^* \mu) \quad \Big(= \Big\langle x_n^*, \int_S h_n \, d\mu \Big\rangle\Big),$$

gives a bounded linear projection P from $cabv(\Sigma, m; X)$ onto $J(l_\infty)$, and $P|ccabv(\Sigma, m; X)$ is a projection onto $J(c_0)$.

Proof. In view of Lemma 2.1 it is clear that J acts as a linear operator from l_∞ into $cabv(\Sigma, m; X_0) \subset cabv(\Sigma, m; X)$, where $X_0 = [(x_n)] \simeq c_0$. Let $c, C > 0$ be constants such that

$$c \, \|(t_n)\|_\infty \le \Big\| \sum_{n=1}^{\infty} t_n x_n \Big\| \le C \, \|(t_n)\|_\infty \quad \text{for all } (t_n) \in c_0.$$

Then, given any $a \in l_\infty$ and $E \in \Sigma$, we have

$$\|(Ja)(E)\| \le C \sup_n \Big(|a_n| \Big| \int_E f_n \, dm \Big| \Big) \le C \, \|a\|_\infty \int_E F \, dm$$

from which it follows that

$$\|Ja\|_1 \le \Big(C \int_S F \, dm \Big) \cdot \|a\|_\infty \, .$$

On the other hand, for every n, since $\int_S |f_n| \, dm = 1$, we can find E_n in Σ with

$$\Big| \int_{E_n} f_n \, dm \Big| \ge \frac{1}{2}$$

($\ge \frac{1}{4}$ in the complex case). Then

$$\|(Ja)(E_n)\| \ge c \cdot \sup_n \Big(|a_n| \Big| \int_{E_n} f_n \, dm \Big| \Big) \ge \frac{c}{2} |a_n| \, ,$$

hence

$$\|Ja\|_1 \ge \|Ja\| \ge \frac{c}{2} \|a\|_\infty \, .$$

Thus $J : l_\infty \to cabv(\Sigma, m; X)$ is an isomorphic embedding.

Obviously,

$$J(c_0) \subset L^1(m; X).$$

Now take any $a = (a_n) \in l_\infty \setminus c_0$; so $|a_n| > \varepsilon$ for some $\varepsilon > 0$ and infinitely many n. Then for every N,

$$\sup_{E \in \Sigma} \Big\| \sum_{n=N}^{\infty} a_n \Big(\int_E f_n \, dm \Big) \cdot x_n \Big\| \ge c \cdot \sup_{n \ge N} \Big(|a_n| \Big| \int_{E_n} f_n \, dm \Big| \Big) > \frac{c}{2} \varepsilon$$

so that the leftmost quantity does not tend to zero as $N \to \infty$, which means $(Ja)(\Sigma)$ is not relatively norm compact in $X_0 \subset X$. This establishes (ii).

Now we proceed to the part of the proposition involving P. Let $L = \sup_n \|h_n\|_\infty \cdot \sup_n \|x_n^*\| < \infty$. If $\mu \in cabv(\Sigma, m; X)$, then $|a_n(\mu)| \le L\|\mu\| \le L\|\mu\|_1$. Hence P is a bounded linear operator from $cabv(\Sigma, m; X)$ into $J(l_\infty)$, and it is easily verified that P is a projection onto $J(l_\infty)$.

Next let $\mu \in cca(\Sigma, m; X)$. Then, since $\mu(\Sigma)$ is relatively norm compact, an easy direct argument shows that

$$K = \{x^*\mu : x^* \in X^*,\ \|x^*\| \le 1\}$$

is a compact subset of $ca(\Sigma, m) \cong L^1(m)$. Let $g_n = dx_n^*\mu/dm$. By the preceding observation, the set $\{g_n : n \in \mathbb{N}\} \subset \mathrm{const} \cdot K$ is relatively compact in $L^1(m)$. Hence, since the sequence $(h_n) \subset L^\infty(m) \cong L^1(m)^*$ is weak* null, we have $a_n(\mu) \to 0$ as $n \to \infty$. It follows that the restriction of P to $ccabv(\Sigma, m; X)$ is a projection onto $J(c_0)$. ■

2.4. Remark. Any Rademacherlike sequence (f_n) over (S, Σ, m) (that is, an orthonormal sequence such that $m(f_n = 1) = m(f_n = -1) = \frac{1}{2}$) satisfies conditions (1) to (3) and admits a sequence (h_n) as specified above.

