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SOME MORE BANACH SPACES WITH THE (NRNP)

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We show that certain quotients of Banach spaces with the (NRNP) have the same property.

In the paper [6] R. Kaufman, M. Petrakis, L.H. Riddle, J.J. Uhl introduced the *Near Radon-Nikodym Property* (in symbols (NRNP)) and gave several examples of Banach spaces possessing this property. The aim of this short note is to give more examples of Banach spaces possessing the (NRNP), showing that special quotients of spaces with the (NRNP) inherit the same property.

Throughout the paper X, Y, Z will denote Banach spaces and μ the Lebesgue measure on $[0, 1]$. Before starting we remind the following definitions.

Definition 1. *A bounded linear operator $T : L_1[0, 1] \to X$ is Bochner representable if there is a bounded measurable function $g : [0, 1] \to X$ such that*

$$Tf = \int_{[0,1]} fg\, d\mu$$

for all $f \in L_1[0, 1]$.

Definition 2. *A bounded linear operator $T : L_1[0, 1] \to X$ is nearly representable if for each Dunford–Pettis operator $D : L_1[0, 1] \to L_1[0, 1]$ the composition $T \circ D : L_1[0, 1] \to X$ is Bochner representable.*

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Definition 3. *A Banach space X has the (NRNP) if every nearly representable operator $T : L_1[0, 1] \to X$ is Bochner representable.*

For other definitions as well as for standard notations concerning vector measures or Banach spaces of vector measures we refer to [2].

In [6] it is proved that Banach spaces containing c_0 do not have the (NRNP), whereas Banach spaces with the (RNP), Banach lattices not containing copies of c_0 and the Banach space $L_1[0, 1]/H_0^1$ enjoy the (NRNP).

In this short note we want to point out how to construct more Banach spaces with the considered property just taking suitable quotients of spaces with the (NRNP). In particular we shall generalize Corollary 19 from [6].

Our main result is the following theorem

Theorem 1. *Let X be a Banach space with the (NRNP), Z be a closed subspace of X with the (RNP) and $Y = X/Z$. If for each countably additive measure with bounded variation $\nu : Bo[0, 1] \to Y$ such that $\|\nu(E)\| \leq \mu(E)$, for each $E \in Bo[0, 1]$, there is a countably additive measure with bounded variation $\tilde{\nu} : Bo[0, 1] \to X$ such that $\|\tilde{\nu}(E)\| \leq \mu(E)$ and $Q[\tilde{\nu}(E)] = \nu(E)$, for each $E \in Bo[0, 1]$, where Q denotes the quotient map of X onto Y, then Y has the (NRNP).*

Proof. Let $S : L_1[0, 1] \to Y$ be a nearly representable operator. Without loss of generality we may assume that $\|S\| = 1$. It is well known, [2], that the vector measure $\nu(E) = S(\chi_E)$ is countably additive with bounded variation on $Bo[0, 1]$ and $\|\nu(E)\| \leq \mu(E)$, for each $E \in Bo[0, 1]$. Since it has values in Y we can apply our assumption to get a countably additive measure $\tilde{\nu} : Bo[0, 1] \to X$ with bounded variation such that $\|\tilde{\nu}(E)\| \leq \mu(E)$ and $Q[\tilde{\nu}(E)] = \nu(E)$, for each $E \in Bo[0, 1]$. Hence $\tilde{\nu}$ easily defines a norm one operator $\tilde{S} : L_1[0, 1] \to X$ by putting

$$\tilde{S}(f) = \int_{[0,1]} f \, d\tilde{\nu} \qquad f \in L_1[0, 1]$$

(start defining $\tilde{S}$ linearly on simple functions by the above formula; then extend by continuity to all of $L_1[0, 1]$). Clearly, we have $\tilde{\nu}(E) = \tilde{S}(\chi_E)$ for each $E \in Bo[0, 1]$ and hence $S = Q \circ \tilde{S}$. We want to show that $\tilde{S}$ is nearly Bochner representable (this part of the proof uses the ideas contained in the proof of Corollary 24 in [6]). Let us consider a Dunford–Pettis operator $D : L_1[0, 1] \to L_1[0, 1]$. Since the operator $S \circ D$ is Bochner representable, it factorizes through l_1 (see [7], Lewis–Stegall Theorem), i.e. there are operators $A : L_1[0, 1] \to l_1$ and $B : l_1 \to Y$ so that $S \circ D = B \circ A$. By virtue of the lifting property of l_1 we can find an operator $\tilde{B} : l_1 \to X$ so that $B = Q \circ \tilde{B}$. Hence we get

$$Q \circ \tilde{S} \circ D = S \circ D = Q \circ \tilde{B} \circ A.$$

It follows that

$$(\tilde{S} \circ D - \tilde{B} \circ A)(f) \in \ker Q = Z \quad \text{for all } f \in L_1[0, 1];$$

hence, $\tilde{S} \circ D$ is Bochner representable, since $\tilde{B} \circ A$ is (see [7], Lewis–Stegall Theorem) and Z has the (RNP). The arbitrariness of D gives that $\tilde{S}$ is nearly Bochner representable and hence Bochner representable. It is then clear that $S = Q \circ \tilde{S}$ is Bochner representable, too. We are done.

