THE POSITION OF $\mathcal{K}(X,Y)$ IN $\mathcal{L}(X,Y)$.

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ABSTRACT. In this note, we investigate the nature of family of pairs of separable Banach spaces (X,Y) such that $\mathcal{K}(X,Y)$ is complemented in $\mathcal{L}(X,Y)$. It is proved that the family of pairs (X,Y) of separable Banach spaces such that $\mathcal{K}(X,Y)$ is complemented in $\mathcal{L}(X,Y)$ is not Borel, endowed with the Effros-Borel structure.

1. INTRODUCTION

Let X and Y be two infinite dimensional real Banach spaces. It has been a long standing question the following (see [18] and [3]).

Question 1.1. *Are the following properties equivalent?*

(a) *There exists a projection from the the space $\mathcal{L}(X,Y)$ of continuous linear operators onto the space $\mathcal{K}(X,Y)$ of compact linear operators;*
(b) $\mathcal{L}(X,Y) = \mathcal{K}(X,Y)$.

Many results have been found about this question. In [19], A.E. Tong and D.R. Wilken showed that if X has an unconditional basis, then the equivalence in the above question is true. Some years later, N.J. Kalton (see [13]) extended this result showing the following

Theorem 1.2. *Let X be a Banach space with an unconditional finite dimensional expansion of the identity. If Y is any infinite-dimensional Banach space the following are equivalent.*

(i) $\mathcal{K}(X,Y)$ *is complemented in* $\mathcal{L}(X,Y)$*;*
(ii) $\mathcal{L}(X,Y) = \mathcal{K}(X,Y)$*;*
(iii) $\mathcal{K}(X,Y)$ *contains no copy of* c_0*;*
(iv) $\mathcal{L}(X,Y)$ *contains no copy of* ℓ_∞.

In [10] and [11], G. Emmanuele proved that, without assumption of unconditional finite dimensional expansion of the identity, we still have some implication of the above theorem; i.e., if c_0 embeds in

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$\mathcal{K}(X,Y)$, then $\mathcal{K}(X,Y)$ is uncomplemented in $\mathcal{L}(X,Y)$. Moreover, he also showed that the classical Bourgain Delbaen's space $X_{a,b}$ (see [6]) is such that $\mathcal{K}(X_{a,b})$ contains no copy of c_0, despite $\mathcal{L}(X_{a,b}) \neq \mathcal{K}(X_{a,b})$.

Recently, S.A. Argyros and R.G. Haydon [2], in a truly spectacular way, have solved the Question 1.1 above. Indeed, using a mixed Tsirelson trick, they constructed a space $\mathfrak{X}_K$, in the wake of Bourgain Delbaen's space (see [5, 6]), such that

$\mathcal{K}(\mathfrak{X}_K)$ contains no copy of c_0;
$\mathcal{L}(\mathfrak{X}_K) = \mathcal{K}(\mathfrak{X}_K) \oplus \mathbb{R}I$,

where I denotes the identity map. In particular $\mathcal{K}(\mathfrak{X}_K)$ is non-trivially complemented in $\mathcal{L}(\mathfrak{X}_K)$.

See also the other interesting paper [12], where the authors extend the Argyros-Haydon construction in terms of totally incomparable spaces.

In what follows, we want to study the descriptive set nature of such spaces: the family of separable Banach spaces, endowed with the Effros-Borel structure, such that $\mathcal{K}(X)$ is non-trivially complemented in $\mathcal{L}(X)$. In particular we are interested to study the following

Question 1.3. *Let $\mathcal{A}$ be the family of all couple of separable Banach spaces (X,Y) such that $\mathcal{K}(X,Y)$ is complemented in $\mathcal{L}(X,Y)$. Is $\mathcal{A}$ Borel?*

As standard notation, we shall consider $\mathcal{L}(X,Y)$ the space of all bounded linear operator between the Banach spaces X and Y, endowed by the classical norm

$$\|T\| = \sup_{\|x\|\leq 1} \|Tx\|_Y.$$

We shall denote by $\mathcal{K}(X,Y)$ the closed subspace of $\mathcal{L}(X,Y)$ of all compact operators. In case $X=Y$ briefly $\mathcal{L}(X)$ and $\mathcal{K}(X)$ will stand for $\mathcal{L}(X,X)$ and $\mathcal{K}(X,X)$ respectively. We refer the reader any classical functional analysis 's book for any notation (i.e., see [1, 8, 16]).

