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Answer to a Question by M.Feder About K(X,Y)

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ABSTRACT. We show that a Banach space E constructed by Bourgain-Delbaen in 1980 answers a question put by Feder in 1982 about spaces of compact operators.

Let X, Y be two Banach spaces. By $K(X, Y)$, $W(X, Y)$, $L(X, Y)$ we denote the Banach spaces of all compact, weakly compact and bounded linear operators from X into Y, respectively. In the paper [4] Feder put a question that in light of recent results in [3] can be reformulated as it follows

Question. *Do Banach spaces X and Y exist so that $K(X, Y) \neq L(X, Y)$ and however $K(X, Y)$ does not contain a copy of c_0?*

Feders's question is related to the following problem: let us assume X, Y are such that $L(X, Y) \neq K(X, Y)$; is $K(X, Y)$ uncomplemented in $L(X, Y)$?

The results in [3] and [4] show that the best result known is the following one: if c_0 embeds into $K(X, Y)$, then $K(X, Y)$ is uncomplemented in $L(X, Y)$; so it remains to study the case of $K(X, Y)$ containing no copy

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of c_0, if such two spaces exist (i.e. if Feder's question has a positive answer).

Up to now, no answer to Feder's question has been given as far as we know.

In this short note we want to show that a Banach space constructed by Bourgain and Delbaen in [1] (before Feder's paper appeared) answers positively to the above question. The space E constructed by Bourgain and Delbaen is a $\mathscr{L}_\infty$-space with the Radon-Nikodym property such that E^* is isomorphic to ℓ^1. If we take $X=Y=E$ we clearly get $K(X, Y)=W(X, Y)\neq L(X, Y)$. Now, let us assume that c_0 lives inside $K(X, Y)$. We recall that $K(X, Y)\simeq K_{W^*}(X^{**}, Y)$ and $W(X, Y)\simeq L_{W^*}(X^{**}, Y)$ where $K_{W^*}(X^{**}, Y)$, $L_{W^*}(X^{**}, Y)$ denote the spaces of all w^*-w continuous compact, bounded operators from X^{**} into Y, respectively. So we can act in $K_{W^*}(X^{**}, Y)$. Let (T_n) be a copy of the unit vector basis of c_0 in $K_{W^*}(X^{**}, Y)$. For $x^{**}\in X^{**}$, the series $\Sigma T_n(x^{**})$ is unconditionally converging in Y and so, for any $\xi\in\ell_\infty$, the series $\Sigma\xi_n T_n(x^{**})$ is also unconditionally converging. It is not difficult to see that the map $\xi\to\Sigma\xi_n T_n$ defines a bounded, linear operator from ℓ_∞ into $L(X^{**}, Y)$. We shall prove that, actually, $\Sigma\xi_n T_n\in L_{W^*}(X^{**}, Y)$. Let (x_α^{**}) be a w^*-null net in $B_{X^{**}}$ and $y^*\in Y^*$. If we denote by φ_ξ the operator $\Sigma\xi_n T_n$, we have to show that

$$\lim_\alpha |\varphi_\xi(x_\alpha^{**})(y^*)|=0$$

Since $\Sigma\xi_n T_n^*(y^*)$ is unconditionally converging, we have

$$\lim_n \sup_{B_{X^{**}}} \sum_{p=n}^{\infty} |\xi_p T_p^*(y^*)(x^{**})|=0.$$

So, given $\varepsilon>0$, there is $n_0\in\mathbb{N}$ such that $\sum_{p=n_0}^{\infty}|\xi_p T_p^*(y^*)(x_\alpha^{**})|<\varepsilon/2$ for all α; since $x_\alpha^{**}\xrightarrow{w^*}\theta$, it is obvius that

$$\lim_\alpha \sum_{p=1}^{n_0-1} |\xi_p T_p^*(y^*)(x_\alpha^{**})|=0,$$

and so for α sufficiently large we have

$$\sum_{p=1}^{n_0-1}|\xi_p T_p^*(y^*)(x_\alpha^{**})|<\varepsilon/2.$$

Hence, for α sufficiently large, we get

$$\sum_{p=1}^{\infty} |\xi_p T_p^*(y^*)(x_\alpha^{**})| < \varepsilon,$$

which means that

$$\lim_\alpha |\varphi_\xi(x_\alpha^{**})(y^*)| = 0.$$

Hence, $\Sigma \xi_n T_n \in L_{W^*}(X^{**}, Y)$. In this way, we have defined a bounded, linear operator from ℓ_∞ into $L_{W^*}(X^{**}, Y) \simeq W(X, Y)$ such that the unit vector basis of c_0 is mapped onto a not relatively compact sequence. A result due to Rosenthal ([5]) implies that ℓ_∞ must live inside $W(X, Y)$. Since $W(X, Y) = K(X, Y)$, ℓ_∞ embeds into $K(X, Y)$, too. But this contradicts a corollary of the main result of [2].

We also observe that in the paper [1] another class of Banach spaces F has been introduced; as remarked in the NOTES ADDED to our paper [3] if $X = Y = F$ we get a second example of Banach spaces answering positively Feder's question; even in that case $W(X, Y) = K(X, Y)$. So we can conclude this note with the following questions

Question A. *Do Banach spaces X, Y exist so that $K(X, Y) \neq W(X, Y)$ and c_0 does not embed into $K(X, Y)$?*

Question B. *Let $X = Y = E$ (or F) be the Bourgain-Delbaen space. Is $K(X, Y)$ uncomplemented in $L(X, Y)$?*

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