In general, given a sequence (f_n) in $L^1(m)$ with properties (1) and (2), there exists a subsequence (f_{k_n}) for which it is possible to find a weak* null sequence (h_n) in $L^\infty(m)$ satisfying $\int_S h_n f_{k_n}\, dm = 1$ for all n. Indeed, since $L^1(m)$ is a Gelfand–Phillips space (see [3] or [8] for more information), such a sequence (f_n) cannot be limited, that is, there must exist a weak* null sequence $(g_n) \subset L^\infty(m)$ for which $\sup_k |\int_S f_k g_n\, dm| \nrightarrow 0$ as $n \to \infty$. From this our assertion follows easily.

In consequence, for any isomorphic embedding J given by the above proposition, we can always find an infinite subset M of $\mathbb{N}$ (independent of (x_n)) such that the subspace $J(c_0(M)) \simeq c_0$ is complemented in $ccabv(\Sigma, m; X)$. Here $c_0(M) = \{ a = (a_n) \in c_0 : a_n = 0 \text{ for } n \notin M \} \cong c_0$.

Finally, let us note that from the estimates of $\|Ja\|$ given in the above proof it follows that the operator $J : l_\infty \to cabv(\Sigma, m; X)$ is an isomorphic embedding even when $cabv(\Sigma, m; X)$ is considered with the (weaker) norm $\|\cdot\|$ induced from $ca(\Sigma, X)$. In addition, it should also be clear that the operator P can be considered as being defined on all of $ca(\Sigma, m; X)$, and that in that case it is still a bounded projection onto $J(l_\infty)$.

2.5. THEOREM. *Suppose the Banach space X contains a subspace X_0 isomorphic to c_0. Then there exists an isomorphic embedding*

$$J : l_\infty \to ccabv(\Sigma, m; X_0) \subset ccabv(\Sigma, m; X)$$

such that

(j) $J(c_0) \subset L^1(m; X_0)$;
(jj) $J(c_0) = J(l_\infty) \cap L^1(m; X)$, *and*
(jjj) $J(c_0)$ *is complemented in* $L^1(m; X_0)$.

This follows immediately from the following more precise result (see also Remark 2.7 below).

2.6. PROPOSITION. *Let (x_n) be a c_0-sequence in X, and let (f_n) be a sequence in $L^1(m)$ satisfying the following conditions*:

(1) $\|f_n\|_1 \to 0$;
(2) $f_n \not\to 0$ *m-a.e., and*
(3) $\sup_n |f_n| =: F \in L^1(m)$.

Then there exists a strictly increasing sequence (N_n) in $\mathbb{N}$, depending only on the sequence (f_n), such that if

$$\eta_n(\cdot) := \sum_{k=N_n}^{N_{n+1}-1} \int_{(\cdot)} f_k \, dm \cdot x_k \,,$$

then the formula

$$(Ja)(E) = \sum_{n=1}^{\infty} a_n \eta_n(E)$$

defines an isomorphic embedding $J : l_\infty \to ccabv(\Sigma, m; X_0)$ satisfying conditions (j) *to* (jjj) *of Theorem* 2.5. *Moreover, there exists a bounded linear projection P from $cabv(\Sigma, m; X)$ onto $J(l_\infty)$ such that $P|L^1(m; X_0)$ is a projection onto $J(c_0)$.*

Proof. Since $f_n \not\to 0$ a.e., there exists $r > 0$ such that the set

$$\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} \{ s : |f_k(s)| \ge r \}$$

is of strictly positive m measure. It is then easily seen that we can find a sequence $1 = N_1 < N_2 < \ldots$ such that also the set

$$B = \bigcap_{n=1}^{\infty} \bigcup_{k\in\Delta_n} \{ s : |f_k(s)| \ge r \} = \bigcap_{n=1}^{\infty} \{ s : \sup_{k\in\Delta_n} |f_k(s)| \ge r \} ,$$

where $\Delta_n = \{ k \in \mathbb{N} : N_n \le k < N_{n+1} \}$, is of strictly positive m measure.