Now, we present three concrete applications of Theorem 1 showing (together with the results from [6]) that quite large families of Banach spaces enjoy the (NRNP).

Corollary 2. *Let, in Theorem 1, X be a dual Banach space with the (NRNP), Z a w^*-closed subspace of X with the (RNP), $Y = X/Z$. Then Y has the (NRNP).*

Proof. In [3], Corollary 11, it is shown that given a countably additive measure with bounded variation $\nu : Bo[0, 1] \to Y$ such that $\|\nu(E)\| \leq \mu(E)$ for each $E \in Bo[0, 1]$, there is a countably additive measure with bounded variation $\tilde{\nu} : Bo[0, 1] \to X$ such that $\tilde{\nu}$ is absolutely continuous with respect to μ and $Q[\tilde{\nu}(E)] = \nu(E)$ for each $E \in Bo[0, 1]$. Now, we observe that both ν and $\tilde{\nu}$ are regular; so $\tilde{\nu}$ belongs to $rcabv(Bo[0, 1], X)$ a dual Banach space ([2]), that admits $rcabv(Bo[0, 1], Y)$, in which ν lives, as a quotient space (because $Y = X/Z = (Z^{\perp})^*$ since Z is w^*-closed in X),the quotient map $\tilde{Q}$ being defined by $[\tilde{Q}(\tilde{\nu})](E) = Q[\tilde{\nu}(E)]$, $E \in Bo[0, 1]$ (see Corollary 11 in [3] where it is also remarked that $\|\tilde{Q}\| = \|Q\| = 1$). Since $\tilde{Q}$ is $w^* - w^*$-continuous (as proved in [3], Corollary 11), using also the w^*-lower semicontinuity of a dual norm, we can find $\tilde{\nu}$ such that $\|\tilde{\nu}\| = \|\nu\|$ and $\tilde{Q}(\tilde{\nu}) = \nu$. We want to show that $\|\tilde{\nu}(E)\| \leq \mu(E)$ for each $E \in Bo[0, 1]$. To this purpose observe that, for $E \in Bo[0, 1]$, we have

$$\|\nu\| = \|\nu\|(E) + \|\nu\|(E^c) \leq \|\tilde{\nu}\|(E) + \|\tilde{\nu}\|(E^c) = \|\tilde{\nu}\| = \|\nu\| \tag{1}$$

where the inequality is due to the fact that, for each $E \in Bo[0, 1]$, $\|\nu(E)\| \leq \|\tilde{\nu}(E)\|$ and hence $\|\nu\|(E) \leq \|\tilde{\nu}\|(E)$. From (1) we derive that

$$\|\nu\|(E) = \|\tilde{\nu}\|(E) \qquad E \in Bo[0, 1]. \tag{2}$$

From (2) we get

$$\|\tilde{\nu}(E)\| \leq \|\tilde{\nu}\|(E) = \|\nu\|(E) \leq \mu(E) \in Bo[0, 1]$$

where the last inequality follows easily from the fact that $\|\nu(E)\| \leq \mu(E)$ for each $E \in Bo[0, 1]$. Since, now, all of the assumptions of Theorem 1 are verified we may apply that result to get our thesis.

Corollary 2 may be applied, for instance, with $X = (C(\partial D))^*$ and $Z = H_0^1$ to get that A^*, isomorphic to $(C(\partial D))^*/H_0^1$ (see [8]), has the (NRNP), a fact that doesn't seem to follow from the results in [6].

Corollary 3. *Let, in Theorem 1, X be a Banach space with the (NRNP) and Z be a reflexive subspace of X, $Y = X/Z$. Then Y has the (NRNP).*

Proof. Let us look at Z as a w^*-closed subspace of X^{**}. The proof of Corollary 2 works to get that for each countably additive measure with bounded variation $\nu : Bo[0,1] \to Y$ such that $\|\nu(E)\| \leq \mu(E)$ for each $E \in Bo[0,1]$, there is a countably additive measure with bounded variation $\tilde{\nu} : Bo[0,1] \to X^{**}$ such that $\|\tilde{\nu}(E)\| \leq \mu(E)$ and $Q^{**}[\tilde{\nu}(E)] = \nu(E)$ for each $E \in Bo[0,1]$. Repeating the proof of Corollary 12 in [3] we can prove that $\tilde{\nu}$ takes its values in X. Again an application of Theorem 1 concludes the proof.

