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Automatic continuity of orthogonality or disjointness preserving bijections

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Abstract Elements a and b of a non-commutative $L^p(M,\tau)$ space associated to a von Neumann algebra, M, equipped with a normal semi-finite faithful trace τ, are called orthogonal if $l(a)l(b)=r(a)r(b)=0$, where $l(x)$ and $r(x)$ denote the left and right support projections of x. A linear map T from $L^p(M,\tau)$ to a normed space X is said to be orthogonality-to-p-orthogonality preserving if $\|T(a)+T(b)\|^p=\|a\|^p+\|b\|^p$ whenever a and b are orthogonal. In this paper, we prove that an orthogonality-to-p-orthogonality preserving linear bijection from $L^p(M,\tau)$ to a Banach space is automatically continuous if $1\le p<\infty$, and M is either an abelian von Neumann algebra or a discrete von Neumann algebras. Furthermore, any complete p-additive norm on such $L^p(M,\tau)$ is equivalent to the canonical norm.

Keywords Non-commutative L^p spaces · Banach lattices · Von Neumann algebras · Orthogonality preservers · p-orthogonality

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1 Introduction: orthogonality and p-orthogonality

Suppose a Banach space X is equipped with an algebraic structure (for instance, X is a C*-algebra). How are the metric and algebraic properties of X related? This paper approaches this problem by examining different notions of orthogonality.

We define the "metric" orthogonalities first. Suppose $1 \le p < \infty$. The elements x, y of a normed space X are said to be *p-orthogonal* (and we write $x \perp^p y$) if $\|x + y\|^p = \|x\|^p + \|y\|^p$, and *semi-$p$-orthogonal* ($x \perp_S^p y$) if $\|x + y\|^p \ge \|x\|^p + \|y\|^p$. It is customary to refer to 1-orthogonality as L-orthogonality, and to use the notation $\perp_L$. When $\|x + y\| = \max\{\|x\|, \|y\|\}$ (resp., $\|x + y\| \ge \max\{\|x\|, \|y\|\}$) we say that x and y are M-orthogonal (resp., semi-M-orthogonal), and we write $x \perp_M y$ (resp., $x \perp_{SM} y$). Some "natural" pairs of p-orthogonal elements are presented below.

On the algebraic side, suppose first A is a C*-algebra. The elements a, b in A are said to be (*algebraically*) *orthogonal* (written $a \perp b$) if $ab^* = a^*b = 0$. It is well known that orthogonal elements in A are (geometrically) M-orthogonal, while the converse is not, in general, true.

Now suppose M is a von Neumann algebra, equipped with a normal semi-finite faithful trace τ, and acting on a Hilbert space H. Following [21] (see also [12, 25]), we say that a closed densely defined (in general, unbounded) operator a is *affiliated with M* if it commutes with M' (the commutant of M). The *left* and *right support projections* of a (denoted by $l(a)$ and $r(a)$) are defined as the orthogonal projections onto the closure of the range of a, and the orthogonal complement of the kernel of a, respectively. Equivalently, $l(a)$ (resp., $r(a)$) is the smallest projection e (resp., f) with the property that $ea = a$ (resp., $af = a$). These projections belong to M. Following [26, Sect. 1], we say that two operators a and b, affiliated with M, are *orthogonal* ($a \perp b$) if $l(a)l(b) = r(a)r(b) = 0$.

We denote by $\mathfrak{S}$ the set of all linear combinations of positive elements $a \in M$, satisfying $\tau(a) < \infty$. If $|x|^p \in \mathfrak{S}$ (where $|x| = (x^*x)^{1/2}$), define $\|x\|_p = (\tau(|x|^p))^{1/p}$. The space $L^p(\tau)$ (also denoted by $L^p(M, \tau)$) is defined as the completion of $\mathfrak{S}$ in the norm $\|\cdot\|_p$. By [26, Fact 1.3], the elements $a, b \in L^p(\tau)$ are orthogonal if and only if $\|a+b\|_p^p = \|a-b\|_p^p = \|a\|_p^p + \|b\|_p^p$, that is, if and only if a is p-orthogonal to both b and $-b$.

As an example, consider $M = L^\infty(\mu)$, where μ is a σ-finite measure. Define a trace via $\tau(f) = \int f\,d\mu$. Then $L^p(\tau)$ is the classical space $L^p(\mu)$.