Let us now define the sequence of measures (η_n), and next the operator J, as specified in the proposition. Let the positive constants c, C be as in the proof of Proposition 2.3. That J is an isomorphic embedding follows from the following inequalities:

$$\|Ja\|_1 = |Ja|(S) \le C \int_S \sup_n(|a_n| \sup_{k\in\Delta_n} |f_k|) \, dm \le C \int_S F \, dm \cdot \|a\|_\infty$$

and

$$\|Ja\|_1 \ge c \int_B \sup_n(|a_n| \sup_{k\in\Delta_n} |f_k|) \, dm \ge cr \cdot m(B) \cdot \|a\|_\infty \,.$$

Evidently, $J(c_0) \subset L^1(m; X_0)$. If $a = (a_n) \in l_\infty \setminus c_0$ so that $|a_n| > \varepsilon$ for infinitely many n and some $\varepsilon > 0$, then for those n we have

$$\Big| a_n \sum_{k \in \Delta_n} f_k \Big| > \varepsilon r \quad \text{on } B .$$

Hence the sequence of $L^1(m)$ functions which represents Ja (in the sense of Lemma 2.1) does not tend to 0 a.e. Thus the measure Ja is not representable as the indefinite Bochner integral of an X_0-, nor even X-valued function. This proves equality (jj).

Finally, we are going to construct a (bounded linear) projection from $L^1(m; X_0)$ onto its subspace $J(c_0) = [(\eta_n)]$. To simplify the notation, we may clearly assume here that $X_0 = c_0$ and that the c_0-sequence (x_n) is simply the standard basis (e_n) of c_0. For any fixed n consider the function

$$H_n = \sum_{k \in \Delta_n} f_k e_k \in L^1(m; Z_n) ,$$

where $Z_n = [e_k : k \in \Delta_n] \subset c_0$. Then

$$\begin{aligned} \|H_n\|_1 &= \int_S \|H_n(s)\|_{Z_n} \, dm(s) = \int_S \sup_{k \in \Delta_n} |f_k(s)| \, dm(s) \\ &\geq r m(B) =: K^{-1} > 0 . \end{aligned}$$

By the Hahn–Banach theorem, there exists a functional H_n^* in $L^1(m; Z_n)^* \cong L^\infty(m; Z_n^*)$, where $Z_n^* \cong [e_k : k \in \Delta_n] \subset l_1$, which we can represent in the form

$$H_n^* = \sum_{k \in \Delta_n} h_k e_k, \qquad \text{where } h_k \in L^\infty(m) ,$$

such that $\|H_n^*\| = (\|H_n\|_1)^{-1}$ and $\langle H_n^*, H_n \rangle = 1$. Thus

$$\|H_n^*\| = \|H_n^*\|_\infty = \operatorname*{ess\,sup}_{s \in S} \|H_n^*(s)\|_{Z_n^*} = \operatorname*{ess\,sup}_{s \in S} \sum_{k \in \Delta_n} |h_k(s)| \leq K$$

and

$$\langle H_n^*, H_n \rangle = \int_S \langle H_n^*(s), H_n(s) \rangle \, dm(s) = \int_S \sum_{k \in \Delta_n} h_k(s) f_k(s) \, dm(s) = 1 .$$

Now, let a measure $\gamma \in L^1(m; c_0)$ be represented by a sequence $(g_n) \subset L^1(m)$; thus

$$\gamma(\cdot) = \sum_{n=1}^{\infty} \int_{(\cdot)} g_n \, dm \cdot e_n, \qquad G := \sup_n |g_n| \in L^1(m)$$

and $g_n \to 0$ a.e. Define

$$a_n(\gamma) = \int_S \sum_{k\in\Delta_n} h_k g_k \, dm \qquad (n = 1, 2, \dots).$$

Since

$$\begin{aligned}\Big|\sum_{k\in\Delta_n} h_k(s) g_k(s)\Big| &\le \sum_{k\in\Delta_n} |h_k(s)| \cdot \sup_{k\in\Delta_n} |g_k(s)| \\ &\le K \sup_{k\in\Delta_n} |g_k(s)| \qquad \text{a.e.} \\ &\le KG(s)\end{aligned}$$

and $g_n \to 0$ a.e. (so that also $\sup_{k\in\Delta_n} |g_k| \to 0$ a.e.), we see that $a_n(\gamma) \to 0$ as $n \to \infty$. Moreover, for every n,

$$|a_n(\gamma)| \le \int_S \Big|\sum_{k\in\Delta_n} h_k g_k\Big| \, dm \le K \int_S G \, dm = K \, \|\gamma\|_1$$

so $\|(a_n(\gamma))_{n=1}^{\infty}\|_\infty \le K \, \|\gamma\|_1$. Since $a_n(\eta_m) = \delta_{nm}$ for all n, $m \in \mathbb{N}$, it follows that the formula

$$P\gamma = \sum_{n=1}^{\infty} a_n(\gamma) \cdot \eta_n$$

defines a required projection from $L^1(m; c_0)$ onto its subspace $J(c_0)$.

