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EXISTENCE OF SOLUTIONS OF A FUNCTIONAL-INTEGRAL EQUATION IN INFINITE DIMENSIONAL BANACH SPACES[1]

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Let Ω be a bounded closed subset of $\mathbb{R}^n$. If f, k, g are functions defined, respectively, in $\Omega \times E$, $\Omega \times \Omega$, $\Omega \times E$ (E a Banach space) with values into, respectively, E, $L(F, E)$, F (F a Banach space, $L(F, E)$ the space of all linear continuous operators from F into E), we consider the functional-integral equation

$$x(t) = f\Big(t, r\int_\Omega k(t,s)g(s,x(s))\,\mathrm{d}s\Big), \qquad t \text{ a.e. in } \Omega \tag{1}$$

and look for solutions of (1) lying in $L^1(\Omega, E)$, the usual Bochner function space on $(\Omega, \mathscr{L}, m)$, the usual Lebesgue measure space. The equation (1) is quite general, because for $f(t,x) = x$ we get the Hammerstein integral equation, whereas if $g(t,x) = x$ we get an equation recently considered in [2] and in [4]. (In particular, we improve the result in [4] because, in the case of $E = F = \mathbb{R}$ and $\Omega = [0,1] \subset \mathbb{R}$, we are able to dispense with one of the hypotheses used in that paper.) For several applications of the Hammerstein integral equation to partial differential equations we refer to [3] and [8].

The technique we use in the main theorem is the usual one: we construct an operator A mapping continuously a suitable bounded, closed and convex subset Q of $L^1(\Omega, E)$ into itself, and prove that $A(Q)$ is relatively compact. Hence the Schauder fixed point theorem can be applied. The choice of the set Q is such that it allows us to avoid the use of certain monotonicity assumptions contained in [2] (see also results and examples in [3] and in [8]) that are not always extendible to the case of functions with values in infinite dimensional Banach spaces; the hypotheses we consider are quite general and "natural" in the sense that they are necessary and sufficient for certain operators to take $L^1(\Omega, E)$ continuously into itself (see [7]).

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One of the main tools we use is the following generalization of the Ascoli-Arzelà theorem to the case of vector-valued continuous functions.

Theorem 1 (see [1], for instance). *Let T be a compact metric space and let $(f_n) \subset C^0(T, E)$ be a sequence of equicontinuous and equibounded functions. If for each $t \in T$, $\{f_n(t)\}$ is relatively compact in E, then (f_n) is relatively compact in $C^0(T, E)$. Moreover, the set $\{f_n(t) \colon t \in T,\ n \in \mathbb{N}\}$ is relatively compact in E, too.*

We even need the following extension of a theorem of Scorza-Dragoni to be found in [9].

Theorem 2 ([9]). *Let T be a compact metric space with a Radon measure defined on it, E a separable metric space, F a Banach space. If $f \colon T \times E \to F$ is a function verifying the Carathéodory hypotheses, i.e. f is measurable with respect to $t \in T$ for all $x \in E$ and continuous with respect to $x \in E$ for almost all $t \in T$, given $\varepsilon > 0$ there is a measurable closed subset T_ε of T with $m(T \setminus T_\varepsilon) < \varepsilon$ and $f|_{T_\varepsilon \times E}$ continuous.*

We shall make use of the following result concerning compact subsets of separable Banach spaces (see [5]).

Theorem 3 ([5]). *Let M be a bounded subset of a separable Banach space E. M is relatively compact if and only if for any w^*-null sequence $(x_n^*) \subset E^*$ one has* $\lim_n \sup_{x \in M} |x_n^*(x)| = 0$.