Let us recall the following

Definition 1.4 ([14])**.** Let $1 \leq p < \infty$. A separable Banach space X is said to have *property* (m_p) if

$$\limsup_{n\to\infty} \|x + x_n\|^p = \|x\|^p + \limsup_{n\to\infty} \|x_n\|^p$$

whenever $x_n \to 0$ weakly.

Such a property has been intensively studied in [14], where it was proved that a Banach space X has property (m_p) if and only if X is almost isometric to a subspace of some ℓ_p-sum of finite-dimensional spaces.

2. Preliminaries and Notation

Let X be a separable Banach space. We endow the set $\mathcal{F}(X)$ of all closed subsets of X with the *Effros-Borel* structure, i.e. the structure generated by the family

$$\{\{F \in \mathcal{F}(X) \ : \ F \cap O \neq \emptyset\} \ : \ O \text{ is an open subset of } X\}.$$

We denote by $\mathcal{SB}(X)$ the subset of $\mathcal{F}(X)$ consisting of all linear closed subspaces of X endowed with the relative Effros-Borel σ-algebra. If X is $C(2^\omega)$ (where $2^\omega = \{0,1\}^\omega$ is a compact Polish space endowed with the product topology), we denote briefly $\mathcal{SB}(X)$ by $\mathcal{SB}$. It is well known that, if X is a Polish space, then $\mathcal{F}(X)$ with the Effros Borel structure is a standard Borel space. We refer the reader to the recent book [9].

We denote by $\omega = \{0, 1, \ldots\}$ the first infinite ordinal, and let $\omega^{<\omega}$ be the tree of all finite sequences in ω. Let $\mathcal{T}$ be the set of all trees on ω. If $s = (s(0), ..., s(n-1))$ is a sequence of ω, we denote its length n by $|s|$. In particular the empty sequence $\emptyset$ has length 0.

For $s = (s(0), ..., s(n-1)), t = (t(0), ..., t(k-1))$ the concatenation $s \frown t$ is defined by

$$s \frown t = (s(0), ..., s(n-1), t(0), ..., t(k-1)).$$

For a tree θ, a *branch* through θ is an $\varepsilon \in \omega^\omega$ such that for all $n \in \omega$,

$$\varepsilon|n = (\varepsilon(0), ..., \varepsilon(n-1)) \in \theta.$$

We denote by

$$[\theta] = \{\varepsilon \in \omega^\omega \ : \ \varepsilon \text{ is a branch through } \theta\}$$

the *body* of θ.

We call θ *well founded* if $[\theta] = \emptyset$, i.e. θ has no branches. Otherwise, we will call θ *ill founded*. We will denote by $\mathcal{WF}$ (resp. $\mathcal{IF}$) the set of well-founded trees (resp. ill founded trees) on ω.

For a tree $\theta \in \mathcal{T}$, roughly speaking the high of θ (denoted by $ht(\theta)$) is the supremum of the lengths of its elements (see [15] for the definition).

We refer the reader the book [15] for all notion and notation in Descriptive set theory.

Let us recall the constructive space of [17, Theorem 1] with normalized unconditional basis which is universal for all spaces with unconditional basis (some time called Pelczynski's space $\mathcal{U}$).