It is now easy to extend P to a projection from $cabv(\Sigma, m; X)$ onto $J(l_\infty)$: Let (x_n^*) be a bounded sequence in X^* which is biorthogonal to (x_n); define $b = \sup_n \|x_n^*\|$. For every $\mu \in cabv(\Sigma, m; X)$ and $n \in \mathbb{N}$ set

$$a_n(\mu) = \sum_{k\in\Delta_n} \int_S h_k \, dx_k^* \mu .$$

Then

$$|a_n(\mu)| \le \sum_{k\in\Delta_n} \int_S |h_k| \, d|x_k^*\mu| \le b \sum_{k\in\Delta_n} \int_S |h_k| \, d|\mu| \le b\, K \, \|\mu\|_1 ,$$

hence $\|(a_n(\mu))_{n=1}^{\infty}\|_\infty \le b\, K \, \|\mu\|_1$. It is now clear that the same formula as above defines a bounded linear projection P from $cabv(\Sigma, m; X)$ onto $J(l_\infty)$ which extends the previously constructed projection from $L^1(m; X_0)$ onto $J(c_0)$. ■

2.7. Remark. The simplest example of a sequence $(f_n) \subset L^1(m)$ satisfying conditions (1) to (3) from Proposition 2.6 can be obtained as follows: For $i = 1, 2, \dots$ let $d_i = 2^0 + \dots + 2^{i-1}$. Let $\{A_{i,j} : 0 \le j < 2^i\}$, $i = 1, 2, \dots$, be a sequence of consecutive dyadic Σ-partitions of S. If $n \in \mathbb{N}$ and $n = d_i + j$ for some $i \in \mathbb{N}$ and $0 \le j < 2^i$, let f_n be the characteristic function of the set $A_{i,j}$. Then the sequence (f_n) is as required. Moreover, the construction from the proof of the proposition works with $\Delta_n = \{d_i + j : 0 \leq j < 2^i\}$ and $h_n = f_n$.

3. Uncomplementability of $L^1(m;X)$ in $cabv(\Sigma,m;X)$ and in $ccabv(\Sigma,m;X)$. Our first result here follows directly from Theorem 2.2 and the well known fact that c_0 is not complemented in l_∞.

3.1. THEOREM. *If $X \supset c_0$, then neither $L^1(m;X)$ nor $ccabv(\Sigma,m;X)$ is complemented in $cabv(\Sigma,m;X)$.*

Similarly, from Theorem 2.5 it follows that if $X \supset X_0 \simeq c_0$, then $L^1(m;X_0)$ is not complemented in $ccabv(\Sigma,m;X)$. In Theorem 3.3 below we will see that also $L^1(m;X)$ is uncomplemented in $ccabv(\Sigma,m;X)$, but the proof of this will not be as quick as above. Among other things we will need the following

3.2. LEMMA. *Let, as everywhere above, (S,Σ,m) be an atomless probability measure space, and let X be any Banach space. Furthermore, let $([0,1],\mathcal{B},\lambda)$ be the Borel–Lebesgue measure space. If $L^1(m;X)$ is complemented in $ccabv(\Sigma,m;X)$, then $L^1(\lambda;X)$ is complemented in $ccabv(\mathcal{B},\lambda;X)$.*

Proof. Choose a countably generated sub-σ-algebra $\Sigma_0 \subset \Sigma$ so that the measure $m_0 = m|\Sigma$ is atomless. Let the operator $T : ca(\Sigma_0,m_0;X) \to ca(\Sigma,m;X)$ be given by the formula

$$(T\mu_0)(A) = \int_S \mathbb{E}(\chi_A \mid \Sigma_0)\, d\mu_0\,,$$

where $\mathbb{E}(\cdot \mid \Sigma_0)$ is the conditional expectation operator from $L^1(m)$ onto $L^1(m_0)$ (cf. [2] and, for more details, [7]). Then T is a linear isometric embedding of $ccabv(\Sigma_0,m_0;X)$ into $ccabv(\Sigma,m;X)$, $(T\mu_0)|\Sigma_0 = \mu_0$ for all $\mu_0 \in ca(\Sigma_0,m_0;X)$, and if $\mu_0(E) = \int_E f\, dm_0$ $(E \in \Sigma_0)$ for $f \in L^1(m_0;X)$, then