We are now ready to give the proof of our result for which we need an easy lemma:

Lemma 4. *Let us assume*
(k$_1$) $h_1, h_2 \in L^1(\Omega, \mathbb{R})$, $h_1(t) \geqslant 0$, $h_2(t) \geqslant 0$ a.e. *on* Ω;
(k$_2$) $\psi \colon \Omega \times \Omega \to \mathbb{R}_+$ *verifies the Carathéodory hypotheses and the linear operator*

$$(\Psi z)(t) = \int_\Omega \psi(t, s) z(s)\, \mathrm{d}s$$

maps $L^1(\Omega, \mathbb{R})$ into $L^1(\Omega, \mathbb{R})$ (in this case Ψ is continuous ([10]) and $\|\Psi\|$ denotes its norm);
(k$_3$) *$b_1, b_2, r \geqslant 0$ are such that $r b_1 b_2 \|\Psi\| < 1$.*
Then there is a nonnegative $\varphi_0 \in L^1(\Omega, \mathbb{R})$ such that

$$\varphi_0(t) = h_1(t) + r b_1 \int_\Omega \psi(t, s)\big(h_2(s) + b_2 \varphi_0(s)\big)\, \mathrm{d}s\,, \qquad t \text{ a.e. in } \Omega.$$

Proof. Let us put $p = \frac{\|h_1\| + \|a\|}{1 - rb_1b_2\|\Psi\|}$ where $a(t) = rb_1 \int_\Omega \psi(t,s)h_2(s)\,\mathrm{d}s \in L^1(\Omega,\mathbb{R})$. It is easy to see that $Mx \in B_p = \{x\colon x \in L^1(\Omega,\mathbb{R}),\ \|x\| \leqslant p\}$ whenever $x \in B_p$, where

$$Mx(t) = h_1(t) + rb_1 \int_\Omega \psi(t,s)\big(h_2(s) + b_2x(s)\big)\,\mathrm{d}s\,, \qquad t \in \Omega.$$

It is also clear that $Mx(t) \geqslant 0$ a.e. on Ω when $x(t) \geqslant 0$ a.e. on Ω and so $M(B_p^+) \subset B_p^+$ with $B_p^+ = B_p \cap \{x\colon x \in L^1(\Omega,\mathbb{R}),\ x(t) \geqslant 0 \text{ a.e. on } \Omega\}$; furthermore, B_p^+ is a complete metric space. It is also easy to prove that M is a contraction when restricted to B_p^+. Then the Banach-Caccioppoli fixed point theorem applies to give the result. We are done. □

Theorem 5. *Let E be a separable Banach space, F an arbitrary Banach space and Ω a bounded, closed subset of $\mathbb{R}^n$. Let us assume*

(h_1) *$f\colon \Omega \times E \to E$ verifies the Carathéodory hypotheses and, moreover, there exist $h_1 \in L^1(\Omega,\mathbb{R})$ and $b_1 \geqslant 0$ such that*

$$\|f(t,x)\|_E \leqslant h_1(t) + b_1\|x\|_E \qquad \text{for a.a. } t \in \Omega \text{ and all } x \in E;$$

(h_2) *$k\colon \Omega \times \Omega \to C(F,E)$ (the Banach space of linear compact operators from F into E with the usual operator norm) verifies the Carathéodory hypotheses and the linear operator K defined by*

$$(Kz)(t) = \int_\Omega \|k(t,s)\|_{C(F,E)} z(s)\,\mathrm{d}s\,, \qquad t \text{ a.e. in } \Omega$$

maps $L^1(\Omega,\mathbb{R})$ into itself (this last fact implies that K is continuous, see [10]; let $\|K\|$ denote its norm);

(h_3) *$g\colon \Omega \times E \to F$ verifies the Carathéodory hypotheses and, moreover, there exist $h_2 \in L^1(\Omega,\mathbb{R})$ and $b_2 \geqslant 0$ such that*

$$\|g(t,x)\|_F \leqslant h_2(t) + b_2\|x\|_E \qquad \text{for a.a. } t \in \Omega \text{ and all } x \in E;$$

(h_4) $rb_1b_2\|K\| < 1$.

Then the equation (1) has a solution x in $L^1(\Omega,E)$.