Theorem 2.1. *There exists a space $\mathcal{U}$ with a normalized unconditional basis $(u_n)_n$ such that for every seminormalized unconditional basic sequence $(x_n)_n$ in a Banach space X there exists $L = \{l_0 < l_1 < \cdots\} \in [\omega]$ such that $(x_n)_n$ is equivalent to $(u_{l_n})_n$ and the natural projection P_L onto $\overline{span}\{u_n \ : \ n \in L\}$ has norm one. Moreover, if U' is another space with the above properties, then U' is isomorphic to $\mathcal{U}$.*

3. Proof of the main result

For $s \in \omega^{<\omega}$, we denote by $\chi_s : \omega^{<\omega} \longrightarrow \{0,1\}$ the characteristic function of $\{s\}$. For a tree $\theta \in \mathcal{T}$, let $U_p(\theta)$ $(1 < p < \infty)$ be the completion of the $span\{\chi_s \ : \ s \in \theta\}$ under the norm

$$\|y\|_p = \sup \left[\sum_{j=0}^{k} \left\| \sum_{s \in I_j} y(s) \ u_{|s|} \right\|_{\mathcal{U}}^{p} \right]^{\frac{1}{p}}$$

where the supremum is taken over $k \in \omega$ and over all admissible choice of intervals $\{I_j \ : \ 0 \leq j \leq k\}$ (an *admissible choice of intervals* is a finite set $\{I_j \ : \ 0 \leq j \leq k\}$ of intervals of θ such that every branch of θ meets at most one of these intervals.).

Both of the below Lemma's are essentially included in [4].

Lemma 3.1. *For any θ tree on ω, the sequence $\{\chi_{s_i} \ : \ s_i \in \theta\}$ determines an unconditional basis for $U_p(\theta)$.*

Proof. Let $(\lambda_i)_{i \in \omega}$ be a sequence in $\mathbb{R}$, I an interval of θ and $n, m \in \omega$. Let us denote by $c_{\underline{u}}$ the basis constant for the universal basis $\underline{u} = (u_n)_n$ of $\mathcal{U}$.

Let $\mathcal{K} : \omega \longrightarrow \omega^{<\omega}$ be an enumeration of $\omega^{<\omega}$ such that if $s \subsetneq t$ then $\overline{s} < \overline{t}$, where $\overline{s} = \mathcal{K}^{-1}(s)$.

For $s \in T$, $(\sum_{i=0}^{n} \lambda_i \chi_{s_i})(s)$ is equal to $\lambda_{\overline{s}}$ if $\overline{s} \leq n$, and 0 if not. Therefore

$$\begin{aligned} \| \sum_{s \in I} (\sum_{i=0}^{n} \lambda_i \chi_{s_i})(s) \ u_{|s|} \|_{\mathcal{U}} &= \| \sum_{\substack{s \in I \\ \overline{s} \leq n}} \lambda_{\overline{s}} u_{|s|} \|_{\mathcal{U}} \leq c_{\underline{u}} \ \| \sum_{\substack{s \in I \\ \overline{s} \leq n+m}} \lambda_{\overline{s}} u_{|s|} \|_{\mathcal{U}} \\ &= c_{\underline{u}} \ \| \sum_{s \in I} (\sum_{i=0}^{n+m} \lambda_i \chi_{s_i})(s) \ u_{|s|} \|_{\mathcal{U}} \end{aligned}$$

since for $s, t \in I$, then $t \supseteqq s$ if and only if $\overline{t} \geq \overline{s}$.

Let $\{I_j \ : \ 0 \leq j \leq k\}$ be an admissible choice of intervals. We have

$$\sum_{j=0}^{k} \| \sum_{s\in I_j} (\sum_{i=0}^{n} \lambda_i \chi_{s_i})(s)\ u_{|s|} \|_{\mathcal{U}}^p \leq c_{\underline{u}}^p \ \sum_{j=0}^{k} \| \sum_{s\in I} (\sum_{i=0}^{n+m} \lambda_i \chi_{s_i})(s)\ u_{|s|} \|_{\mathcal{U}}^p.$$

Thus $\| \sum_{i=0}^{n} \lambda_i \chi_{s_i} \|_p \leq c_{\underline{u}} \ \| \sum_{i=0}^{n+m} \lambda_i \chi_{s_i} \|_p$ and $\{\chi_{s_i} : i \in \omega\}$ is a basic sequence.

Using the unconditionality of $(u_n)_n$, the same argument as above shows that $\{\chi_{s_i} \ : \ s_i \in \theta\}$ is actually an unconditional basis for $U_p(\theta)$.