If τ is the canonical trace on $B(H)$, the construction described above produces the *Schatten space* $\mathcal{S}^p(H)$. To describe it more explicitly, consider a compact operator a in $B(H)$. Following the terminology of [30, Sect. 1.2], denote by $\mathbf{s}_1(a) \ge \mathbf{s}_2(a) \ge \cdots \ge 0$ the singular numbers or values of a. Then a can be written (essentially uniquely) in the form

$$a = \sum_{n=1}^{\infty} \mathbf{s}_n(a) h_n \otimes k_n,$$

where (h_n) and (k_n) are orthonormal systems in H. Here and below, we use the notation $h \otimes k$ to denote the rank one operator $\xi \mapsto (\xi|k)h$. Then $\mathcal{S}^p(H)$ is the space of all compact operators a with $\sum_i \mathbf{s}_i(a)^p < \infty$. The norm $\|\cdot\|_p$ is defined by

$$\|a\|_p = \left(\sum_i \mathbf{s}_i(a)^p\right)^{1/p} \quad \text{(cf. [30, Sect. 2]).}$$

A detailed description of Schatten spaces can be found, for instance, in [14, 30].

Now suppose A is either a C*-algebra, or a non-commutative L^p space. As in [22], we say that a norm $\|\cdot\|$ on A is a *(semi-)M-norm* if $\|a \pm b\| = \max\{\|a\|, \|b\|\}$ (resp., $\|a \pm b\| \geq \max\{\|a\|, \|b\|\}$) whenever $a \perp b$ in A. When $a \perp b$ in A implies $\|a \pm b\|^p = \|a\|^p + \|b\|^p$, we shall say that $\|\cdot\|$ is a *p-norm* (or a *p-additive norm*) on A. As noted above, the canonical norm on a C*-algebra (resp., on a non-commutative L^p space) is an M-norm (resp., a p-norm). Due to the connections to the theory of L-ideals, 1-norms are sometimes referred to as L-norms.

In [22], M. Ramírez and the first two authors study the (isomorphic) uniqueness of a complete M-norm on a C*-algebra (we call a norm $\|\cdot\|$ on a vector space X *complete* if the pair $(X, \|\cdot\|)$ is complete as a normed space).

Conjecture 1.1 Every complete (semi-)M-norm on a C*-algebra A is equivalent to its original C*-norm.

In [22], this conjecture is proved in several cases—for instance, when A is a von Neumann algebra, or a compact C*-algebra. The general case still open.

This paper is devoted to a related question.

Conjecture 1.2 Every complete p-norm on a non-commutative L^p space is equivalent to the original norm of that space.

This conjecture can be thought of as an automatic continuity question. In functional analysis, the term "automatic continuity" refers to the situation when the continuity of a map T follows from a different (ostensibly weaker) condition on T. For instance, in many cases, a homomorphism between Banach algebras is automatically continuous (see e.g. [10]). On the other hand, the maps preserving some metric properties, such as the Birkhoff-James orthogonality, are automatically continuous [6].

In this paper, we investigate the continuity of linear maps $T : A \to X$, for which the images of "disjoint" elements of A satisfy certain metric conditions. More specifically, suppose A is a C*-algebra or a non-commutative L^p space, and X is a normed space. We say that a linear map $T : A \to X$ is *orthogonality-to-p-orthogonality* (*O-p-O* for short) *preserving* if $T(x) \perp^p T(y)$ whenever $x \perp y$ (equivalently, $\|T(x) \pm T(y)\|^p = \|T(x)\|^p + \|T(y)\|^p$ whenever $x \perp y$).

Conjecture 1.3 Every orthogonality-to-p-orthogonality preserving linear bijection from a non-commutative L^p space to a Banach space is continuous.

It is easy to see that Conjectures 1.2 and 1.3 are equivalent. Henceforth, denote by $\|\cdot\|_p$ the canonical norm on $L^p(\tau)$. Suppose $\|\cdot\|'$ is a complete p-norm on $L^p(\tau)$. If Conjecture 1.3 holds, then the formal identity map from $(L^p(\tau), \|\cdot\|_p)$ to

$(L^p(\tau), \|\cdot\|')$ is continuous. By Banach Isomorphism Principle, its inverse is also continuous. Thus, the norms $\|\cdot\|_p$ and $\|\cdot\|'$ are equivalent, and Conjecture 1.2 is true. Conversely, suppose X is a Banach space, and $T : L^p(\tau) \to X$ is a O-p-O preserving linear bijection. Then $\|\cdot\|' = \|T(\cdot)\|$ is a complete p-norm on $L^p(\tau)$. If Conjecture 1.2 is true, then the norms $\|\cdot\|_p$ and $\|\cdot\|'$ are equivalent, hence T is continuous.