Proof. Putting $\psi(t,s) = \|k(t,s)\|_{C(E,F)}$ in Lemma 4, we get that there is a nonnegative $\varphi_0 \in L^1(\Omega,\mathbb{R})$ such that

$$\varphi_0(t) = h_1(t) + rb_1 \int_\Omega \|k(t,s)\|_{C(E,F)}\big(h_2(s) + b_2\varphi_0(s)\big)\,\mathrm{d}s\,, \qquad t \text{ a.e. in } \Omega.$$

First of all, assume $\varphi_0 = \theta_{L^1(\Omega,\mathrm{R})}$. In such a case we easily get that $\theta_{L^1(\Omega,E)}$ is a solution of (1). Indeed, we have

$$\begin{aligned}
&\left\| f\left(t, r \int_\Omega k(t,s) g\big(s, \theta_{L^1(\Omega,E)}(s)\big)\,\mathrm{d}s\right)\right\| \\
&\qquad \leqslant h_1(t) + b_1 \left\| r \int_\Omega k(t,s) g\big(s, \theta_{L^1(\Omega,E)}(s)\big)\,\mathrm{d}s \right\| \\
&\qquad \leqslant h_1(t) + b_1 r \int \big\|k(t,s)\big\|_{C(E,F)} \big\|g\big(s, \theta_{L^1(\Omega,E)}(s)\big)\big\|\,\mathrm{d}s \\
&\qquad \leqslant h_1(t) + b_1 r \int \big\|k(t,s)\big\|_{C(E,F)} \big(h_2(s) + b_2 \big\|\theta_{L^1(\Omega,E)}(s)\big\|\big)\,\mathrm{d}s \\
&\qquad \leqslant h_1(t) + b_1 r \int \big\|k(t,s)\big\|_{C(E,F)} \big(h_2(s) + b_2 \theta_{L^1(\Omega,R)}(s)\big)\,\mathrm{d}s \\
&\qquad\quad = \theta_{L^1(\Omega,R)}(t), \qquad\qquad t \text{ a.e. in } \Omega,
\end{aligned}$$

which means that

$$f\left(t, r \int_\Omega k(t,s) g\big(s, \theta_{L^1(\Omega,E)}(s)\big)\,\mathrm{d}s\right) = \theta_{L^1(\Omega,E)}(t), \qquad t \text{ a.e. in } \Omega.$$

So let us assume $\varphi_0 \neq \theta_{L^1(\Omega,\mathrm{R})}$ and consider the following subset of $L^1(\Omega, E)$:

$$Q = \{x \colon x \in L^1(\Omega, E),\ \|x(t)\|_E \leqslant \varphi_0(t) \quad \text{a.e. on } \Omega\}.$$

Q is clearly bounded, closed and convex in $L^1(\Omega, E)$; furthermore, Q is uniformly integrable, i.e. $\lim_{m(S)\to 0} \int_S \|x(s)\|\,\mathrm{d}s = 0$ uniformly on Q. We consider the operator

$$(Ax)(t) = f\left(t, r \int_\Omega k(t,s) g\big(s, x(s)\big)\,\mathrm{d}s\right).$$

We shall prove that

(i) $A\big(L^1(\Omega, E)\big) \subset L^1(\Omega, E)$,

(ii) $A(Q) \subset Q$,

(iii) $A|_Q$ is continuous,

(iv) $A(Q)$ is relatively compact.

Hence an easy application of the Schauder fixed point theorem will give the existence of a solution of (1). That (i) is true is an easy consequence of our assumptions

(h_1), (h_2), (h_3). Let us show (ii). If $x \in Q$ we have, for a.a. $t \in \Omega$,

$$\begin{aligned}
\|(Ax)(t)\|_E &= \Big\|f\Big(t, r\int_\Omega k(t,s)g\big(s,x(s)\big)\,\mathrm{d}s\Big)\Big\|_E \\
&\leqslant h_1(t) + b_1 r\Big\|\int_\Omega k(t,s)g\big(s,x(s)\big)\,\mathrm{d}s\Big\|_E \\
&\leqslant h_1(t) + b_1 r\int_\Omega \|k(t,s)\|_{C(F,E)}\big\|g\big(s,x(s)\big)\big\|_F\,\mathrm{d}s \\
&\leqslant h_1(t) + b_1 r\int_\Omega \|k(t,s)\|_{C(F,E)}\big(h_2(s) + b_2\|x(s)\|_E\big)\,\mathrm{d}s \\
&\leqslant h_1(t) + b_1 r\int_\Omega \|k(t,s)\|_{C(F,E)}\big(h_2(s) + b_2\varphi_0(s)\big)\,\mathrm{d}s = \varphi_0(t)
\end{aligned}$$