□

Lemma 3.2. *Let $(A_i)_{i\in\omega}$ be a sequence of subsets of θ such that every branch meets at most one of these subsets. Then the spaces*

$$U_p(\bigcup_{i\in\omega} A_i) \text{ and } (\bigoplus_{i\in\omega} U_p(A_i))_{\ell_p} \text{ are isometric}$$

Proof. Pick $y \in span \left\{\chi_s \ : \ s \in \bigcup_{i\in\omega} A_i\right\}$. We let $y_i = \sum_{s\in A_i} y(s)\chi_s$. Since the set $\{y_i : i \in \omega \text{ and } y_i \neq 0\}$ is finite, there is $m \in \omega$ such that $y = \sum_{i=0}^{m} y_i$. To finish the proof, it is enough to show the following

Claim $\|y\|_p^p = \sum_{i=0}^{m} \|y_i\|_p^p$.

Indeed, let $\{I_j \ : \ 0 \leq j \leq k\}$ be an admissible choice of intervals. We set, for $0 \leq j \leq k$ and $0 \leq i \leq m$, $I_j(y) = \sum_{s\in I_j} y(s)u_{|S|}$ and $M_i = \{j \in \omega : 0 \leq j \leq k, \ I_j \cap A_i \neq \emptyset\}$. The largest interval with ends in $I_j \cap A_i$ is denoted by J_j^i. For any $i \in \omega$, $\{J_j^i : j \in M_i\}$ is an admissible choice of intervals, thus

$$\sum_{j=0}^{k} \|I_j(y)\|^p = \sum_{i=0}^{m} \sum_{j\in M_i} \|J_j^i(y_i)\|^p \leq \sum_{i=0}^{m} \|y_i\|_p^p.$$

It follows by taking the supremum over admissible choices of intervals that

$$\|y\|_p^p \leq \sum_{i=0}^{m} \|y_i\|_p^p.$$

Now, for any $0 \leq i \leq m$, let $\{I_j^i : 0 \leq j \leq_i\}$ be an admissible choice of intervals. We denote by $\widetilde{I}_j^i$ the largest interval with ends in $I_j^i \cap A_i$. Then $\{\widetilde{I}_j^i \ : \ 0 \leq i \leq m, \ 0 \leq j \leq k_i\}$ is an admissible choice of intervals, because every branch of T meets at most one of the A_i's. For any i,

$$\sum_{j=0}^{k_i} \|I_j^i(y_i)\|^p = \sum_{j=0}^{k_i} \|\widetilde{I}_j^i(y_i)\|^p = \sum_{j=0}^{k_i} \|I_j^i(y)\|^p,$$

$$\sum_{i=0}^{m}\sum_{j=0}^{k_i}\|I_j^i(y_i)\|^p = \sum_{i=0}^{m}\sum_{j=0}^{k_i}\|\widetilde{I}_j^i(y)\|^p \le \|y\|_p^p$$

thus

$$\sum_{i=0}^{m}\|y_i\|_p^p \le \|y\|_p^p.$$

□

Theorem 3.3. *Let $\theta \in \mathcal{T}$, and let $1 < q < p < \infty$.*

(i) *If θ is ill founded, then $\mathcal{K}(U_p(\theta), U_q(\theta))$ is uncomplemented in $\mathcal{L}(U_p(\theta), U_q(\theta))$;*

(ii) *If θ is well founded, then $\mathcal{K}(U_p(\theta), U_q(\theta))$ is complemented in $\mathcal{L}(U_p(\theta), U_q(\theta))$.*

Proof. (i) We actually show that if θ is ill founded, then $U_p(\theta)$ is isomorphic to $\mathcal{U}$. Since both spaces $U_p(\theta)$ and $U_q(\theta)$ are isomorphic, we get that $\mathcal{K}(U_p(\theta), U_q(\theta)) \neq \mathcal{L}(U_p(\theta), U_q(\theta))$. Since $\mathcal{U}$ has an unconditional basis, the thesis follows by [19, Theorem 6].

Suppose θ is ill founded, and let $b \in [\theta]$ a branch of θ. Let

$$U_p(b) = U_p(\{s \in \theta \ : \ s \subseteq b\})$$

We show that actually, $U_p(b)$ is isomorphic to $\mathcal{U}$.