In this paper, we prove Conjecture 1.2 for non-commutative L^p spaces arising from commutative von Neumann algebras (Proposition 3.3), and from discrete von Neumann algebras (Theorem 4.1).

We also consider linear maps on Banach lattices. Recall that elements x and y in a Banach lattice E are *disjoint* if $|x| \wedge |y| = 0$. As noted above, $x, y \in L^p(\mu)$ $(1 \leq p < \infty)$ are disjoint if and only if they are p-orthogonal. We say that a map T from a Banach lattice E to a normed space X is called *disjointness to semi-M-orthogonality preserving* (*DSMO preserving* for short) if $T(x) \perp_{SM} T(y)$ whenever x and y are disjoint. The class of disjointness to p-orthogonality preserving maps is defined similarly. Theorem 3.1 shows that any DSMO preserving bijection from an order continuous Banach lattice E to a Banach space is automatically continuous. Theorem 3.2 describes the general form of such bijections between function spaces, satisfying certain conditions.

The notion of orthogonality also makes sense in the predual of a general (not necessarily tracial) von Neumann algebra N. Following [31, Page 140], define the *left support projection* of $\phi \in N_*$ (denoted by $l(\phi)$) as the projection $l \in N$ with the property that $\overline{\phi N} = l N_*$. Note that $\overline{\phi N}$ is a right invariant subspace of N_*, hence such an l exists. One can see that $l(\phi)$ is the smallest projection $e \in N$ satisfying $e\phi = \phi$. Therefore, in the tracial case, we obtain the same left support projection. The *right support projection* is defined similarly.

As before, we say that the elements ϕ and ψ of N_* are *orthogonal* ($\phi \perp \psi$) if $l(\phi)l(\psi) = r(\phi)r(\psi) = 0$. By [11, Theorem 5.4] (where a more general result is established) or [24, Lemma 2.1], ϕ and ψ are orthogonal if and only if $\|\phi + \psi\| = \|\phi - \psi\| = \|\phi\| + \|\psi\|$. We can similarly define the notion of *orthogonality-to-p-orthogonality* (O-p-O for short) *preserving* linear mapping from the predual of a von Neumann algebra to a Banach space and the concepts of M-norm and p-additive norm on the predual of a von Neumann algebra. In this general context we can consider the following conjecture:

Conjecture 1.4 Every complete 1-norm on the predual M_* of a von Neumann algebra M is equivalent to the original norm of M_*. Equivalently, every orthogonality-to-1-orthogonality preserving linear bijection from M_* to a Banach space is continuous.

In Sect. 2 we collect a variety of technical results, to be used throughout the paper. There, we show that the original norm on $L^p(M, \tau)$ is not equivalent to any q-norm, for $q \neq p$, unless M is finite dimensional (Proposition 2.5). Similar results are obtained for complete q-norms on C*-algebras (Proposition 2.4) and for complete M-norms on the predual of a von Neumann algebra (Proposition 2.6). We investigate the automatic continuity of DSMO and O-p-O preserving bijections on Banach lattices and von Neumann algebras in Sects. 3 and 4, respectively. Section 5 is devoted to describing orthogonality preserving maps for special co-domains. In particular, we characterize these type of maps on $\mathcal{S}^p(H)$ (Theorem 5.1), and on non-commutative L^p spaces arising from discrete von Neumann algebras (Theorems 5.10 and 5.14, Remark 5.15). Conjecture 1.4 is proved for preduals of commutative von Neumann algebras (Theorem 3.11, Corollary 3.13) and preduals of atomic or discrete von Neumann algebras (Theorem 5.10).

O-p-O preserving linear maps, and p-norms on non-commutative L^p spaces, have not hitherto studied. By contrast, the related topic of orthogonally preserving operators between C*-algebras and JB*-triples, and automatic continuity of M-norms on C*-algebras, have been widely investigated (see e.g. [5, 7–9, 16, 22, 32]).

2 Preliminaries

We deal now with some technical results which are needed later. For a sequence of Banach spaces (Z_i), and $1 \leq p < \infty$, consider the projections P_k on $(\bigoplus_i Z_i)_{\ell_p}$, defined by $P_k(z_1, z_2, \ldots) = (0, \ldots, 0, z_k, z_{k+1}, \ldots)$. We shall say that two elements (x_i) and (y_i) in $(\bigoplus_i Z_i)_{\ell_p}$ have *disjoint supports* if $\|x_i\| \, \|y_i\| = 0$ for every i. Our next proposition is inspired by [22, Proposition 3.6]. The same proof given in the just quoted paper remains valid here.