$$(T\mu_0)(A) = \int_A f\, dm \quad \text{for all } A \in \Sigma\,.$$

Let P be a projection from $ccabv(\Sigma,m;X)$ onto $L^1(m;X)$, and consider the operator Q on $ccabv(\Sigma_0,m_0;X)$ defined by the equality

$$Q = \mathbb{E}(\cdot \mid \Sigma_0) \circ P \circ T\,.$$

It is then easily seen that Q is a projection onto $L^1(m_0;X)$. Since, by a well known result of Carathéodory, (S,Σ_0,m_0) is measure-algebra isomorphic to $([0,1],\mathcal{B},\lambda)$, the proof is complete. ∎

3.3. THEOREM. *If $X \supset c_0$, then $L^1(m;X)$ is not complemented in $ccabv(\Sigma,m;X)$.*

Proof. We split our argument in two parts.

Case 1: *X has no subspace isomorphic to* l_∞. Then, by a result of Mendoza [14], also $L^1(m;X)$ contains no copy of l_∞. Suppose there is a projection Q from $ccabv(\Sigma,m;X)$ onto $L^1(m;X)$. Now, if $J : l_\infty \to ccabv(\Sigma,m;X)$ is an embedding provided by Theorem 2.5, then for the operator $QJ : l_\infty \to L^1(m;X)$ we have $QJe_n = Je_n \nrightarrow 0$. Hence, by Rosenthal's l_∞-theorem (see [15] or [6]), $L^1(m;X)$ must contain an isomorphic copy of l_∞; a contradiction.

Case 2: *X has a subspace isomorphic to* l_∞. In view of Lemma 3.2 we may and will assume that the σ-algebra Σ is countably generated. Moreover, as is easily seen, we may also assume that $X = l_\infty$. By Theorem 2.5 there exists an isomorphic embedding

$$J : l_\infty \to ccabv(\Sigma,m;c_0) \subset ccabv(\Sigma,m;l_\infty)$$

such that $J(c_0) \subset L^1(m;c_0)$ and

$$(*) \qquad J(c_0) = J(l_\infty) \cap L^1(m;l_\infty)\,.$$

Suppose there exists an onto projection $Q : ccabv(\Sigma,m;l_\infty) \to L^1(m;l_\infty)$. Since the operators $QJ : l_\infty \to L^1(m;l_\infty) \subset ccabv(\Sigma,m;l_\infty)$ and J coincide on the sequence (e_n),

$$(**) \qquad (QJ - J)|c_0 = 0\,.$$

Now observe that the space $ccabv(\Sigma,m;l_\infty)$ admits a countable total set of continuous linear functionals. Indeed, if $\mathcal{A}$ is a countable algebra of sets generating Σ and e_n^* ($n \in \mathbb{N}$) are the coordinate functionals on l_∞, then the functionals

$$\mu \mapsto \langle e_n^*, \mu(A)\rangle \qquad (A \in \mathcal{A},\ n \in \mathbb{N})$$

are as required.

It follows that there exists a continuous linear injection of $ccabv(\Sigma,m;l_\infty)$ into l_∞. Hence, by a result of Kalton [12; Prop. 4], $(**)$ implies the existence of an infinite subset M of $\mathbb{N}$ such that $J = QJ$ on $l_\infty(M)$. Hence $J(l_\infty(M)) \subset L^1(m;l_\infty)$, which contradicts $(*)$. ■

The same argument as above establishes the following general fact. (It can be shown that a Banach space E has the property assumed below provided it contains no isomorphic copy of the space $l_\infty \times c_0(2^{\aleph_0})$.)

3.4. Proposition. *Let E be a Banach space such that whenever we have an operator $u : l_\infty \to E$ with $u|c_0 = 0$, then there is an infinite subset M of $\mathbb{N}$ for which $u|l_\infty(M) = 0$. Furthermore, let F be a closed subspace of E and suppose that it is possible to find an isomorphic embedding $J : l_\infty \to E$ such that $F \cap J(l_\infty)$ contains no copy of l_∞. Then F is not complemented in E.*

We conclude the paper with a result involving quotients of the spaces appearing in Theorems 3.1 and 3.3. The following lemma is certainly well known; we sketch its proof for the sake of completeness.