Corollary 3 may be applied, for instance, with $X = L_1[0,1]$ and Z the subspace spanned by the Rademacher functions.

Corollary 4. *Let X be a L-summand in the bidual having the (NRNP) and Z be a subspace of X that is an L-summand in the bidual having the (RNP). Then $Y = X/Z$ has the (NRNP).*

Proof. Let $\nu : Bo[0,1] \to Y$ be a countably additive measure with bounded variation such that $\|\nu(E)\| \leq \mu(E)$, for each $E \in Bo[0,1]$. We can define an operator $T : L_1[0,1] \to Y$, $\|T\| \leq 1$, possessing such a ν as representing measure. Proposition 2.12 on p.177 of [5] gives that there is $\tilde{T} : L_1[0,1] \to X$, $\|\tilde{T}\| \leq 1$, for which $Q \circ \tilde{T} = T$. So the representing measure of $\tilde{T}$ is the $\tilde{\nu}$ required to allow Theorem 1 works. We are done.

Corollary 4 for instance contains, as a special case, Corollary 24 in [6].

The next result is a generalization of Corollary 19 from [6]; it utilizes the results in the present note, but also the same Corollary 19 from [6], so that it can be seen as a consequence of that result.

Corollary 5. *Let X be a complemented subspace of a Banach lattice L and Z a reflexive subspace of X. If $Y = X/Z$, the following facts are equivalent*

(1) *Y does not contain copies of c_0;*
(2) *Y has the (NRNP).*

Proof. It follows from the results in [6] that the implication "(2) $\Rightarrow$ (1)" is always true; so it remains to show just the reverse implication. If Y verifies (1), a result in [1] implies that X does not contain copies of c_0. Now, we may use a theorem from [4] showing that, under our assumptions, X is complemented in a Banach lattice not containing copies of c_0 so that we may suppose that L itself does. Hence L has the (NRNP) (Corollary 19, [6]). Now, let T be a nearly representable operator taking its values into X; it is clear that it can be seen as taking values into L that has the (NRNP). So T can be represented by a L-valued bounded measurable function. Since X is complemented in L, it is trivial to see that T can be represented by a X-valued bounded measurable function. Hence X has the (NRNP). It is now enough to apply Corollary 3 to get (2).

The last result is a consequence of Corollary 2

Corollary 6. *Let X be a dual Banach space complemented in a Banach lattice L, Z a w^*-closed subspace of X with the Radon-Nikodym property, $Y = X/Z$. Then the following facts are equivalent*

(1) *Y does not contain copies of c_0;*

(2) *Y has the (NRNP).*

We do not give the proof of this result that is similar to that of Corollary 5.

REFERENCES

[1] J. Bourgain - G. Pisier, *A construction of $\mathscr{L}_\infty$-spaces and related Banach spaces,* Bol. Soc. Bras. Mat., 14 (2) (1983), pp.109–123.

[2] J. Diestel - J.J. Uhl jr., *Vector Measures,* Math. Surveys 15, Amer. Math. Soc. Providence, Rhode Island, 1977.

[3] G. Emmanuele, *Remarks on the complementability of spaces of Bochner integrable functions in spaces of vector measures*, to appear.

[4] T. Figiel - W.B. Johnson - L. Tzafriri, *On Banach lattices and spaces having local unconditional structure, with applications to Lorentz spaces,* J. Approx. Theory, 13 (1975), pp. 395–412.

[5] P. Harmand - D. Werner - W. Werner, *M-ideals in Banach spaces and Banach algebras,* LNM 1547, Springer Verlag, Berlin, Heidelberg, 1993.

[6] R. Kaufman - M. Petrakis - L.H. Riddle - J.J. Uhl jr., *Nearly Representable Operators,* Trans. Amer. Math. Soc., 312 (1989), pp. 315–333.

[7] D.R. Lewis - C. Stegall, *Banach spaces whose duals are isomorphic to $l_1(\Gamma)$,* J. Funct. Analysis, 12 (1973), pp. 177–187.

[8] A. Pelczynski, *Banach spaces of Analytic Functions and Absolutely Summing Operators,* CBMS 30, Amer. Math. Soc., Providence, Rhode Island, 1977.

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