by virtue of Lemma 4. Let us prove (iii). Let $(x_n), (x_0) \subset Q$ with $x_n \to x_0$. This means that $x_n(s) \to x_0(s)$ almost everywhere on Ω (by passing to a subsequence if necessary). Fix $\bar t \in \Omega$; we have $k(\bar t,s)g\big(s,x_n(s)\big) \to k(\bar t,s)g\big(s,x_0(s)\big)$ for a.a. $s \in \Omega$ because of (h_3) and (h_2). Thanks to (h_3) we also have that, for a.a. $s \in \Omega$,

$$\begin{aligned}
&\big\|k(\bar t,s)\big[g\big(s,x_n(s)\big) - g\big(s,x_0(s)\big)\big]\big\|_E \\
&\qquad\leqslant \|k(\bar t,s)\|_{C(E,F)}\big[2h_2(s) + b_2\big(\|x_n(s)\| + \|x_0(s)\|\big)\big] \\
&\qquad\leqslant \|k(\bar t,s)\|_{C(E,F)}2\big(h_2(s) + b_2\varphi_0(s)\big);
\end{aligned}$$

this easily yields

$$\int_\Omega k(\bar t,s)g\big(s,x_n(s)\big)\,\mathrm{d}s \to \int_\Omega k(\bar t,s)g\big(s,x_0(s)\big)\,\mathrm{d}s\,.$$

Hence

$$f\Big(\bar t, r\int_\Omega k(\bar t,s)g\big(s,x_n(s)\big)\,\mathrm{d}s\Big) \to f\Big(\bar t, r\int_\Omega k(\bar t,s)g\big(s,x_0(s)\big)\,\mathrm{d}s\Big),$$

i.e. $Ax_n(\bar t) \to Ax_0(\bar t)$, thanks to (h_1). But $\|Ax_n(t) - Ax_0(t)\|_E \leqslant 2\varphi_0(t)$ and so $\|Ax_n - Ax_0\|_{L^1(\Omega,E)} \to 0$. It remains to show the most difficult (iv). It is clear that we can assume Q countable; so we do it. First of all, we observe that thanks to Theorem 2, given $\sigma > 0$ there is $\Omega_\sigma \subset \Omega$, closed, with $m(\Omega \setminus \Omega_\sigma) < \sigma$, such that $f\big|_{\Omega_\sigma \times E}$, $k\big|_{\Omega_\sigma \times \Omega}$ are continuous. First we shall prove that

(j) $B = \{Hx\big|_{\Omega_\sigma} : x \in Q\} \subset C^0(\Omega_\sigma)$, where $(Hx)(t) = \int_\Omega k(t,s)g\big(s,x(s)\big)\,\mathrm{d}s$, $t \in \Omega$,

(jj) B is relatively compact in $C^0(\Omega_\sigma)$.

Let $t', t'' \in \Omega$. We have

$$\|(Hx)(t') - (Hx)(t'')\|_E \leqslant \int_\Omega \|k(t', s) - k(t'', s)\|_{C(F,E)} \|g(s, x(s))\| \, ds$$
$$\leqslant \int_\Omega \|k(t', s) - k(t'', s)\|_{C(F,E)} (h_2(s) + b_2\varphi_0(s)) \, ds \, .$$

Since $k|_{\Omega_\sigma \times \Omega}$ is uniformly continuous, we get that B is an equicontinuous subset of $C^0(\Omega_\sigma)$; it is also clear that B is an equibounded subset of $C^0(\Omega_\sigma)$. It remains to show that for all $t \in \Omega_\sigma$, $B(t) = \{Ax|_{\Omega_\sigma}(t) : x \in Q\}$ is relatively compact, so that we can apply Theorem 1 to B. Now we use Theorem 3. Let (x_n^*) be a w^*-null sequence in E^*; for $\bar{t} \in \Omega_\sigma$ we have

$$(2) \qquad x_n^* \int_\Omega k(\bar{t}, s) g(s, x(s)) \, ds = \int_\Omega x_n^* k(\bar{t}, s) g(s, x(s)) \, ds \, , \qquad n \in \mathbb{N}.$$

