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About the Existence of Integrable Solutions of a Functional-Integral Equation (1)

G. EMMANUELE

ABSTRACT. We improve (in some sense) a recent theorem due to Banas and Knap ([2]) about the existence of integrable solutions of a functional-integral equation.

1. INTRODUCTION

Let $I=[0,1]$ be. We consider the following functional-integral equation

$$x(t)=g(t)+f\left(t, \int_0^1 k(t,s)x(\varphi(s))\,ds\right) \quad t\in I \tag{1}$$

where $f\colon I\times R\to R^+=[0,+\infty)$, $k\colon I\times I\to R^+$, $g\colon I\to R$ $\varphi\colon I\to I$ are functions verifying special hypotheses (see section 2) and we look-for solutions $x\in L^1(I)$. As remarked in the paper [2] this equation has been considered by a number of authors because of its importance in problems in physics, engineering and economics; further, problems in the theory of partial differential equations lead, sometimes, to the study of the equation (1). Recently, Banas and Knap ([2]) gave a result of existence of integrable solutions to (1). They were forced by the techniques used to consider certain monotonicity assumptions on g, f, k (see hypotheses i), ii) and iv) in [2]), that we are able to eliminate completely here. However, we must observe that Banas and Knap obtain a monotone solution, a fact that doesn't follow from our hypotheses. Prof Banas also observed that under our hypotheses we don't need to use the

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measure of weak noncompactness he considered in [2] because the operator we define following [2] actually has a relatively weakly compact range. So it is enough to apply Tychonoff fixed point Theorem ([5]). We take this opportunity to thank him very much for this remark that made our proof simpler.

2. PRELIMINARIES AND MAIN RESULT

As in the paper [2] we define the following four operators

$$(Kx)(t)=\int_0^1 k(t,s)x(s)\,ds$$

$$(Fx)(t)=f(t,x(t))$$

$$(Hx)(t)=f\left(t,\int_0^1 k(t,s)x(s)\,ds\right)$$

$$x=Ax=g+Hx(\varphi)=g+FKx(\varphi).$$

We consider the following hypotheses

(i) $g\in L^1(I)$.

(ii) $f\colon I\times R\to R^1$ *satisfies Caratheodory hypotheses* (i.e. *f is measurable with respect to* $t\in I$, *for all* $x\in R$, *and continuous in* $x\in R$, *for a.a.* $t\in I$) *and there are* $a\in L^1(I)$, $b\geq 0$ *such that*

$$f(t,x)\leq a(t)+b\,|x| \qquad t\in I, \quad x\in R$$

(this last inequality is a necessary and sufficient condition for F, and so H, to take values in $L^1(I)$ when acting on elements of $L^1(I)$; see Theorem 1 in [2])

(iii) *k verifies Caratheodory hypotheses and there is* $\lambda\in L^1(I)$ *such that*

$$k(t,x)\leq \lambda(t) \qquad t \text{ a.e. in } I,\ x\in R$$

(under (iii) the linear operator K maps $L^1(I)$ into $L^1(I)$ continuously; let us denote by $\|K\|$ its operator norm)

(iv) $\varphi\colon I\to I$ *is absolutely continuous and there exists* $B>0$ *such that* $\varphi'(t)\geq B$ *for a.a.* $t\in I$.

(v) $b\,\|K\|/B<1$.

The technique used in [2] is the following: under the above assumptions A is a weakly continuous operator from a suitable B_s into itself; furthermore there exists $L \in [0, 1]$ such that $\beta(A(Y)) \leq L\beta(Y)$, ($\beta$ the measure of weak noncompactness introduced in [3]), for all nonempty subsets Y of B_s and hence results from [1] and [6] can be applied to get a fixed point of the operator $x \to g + FKx(\varphi)$. The difference between the result in [2] and our Theorem below resides in the technique we use to obtain the weak continuity of A; indeed, Banas and Knap consider some monotonicity hypotheses on g, f, k we are able to dispense with. Further, we do not make use of the measure of weak noncompactness introduced in [3] as remarked in the Introduction.