Proposition 2.1 *Let $(Z_n)_{n\in\mathbb{N}}$ be a sequence of Banach spaces. Suppose $1 \leq p < \infty$, and T is a bijective linear map from $(\bigoplus_n Z_n)_{\ell_p}$ to a Banach space X such that $\|T(x) + T(y)\| \geq \max\{\|T(x)\|, \|T(y)\|\}$ whenever x and y have disjoint supports. Then there exists $k \in \mathbb{N}$ such that TP_k is bounded. In particular, if for each natural n, the mapping $T|_{Z_n}$ is continuous (for example when the spaces Z_n are finite dimensional), then T is bounded.*

Proof The arguments given in the proof of [22, Proposition 3.6] remains valid here line by line. We sketch the proof for the sake of completeness. Note first that there exists $k \in \mathbb{N}$ such that T is bounded on $P_k((\bigoplus_n^{\ell_p} Z_n)_{00}) = (\bigoplus_{n\geq k}^{\ell_p} Z_n)_{00}$, where

$$\left(\bigoplus_{n\geq 1}^{\ell_p} Z_n\right)_{00} = \left\{(z_n) \in \left(\bigoplus_{n\geq 1} Z_n\right)_{\ell_p} : \{n : z_n \neq 0\} \text{ is finite}\right\}.$$

Indeed, otherwise there exist positive integers $k_1 < k_2 < \cdots$, and vectors $x_i \in (\bigoplus_{n=k_i}^{k_{i+1}-1} Z_n)_{\ell_p}$, so that $\|x_i\| < 2^{-i}$, and $\|T(x_i)\| > 2^i$ $(i \in \mathbb{N})$. Consider $x = \sum_i x_i \in (\bigoplus_n Z_n)_{\ell_p}$. Then, for every i, x_i and $x - x_i$ have disjoint supports, hence $\|T(x)\| \geq \|T(x_i)\| > 2^i$, which is impossible.

By scaling, we can assume that, for some $k \in \mathbb{N}$, $\|T|_{P_k((\bigoplus_{n\geq 1}^{\ell_p} Z_n)_{00})}\| \leq 1$.

We shall show that $\|TP_k\| \leq 1$, or equivalently, $\|T(x)\| \leq \|x\|$, for every x in $P_k((\bigoplus_n Z_n)_{\ell_p})$. Let us fix $x \in P_k((\bigoplus_n Z_n)_{\ell_p})$ with $\|x\| \leq 1$. From now on, we denote $P_i^\perp = I - P_i$. The sequence $(TP_m^\perp(x))_{m\geq k}$ is Cauchy, hence it converges in norm to some $x_0 \in X$. Moreover, $\|x_0\| \leq \liminf \|TP_m^{\perp}x\| \leq 1$. By the surjectivity of T, there exists $y \in (\bigoplus_n Z_n)_{\ell_p}$ so that $\lim_m TP_m^{\perp}(x) = T(y)$, for some $y \in (\bigoplus_n Z_n)_{\ell_p}$. We shall prove that $y = x$. For $m \geq n \geq k$, write

$$P_m^{\perp}x - y = P_n(P_m^{\perp}x - y) + P_n^{\perp}(x - y).$$

The two summands in the right hand side are disjointly supported, so $\|TP_m^{\perp}(x) - T(y)\| \geq \|TP_n^{\perp}(x) - TP_n^{\perp}(y)\|$. Since $\lim_m(TP_m^{\perp}(x) - T(y)) = 0$, the injectivity of T implies that $P_n^{\perp}x = P_n^{\perp}y$ for every $n \geq k$, which gives $x = y$. □

If the spaces Z_i in the previous proposition are 1-dimensional, we obtain:

Corollary 2.2 *Every orthogonality to semi-M-orthogonality preserving linear bijection from ℓ_p to a Banach space is continuous.*

This result is generalized below (see Theorem 3.1).

The following result is a version of [22, Proposition 3.8], the same proof given in the just quoted result works here. The details of the proof are left for the reader.

Proposition 2.3 *Let $(Z_i)_{i\in I}$ be a family of Banach spaces. Suppose T is a bijective linear map from $(\bigoplus_i Z_i)_{\ell_p}$ to a Banach space X such that*

$$\|T(x) + T(y)\|^p = \|T(x)\|^p + \|T(y)\|^p,$$

whenever x and y have disjoint supports. Then for each i, $X_i = T(Z_i)$ is closed. Further,

$$\left\|\sum_{k=1}^{n} x_k\right\|^p = \sum_{k=1}^{n} \|x_k\|^p,$$

whenever $x_1 \in X_{i_1}, \ldots, x_n \in X_{i_n}$.