Indeed, it is enough to show that the elements $\{\chi_{b|j} \ : \ j \in \omega\}$ are equivalent to the basis of $\mathcal{U}$.

Note that, if $\lambda \in \ell_\infty$ then

$$\|\sum_{j=0}^{n}\lambda_j\chi_{b|j}\|_p = \sup\left\{\|\sum_{s\in I}(\sum_{j=0}^{n}\lambda_j\chi_{b|j})(s)\ u_{|s|}\| \ : \ I \text{ interval}, \ I \subseteq \{s : s \subsetneq b\}\right\}$$
$$= \sup\{\|\sum_{j=l}^{m}\lambda_j u_j\| \ : \ 0 \le l \le m \le n\}.$$

Thus

$$\|\sum_{j=0}^{n}\lambda_j u_j\|_{\mathcal{U}} \le \|\sum_{j=0}^{n}\lambda_j\chi_{b|j}\|_p \le 2c_{\underline{u}}\ \|\sum_{j=0}^{n}\lambda_j u_j\|_{\mathcal{U}},$$

where $c_{\underline{u}}$ is the unconditional basis constant of the basis of $\mathcal{U}$.

Thus $U_p(b)$ is isomorphic to $\mathcal{U}$.

Let $y = \sum_{i\in\omega} y(s_i)\chi_{s_i}$ be an element of $U_p(\theta)$. We have

$$\|\sum_{\substack{i\in\omega\\ s_i\in b}} y(s_i)\chi_{s_i}\|_p = \sup\left\{\|\sum_{s\in I} y(s)\ u_{|s|}\| \ :\ I \text{ interval},\ I\subseteq\{s : s \subsetneqq b\}\right\}$$
$$\leq \|y\|_p$$

That means $U_p(b) \cong \mathcal{U}$ is complemented in $U_p(\theta)$. By properties of $\mathcal{U}$, we get that $U_p(\theta) \cong \mathcal{U}$.

(ii) Suppose that θ is well founded. Since $U_p(\theta)$ has an unconditional basis, by [19, Theorem 6], it is equivalent to show that

$$\mathcal{K}(U_p(\theta), U_q(\theta)) = \mathcal{L}(U_p(\theta), U_q(\theta)).$$

For $s \in T$ and $i \in \omega$, we define

$$s \frown \theta = \{s \frown t : t \in \theta\}, \qquad \theta_i = \{t \in T : (i) \frown t \in \theta\}.$$

Since $U_p(\theta) = U_p(\emptyset \frown \theta)$, to prove the theorem, it is enough to show the following

Claim If θ is well founded, then for any $s \in T$,

$$\mathcal{K}(U_p(s\frown\theta), U_q(s\frown\theta)) = \mathcal{L}(U_p(s\frown\theta), U_q(s\frown\theta)).$$

Since θ is well founded and since the map $ht : \mathcal{WF} \longrightarrow \omega_1$ is a Π^1_1-rank on $\mathcal{WF}$ (see [15]), we will show the Claim using transfinite induction on $ht(\theta)$.

We assume that for every tree $\tau \in \mathcal{T}$ such that $ht(\tau) < \alpha < \omega_1$,

$$\mathcal{K}(U_p(s\frown\tau), U_q(s\frown\tau)) = \mathcal{L}(U_p(s\frown\tau), U_q(s\frown\tau)).$$

for any $s \in T$.

Let θ such that $ht(\theta) = \alpha$, and for $s \in T$ let

$$N_s = \{i \in \omega : s \frown (i) \in \theta\}.$$

We let $A_i = s \frown (i) \frown \theta_i$ for $i \in N_s$, so that

$$\cup_{i\in N_s} A_i = s \frown (\theta \setminus \{s\})$$

and every branch of T meets at most one of the A_i's. If $i \in N_s$ then $ht(A_i) < \alpha$, thus

$$\mathcal{K}(U_p(A_i), U_q(A_i)) = \mathcal{L}(U_p(A_i), U_q(A_i)).$$

By Lemma 3.2, we have

$$U_r(s \frown (\theta\setminus\{s\})) = U_r(\bigcup_{i\in N_s} A_i) = (\bigoplus_{i\in N_s} U_r(A_i))_{\ell_r},$$

for $r = p, q$ respectively.