Theorem. *Under the assumptions* i)-v) *above the equation* [1] *has at least a solution* $x \in L^1(I)$.

Proof. As in the paper [2] we can prove that $A: B_s \to B_s$, where $s = (\|g\| + \|a\|)/(1 - b\|K\|B^{-1})$. Furthermore, it is not difficult to see that the set $A(B_s)$ is relatively weakly compact ([5]), since it is bounded and uniformly integrable. Hence Tychonoff fixed point Theorem ([5]) will conclude the proof once we have the weak continuity of A. So, we need only to show that A is weakly continuous from B_s into B_s, i.e. A maps weakly convergent nets $(x_\alpha) \subset B_s$ into weakly convergent nets $(A(x_\alpha))$. It is clearly enough to show that H is weakly continuous. So let (x_α), $x_0 \subset B_s$ be with $x_\alpha \xrightarrow{w} x_0$; if we prove that for any $\epsilon > 0$, any $y^* \in L^\infty(I)$, $\|y^*\| \leq 1$ and any subnet (x_{α_β}) of (x_α), there is another subnet $(x_{\alpha_{\beta_\gamma}})$ for which $|\langle H(x_{\alpha_{\beta_\gamma}}) - H(x_0), y^* \rangle| < \epsilon$ we are done (proceeding by contradiction, of course).

To reach our target, we start by noting that the operator $x \to x(\varphi)$ from $L^1(I)$ into itself is bounded and linear; hence it is weakly continuous and so $x_\alpha(\varphi) \xrightarrow{w} x_0(\varphi)$ in $L^1(I)$. Since B_s is bounded in $L^1(I)$, the set $\{x_\alpha(\varphi), x_0(\varphi)\}$ is even bounded in $L^1(I)$, by a number M. Now, given $\epsilon > 0$ choose $\delta > 0$ such that meas$(D) < \delta$, implies $\int_D 2[a(t) + b\lambda(t)]\,dt < \frac{\epsilon}{2}$. Furthermore, choose a closed subset $I_1 \subset I$, meas$(I \setminus I_1) < \frac{\delta}{4}$, with $\lambda|_{I_1}$ continuous (use Lusin Theorem, [4]) $Q = \max_{I_1} \lambda$. Again consider a closed subset $I_2 \subset I$, meas$(I \setminus I_2) < \frac{\delta}{4}$, with $f|_{I_2 \times [-QM, QM]}$ continuous (and so uniformly continuous) and a closed subset $I_3 \subset I$, meas$(I \setminus I_3) < \frac{\delta}{4}$, with $k|_{I_3 \times I}$ continuous (and so uniformly continuous) (use Scorza-Dragoni Theorem, [6]). Put $I_0 = \bigcap_{i=1}^{3} I_i$. I_0 is a closed subset of I. Now, observe that, for t', $t'' \in I_0$, if $\psi_\alpha(t) = \int_0^1 k(t, s)x_\alpha(\varphi(s))\,ds$, $\psi_0(t) = \int_0^1 k(t, s)x_0(\varphi(s))\,ds$, one has

$$|\psi_\alpha(t') - \psi_\alpha(t'')| \leq \int_0^1 |k(t', s) - k(t'', s)| |x_\alpha(\varphi(s))| \, ds$$