For almost all $s \in \Omega$, the set $\{g(s, x(s)) : x \in Q\}$ is bounded by virtue of (h_3) and because of the very definition of Q; hence $\{k(\bar{t}, s) g(s, x(s)) : x \in Q\}$ is compact in E for a.a. $s \in \Omega$ and so $x_n^* k(\bar{t}, s) g(s, x(s)) \to 0$ uniformly on $x \in Q$; furthermore, $|x_n^* k(\bar{t}, s) g(s, x(s))| \leqslant \sup_n \|x_n^*\| \, \|k(\bar{t}, s)\|_{C(F,E)} (h_2(s) + b_2\varphi_0(s))$, which implies

$$\sup_{x \in Q} \int_\Omega x_n^* k(\bar{t}, s) g(s, x(s)) \, ds \to 0.$$

Thanks to (2) we are done: $B(t)$ is relatively compact for all $t \in \Omega$. Hence B is relatively compact in $C^0(\Omega_\sigma)$. Once we have (j) and (jj) for any $\sigma > 0$, we can conclude our proof as follows. Given a sequence $(x_h) \subset Q$, it is easy to get a sequence (Ω_n) of closed subsets of Ω with $m(\Omega \setminus \Omega_n) \to 0$ and a subsequence (y_h) of (x_h) such that (Hy_h) is a Cauchy sequence in any $C^0(\Omega_n)$. Again thanks to Theorem 1 we have that

$$C_n = \{Hy_h(t) : t \in \Omega_n, \ h \in \mathbb{N}\}$$

is a relatively compact subset of E and so $f|_{\Omega_n \times \bar{C}_n}$ is uniformly continuous. It is then very easy to see (use again Theorem 1) that

$$\{f(\cdot, Hy_h(\cdot)) : \Omega_n \to E, \ h \in \mathbb{N}\}$$

is a Cauchy sequence in $C^0(\Omega_n)$ for all $n \in \mathbb{N}$, by passing to a suitable subsequence if necessary. Hence, if $\varepsilon > 0$, let $\sigma > 0$ be such that

$$\sup_{x \in Q} \int_S \|Ax(s)\| \, ds < \frac{\varepsilon}{4} \qquad \text{for all } S \subset \Omega, \ m(S) < \sigma.$$

Choose $\bar{n} \in \mathbb{N}$ with $m(\Omega \setminus \Omega_{\bar{n}}) < \sigma$. We have

$$\begin{aligned}\int_{\Omega} \|Ay_{h'}(t) - Ay_{h''}(t)\|_E \, dt &= \int_{\Omega_{\bar{n}}} \|Ay_{h'}(t) - Ay_{h''}(t)\|_E \, dt \\ &\quad + \int_{\Omega \setminus \Omega_{\bar{n}}} \|Ay_{h'}(t) - Ay_{h''}(t)\|_E \, dt \\ &\leqslant \int_{\Omega_{\bar{n}}} \big\|f\big(t, Hy_{h'}(t)\big) - f\big(t, Hy_{h''}(t)\big)\big\|_E \, dt + \frac{\varepsilon}{2} \\ &\leqslant m(\Omega_{\bar{n}}) \big\|f\big(\cdot, Hy_{h'}(\cdot)\big) - f\big(\cdot, Hy_{h''}(\cdot)\big)\big\|_{C^0(\Omega_{\bar{n}})} + \frac{\varepsilon}{2} \\ &\qquad h', h'' \in \mathbb{N}.\end{aligned}$$

Since $\big(f(\cdot, Hy_h(\cdot))\big)_{h \in \mathbb{N}}$ is a Cauchy sequence in $C^0(\Omega_{\bar{n}})$ we are done. □

Remark. If one of the two spaces E and F is finite dimensional, then any continuous and linear operator from F into E is compact, but this even happens for suitable infinite dimensional Banach spaces; we refer to [6] for a list of such pairs of Banach spaces.

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