3.5. LEMMA. *Let Y and Z be closed subspaces of a Banach space X. Suppose there is a projection P from X onto Z with $P(Y) = Y \cap Z$ (so that also $P|Y : Y \to Y \cap Z$ is a projection). Then $Z/(Y \cap Z)$ is isomorphic to a complemented subspace of X/Y.*

Proof. Let $Q : X \to X/Y$ be the quotient map. We first verify that $Q(Z) \simeq Z/(Y \cap Z)$. Consider the operator

$$T : Z/(Y \cap Z) \to Q(Z), \qquad z + (Y \cap Z) \mapsto z + Y .$$

It is obvious that T is bounded. Let $V = \ker P$; then, clearly, $Y = (Y \cap V) \oplus (Y \cap Z)$. Since

$$\begin{aligned} \|z + Y\| &= \inf\{\, \|z + v + w\| : v \in Y \cap V,\, w \in Y \cap Z\} \\ &\geq \inf\{\, \|P\|^{-1}\|z + w\| : w \in Y \cap Z\} \\ &= \|P\|^{-1}\|z + Y \cap Z\| , \end{aligned}$$

T is an (onto) isomorphism.

Next, it is clear that the operator

$$\mathcal{P} : X/Y \to Q(Z), \qquad x + Y \mapsto Px + Y \quad (= Q(Px)) ,$$

is a projection onto $Q(Z)$. It is also bounded:

$$\begin{aligned} \|Px + Y\| &\leq \inf\{\|P(x + v + w)\| : v \in Y \cap V,\ w \in Y \cap Z\} \\ &\leq \|P\| \cdot \|x + Y\| . \ \blacksquare \end{aligned}$$

3.6. COROLLARY. *If the Banach space X has a subspace X_0 isomorphic to c_0, then each of the quotient spaces*

$$cabv(\Sigma, m; X)/L^1(m; X), \quad cabv(\Sigma, m; X)/ccabv(\Sigma, m; X), \\ ccabv(\Sigma, m; X)/L^1(m; X_0)$$

contains a complemented subspace isomorphic to l_∞/c_0.

Proof. This follows immediately from the above lemma and Theorems 2.2 and 2.5. ∎

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References

[1] J. Bourgain, *Dunford–Pettis operators on L^1 and the Radon–Nikodym property*, Israel J. Math. 37 (1980), 34–47.

[2] S. D. Chatterji, *Martingale convergence and the Radon–Nikodym theorem in Banach spaces*, Math. Scand. 22 (1968), 21–41.

[3] J. Diestel, *Sequences and Series in Banach Spaces*, Graduate Texts in Math. 92, Springer, New York 1984.

[4] J. Diestel and J. J. Uhl, Jr., *Vector Measures*, Math. Surveys 15, Amer. Math. Soc., Providence, R.I., 1977.

[5] P. Domański and L. Drewnowski, *Uncomplementability of the spaces of norm continuous functions in some spaces of "weakly" continuous functions*, Studia Math. 97 (1991), 245–251.

[6] L. Drewnowski, *Un théorème sur les opérateurs de $l_\infty(\Gamma)$*, C. R. Acad. Sci. Paris 281 (1976), 967–969.

[7] —, *Another note on copies of l_∞ and c_0 in $ca(\Sigma, X)$, and the equality $ca(\Sigma, X) = cca(\Sigma, X)$*, preprint, 1990.

[8] L. Drewnowski and G. Emmanuele, *On Banach spaces with the Gelfand–Phillips property. II*, Rend. Circ. Mat. Palermo (2) 38 (1989), 377–391.

[9] G. Emmanuele, *On complemented copies of c_0 in L^p_X, $1 \le p < \infty$*, Proc. Amer. Math. Soc. 104 (1988), 785–786.

[10] —, *About the position of $K_{w^*}(E^*, F)$ inside $L_{w^*}(E^*, F)$*, Atti Sem. Mat. Fis. Univ. Modena, to appear.

[11] M. Feder, *On the non-existence of a projection onto the space of compact operators*, Canad. Math. Bull. 25 (1982), 78–81.

[12] N. J. Kalton, *Spaces of compact operators*, Math. Ann. 208 (1974), 267–278.

[13] J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces I. Sequence Spaces*, Springer, New York 1977.

[14] J. Mendoza, *Copies of l_∞ in $L^p(\mu; X)$*, Proc. Amer. Math. Soc. 109 (1990), 125–127.

[15] H. P. Rosenthal, *On relatively disjoint families of measures, with some application to Banach space theory*, Studia Math. 37 (1970), 13–36.

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