(the same is true for ψ_0). Since $k|_{I_3 \times I}$ is uniformly continuous and $(x_\alpha) \subset B_r$, the set $\{\psi_\alpha, \psi_0\}$ is equicontinuous in $C^0(I_0)$. It is very easy to see that the same set is bounded by QM in the norm of $C^0(I_0)$, hence the Ascoli-Arzelà Theorem can be applied to get a relatively compact subset of $C^0(I_0)$. The net (ψ_{α_β}) admits a converging subnet $(\psi_{\alpha_{\beta_\gamma}})$. On the other hand, for $\bar{t} \in I_0$,

$$\psi_\alpha(\bar{t}) = \int_0^1 k(\bar{t}, s) x_\alpha(\varphi(s)) \, ds \to \psi_0(\bar{t}) = \int_0^1 k(\bar{t}, s) x_0(\varphi(s)) \, ds$$

since $x_\alpha(\varphi) \xrightarrow{w} x_0(\varphi)$ in $L^1(I)$ and $s \to k(\bar{t}, s)$ is in $L^\infty(I)$. Hence $\psi_{\alpha_{\beta_\gamma}} \to \psi_0$ in the C^0 —norm on I_0. Now, recall that $f|_{I_0 \times [-QM, QM]}$ is uniformly continuous and so we have

$$\lim_\gamma f(t, \psi_{\alpha_{\beta_\gamma}}(t)) = f(t, \psi_0(t)) \quad \text{uniformly on } I_0 \tag{2}$$

Now, take $y^* \in L^\infty(I)$, with $\|y^*\|_\infty \leq 1$, calculate this y^* on $(f(\bullet, \psi_{\alpha_{\beta_\gamma}}(\bullet)) - f(\bullet, \psi_0(\bullet)))$

$$\left| \int_0^1 y^*(t) [f(t, \psi_{\alpha_{\beta_\gamma}}(t)) - f(t, \psi_0(t)) \, dt \right| \leq$$

$$\leq \int_{I_0} |y^*(t)| |f(t, \psi_{\alpha_{\beta_\gamma}}(t)) - f(t, \psi_0(t))| \, dt +$$

$$+ \int_{I \setminus I_0} |y^*(t)| |f(t, \psi_{\alpha_{\beta_\gamma}}(t)) - f(t, \psi_0(t))| \, dt \leq$$

$$\leq \int_{I_0} |f(t, \psi_{\alpha_{\beta_\gamma}}(t)) - f(t, \psi_0(t))| \, dt + \int_{I \setminus I_0} 2[a(t) + b\lambda(t)] \, dt.$$

Now, recall that (2) is true and observe that

$$\text{meas}(I \setminus I_0) \leq \sum_{i=1}^3 m(I \setminus I_i) \leq \frac{3}{4} \delta < \delta \text{ so that} \int_{I \setminus I_0} 2[a(t) + b\lambda(t)] \, dt < \frac{\epsilon}{2}.$$

Hence the last member of the chain of inequalities written above is smaller than ϵ for γ sufficiently large. This is what we need to show that H is weakly continuous on B_r. We are done.

References

[1] J. Appell, E. De Pascale, *Su alcuni parametri connessi con la misura di non compattezza di Hausdorff in spazi di funzioni misurabili,* Boll. Un. Mat. It. (6) 3-B (1984) 497-515.
[2] J. Banas; Z. Knap, *Integrable solutions of a functional-integral equation,* Revista Matemática de la Univ. Complutense de Madrid, 2(1) (1989) 31-38.
[3] F. S. De Blasi, *On a property of the unit sphere in a Banach space,* Bull. Math. Soc. Sci. Math. R. S. Roumanie 21 (1977) 259-262.
[4] N. Dinculeanu, *Vector Measures,* Pergamon Press 1967.
[5] N. Dunford, J. T. Schwartz, *Linear operators,* part. I, Interscience 1958.
[6] G. Emmanuele, *Measure of weak noncompactness and fixed point theorems,* Bull. Math. Soc. Sci. Math. R. S. Roumanie 25 (1981) 253-258.
[7] G. Scorza-Dragoni, *Un teorema sulle funzioni continue rispetto ad una e misurabili rispetto ad un'altra variabile,* Rend. Sem. Mat. Univ. Padova 17 (1948) 102-106.

Department of Mathematics,
University of Catania,
Viale A. Doria 6,
95125 Catania,
Italy

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