Since $\{\chi_{s_j}:\ j\in\omega,\ s_j\in s\frown\theta\}$ is a basis of $U_r(s\frown\theta)$ with the first element χ_s and the other element generate $U_r(s\frown(\theta\setminus\{s\}))$. Then, we have that $U_r(s\frown\theta)\cong\mathbb{R}\times U_r(s\frown(\theta\setminus\{s\}))$. Thus the theorem will be complete once we prove the next two Lemma's. □

Lemma 3.4. *Let $1<p<\infty$. For every $\theta\in\mathcal{WF}$, $U_p(\theta)$ is reflexive and it has property (m_p).*

Proof. Since θ is well-founded one can use transfinite induction on $ht(\theta)$. It is clear when $ht(\theta)=1$. Suppose the lemma holds for all trees with highs less than $ht(\theta)$. As before, we can write

$$U_p(\theta)=(\bigoplus_{n\in\omega}U_p(A_n))_{\ell_p},$$

with $ht(A_n)<ht(\theta)$. By induction, since $U_p(A_n)$ has (m_p), whenever we fix x and a weakly null sequence $(w_n)_n$ in $U_p(\theta)$ we get

$$\begin{aligned}\limsup_{n\to\infty}\|x+w_n\|^p_{U_p(\theta)}&=\limsup_{n\to\infty}\sum_{i\in\omega}\|x^i+w^i_n\|^p_{U_p(A_i)}\\&=\sum_{i\in\omega}\limsup_{n\to\infty}\|x^i+w^i_n\|^p_{U_p(A_i)}\\&=\sum_{i\in\omega}\|x^i\|^p_{U_p(A_i)}+\limsup_{n\to\infty}\sum_{i\in\omega}\|w^i_n\|^p_{U_p(A_i)}\\&=\|x\|^p_{U_p(\theta)}+\limsup_{n\to\infty}\|w_n\|^p_{U_p(\theta)}.\end{aligned}$$

The reflexivity of $U_p(\theta)$ follows by standard argument. □

The following Lemma slightly extends a classical Pitt's compactness theorem.

Lemma 3.5. *Let $1\le q<p<\infty$ and let $(X_n)_n$ and $(Y_n)_n$ two sequences of Banach spaces such that*

- *X_n is reflexive and it has property (m_p), for each $n\in\mathbb{N}$,*
- *Y_n has property (m_q), for each $n\in\mathbb{N}$.*

Then

$$\mathcal{K}\left((\bigoplus_n X_n)_{\ell_p},(\bigoplus_n Y_n)_{\ell_q}\right)=\mathcal{L}\left((\bigoplus_n X_n)_{\ell_p},(\bigoplus_n Y_n)_{\ell_q}\right)$$

Proof. The proof is similar to what of [7]. We give a sketch for sake of completeness.

Let

$$T:(\bigoplus_n X_n)_{\ell_p}\longrightarrow(\bigoplus_n Y_n)_{\ell_q}$$

be a norm one operator. Since $(\bigoplus_n X_n)_{\ell_p}$ is reflexive, any bounded sequence has a weak convergent subsequence. Thus, it is enough to show that T is weak-norm continuous.

Let $(h_n) \subseteq (\bigoplus_n X_n)_{\ell_p}$ be a weakly null sequence.

By hypothesis, since $(\bigoplus_n Z_n)_{\ell_r}$ has the property(m_r), where $Z_n = X_n$ (resp. $Z_n = Y_n$) if $r = p$ (resp. $r = q$), for every $x \in (\bigoplus_n Z_n)_{\ell_r}$ and every weakly null sequence $(w_n)_n$ in $(\bigoplus_n Z_n)_{\ell_r}$,

$$\limsup_{n\to\infty} \|x + w_n\|^r = \|x\|^r + \limsup_{n\to\infty} \|w_n\|^r. \tag{3.1}$$

For every $\varepsilon > 0$, let x_ε of norm one such that

$$1 - \varepsilon \leq \|T(x_\varepsilon)\| \leq 1.$$

For all $n \in \omega$ and $t > 0$

$$\|T(x_\varepsilon) + T(th_n)\| \leq \|x_\varepsilon + th_n\|. \tag{3.2}$$

Now, applying (3.1) to the left hand side of (3.2) inequality for $r = q$ and to the right hand side for $r = p$ we get

$$\limsup_{n\to\infty} \|T(h_n)\|^q \leq \frac{1}{t^q}[(1 + t^p M^p)^{\frac{q}{p}} - (1 - \varepsilon)^q],$$

where $M > 0$ is an upper bound for $(\|h_n\|)_n$.

Taking $t = \varepsilon^{\frac{1}{p}}$, we get

$$\limsup_{n\to\infty} \|T(h_n)\|^q \leq \frac{1}{\varepsilon^{\frac{q}{p}}}[1 + \frac{q}{p} M^p \varepsilon - (1 - q\varepsilon) + o(\varepsilon)].$$

Letting $\varepsilon \to 0$ we get that $(T(h_n))_n$ norm converges to zero. $\square$

Remark 3.6. Notice that the above lemma extends Pitt compactness's theorem since we have spaces with property (m_p) which are not isomorphic to ℓ_p. For example, any space with Schur property has property (m_p), for any $1 \leq p < \infty$. Then, in the above ℓ_q-sum, we can mix different kind of spaces.

Theorem 3.7. *For $1 < q < p < \infty$, the map $\varphi_{p,q} : \mathcal{T} \longrightarrow \mathcal{SB} \times \mathcal{SB}$ defined by*

$$\varphi_{p,q}(\theta) = U_p(\theta) \times U_q(\theta)$$

is Borel.

Proof. It is enough to show that the map

$$\theta \longmapsto U_p(\theta)$$

is Borel.

Let O be open subsets of $C(2^\omega)$. It is enough to show that $\Omega = \{\theta \in \mathcal{T} : U_p(\theta) \cap O \neq \emptyset\}$ is Borel.

Since $\{\chi_{s_i} : i \in \omega, s_i \in \theta\}$ defines a basis of $U_p(\theta)$, we have

$$U_p(\theta) \cap O \neq \emptyset \Leftrightarrow \exists \lambda \in \mathbb{Q}^{<\omega} \text{ such that } \sum_{i=0}^{n} \lambda_i \chi_{s_i} \in O \text{ and if } \lambda_i \neq 0 \text{ then } s_i \in \theta.$$

Let $\Lambda = \{\lambda \in \mathbb{Q}^{<\omega} : \sum_{i=0}^{n} \lambda_i \chi_{s_i} \in O\}$. Then

$$\Omega = \bigcup_{\lambda \in \Lambda} \bigcap_{i \in supp(\lambda)} \{\theta \in \mathcal{T} : s_i \in \theta\}$$

thus Ω is Borel since $\{\theta \in \mathcal{T} : s_i \in \theta\}$ is an open and closed subset. □

Theorem 3.8. *The family $\mathcal{A}$ of all couple of separable Banach spaces (X, Y) such that*

$$\mathcal{K}(X, Y) \text{ is complemented in } \mathcal{L}(X, Y)$$

is not Borel in $\mathcal{SB} \times \mathcal{SB}$.

Proof. Suppose $\mathcal{A}$ is even analytic. For $1 < q < p < \infty$, let $\varphi_{p,q}$ be the map defined in Theorem 3.7. Then $\varphi_{p,q}^{-1}(\mathcal{A})$ is analytic containing $\mathcal{WF}$. Since $\mathcal{WF}$ is not analytic, there is some θ_0 in $\varphi_{p,q}^{-1}(\mathcal{A})$ which is ill founded. Therefore, by Theorem 3.3, $\varphi_{p,q}(\theta_0)$ doesn't lie in $\mathcal{A}$. A contradiction. □

We would like to finish this note with the following

Question 3.9. *Let $\mathcal{B}$ be the family of all separable Banach space X such that $\mathcal{K}(X)$ is complemented in $\mathcal{L}(X)$. Is it $\mathcal{B}$ Borel? Is it coanalytic?*

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