Proposition 2.4 *Any C^*-algebra admitting a continuous and complete q-norm $(1 \leq q < \infty)$ is finite dimensional.*

Proof Suppose that $\|\cdot\|$ is a continuous and complete q-norm on a C*-algebra A with $\dim(A) = \infty$. Denote the original C*-norm of A by $\|\cdot\|_\infty$. The identity mapping from $(A, \|\cdot\|_\infty)$ to $(A, \|\cdot\|)$ is a continuous linear bijection, hence, by Open Mapping Theorem, the norms $\|\cdot\|$ and $\|\cdot\|_\infty$ are equivalent. Thus, there exist positive constants m_1, m_2 such that

$$m_1\|\cdot\|_\infty \leq \|\cdot\| \leq m_2\|\cdot\|_\infty.$$

By [17, Exercise 4.6.13] we can find a sequence (a_n) of mutually orthogonal norm-one (positive) elements in A. The series $\sum_{n=1}^{\infty} n^{-1/q}a_n$ is $\|\cdot\|_\infty$-convergent in A (compare [9, Remark 7]). For each natural n, $a_1, 2^{-1/q}a_2, \ldots, n^{-1/q}a_N$, and

$\sum_{k=N+1}^{\infty} k^{-1/q} a_k$ are mutually orthogonal elements in A. It follows from the assumptions that

$$m_2^q \left\| \sum_{n=1}^{\infty} n^{-1/q} a_n \right\|_{\infty}^q \geq \left\| \sum_{n=1}^{\infty} n^{-1/q} \ a_n \right\|^q \geq \sum_{n=1}^{N} \frac{1}{n} \ \|a_n\|^q \geq m_1^q \sum_{n=1}^{N} \frac{1}{n}$$

for every natural N, which is impossible. □

In a similar fashion, one can prove:

Proposition 2.5 *Suppose $p \in [1, \infty)$, $q \in [1, \infty]$, M is an infinite dimensional von Neumann algebra with a faithful normal semi-finite trace τ, and $L^p(\tau)$ admits a continuous complete q-norm. Then $p = q$.*

Proof Denote the original norm of $L^p(\tau)$, and the q-norm, by $\| \cdot \|_p$ and $\| \cdot \|$, respectively. These norms must be equivalent—that is, there exists $m_1, m_2 \in (0, \infty)$ so that $m_1 \| \cdot \|_p \leq \| \cdot \| \leq m_2 \| \cdot \|_p$. Find a sequence of mutually orthogonal projections $r_i \in M$ $(i \in \mathbb{N})$, with finite trace. Let $\alpha_i = \tau(r_i)$, and $a_i = \alpha_i^{-1/p} r_i$. Then $\|a_i\|_p = 1$.

First rule out the possibility of $q < p$. Pick $\gamma \in (1/p, 1/q)$, and consider $a = \sum_{k=1}^{\infty} k^{-\gamma} a_k$ (this series converges in $\| \cdot \|_p$. As in Proposition 2.4, we conclude that, for every $N \in \mathbb{N}$,

$$\|a\|^q \geq \sum_{k=1}^{N} \|k^{-\gamma} a_k\|^q \geq m_1^q \sum_{k=1}^{N} k^{-\gamma q}.$$

This, however, is impossible, as $\sum_{k=1}^{\infty} k^{-\gamma q} = \infty$.

The possibility of $q > p$ is ruled out in a similar manner. Pick $\gamma \in (1/q, 1/p)$, and consider the sequence $b_N = \sum_{k=1}^{N} k^{-\gamma} a_k$. This sequence is Cauchy in $\| \cdot \|$. Indeed, for $M > N$,

$$\|b_M - b_N\|^q = \sum_{k=N+1}^{M} \|k^{-\gamma} a_k\|^q \leq m_2^q \sum_{k=N+1}^{M} k^{-\gamma q}.$$

The convergence of $\sum_k k^{-\gamma q}$ yields $\lim_{N,M\to\infty} \|b_N - b_M\| = 0$. As the norms $\| \cdot \|$ and $\| \cdot \|_p$ are equivalent, we must also have $\lim_{N,M\to\infty} \|b_N - b_M\|_p = 0$. However, $\|b_M - b_N\|_p^p = \sum_{k=N+1}^{M} k^{-\gamma p}$, and the divergence of $\sum_k k^{-\gamma p}$ leads to a contradiction. □

Our next result shows that the predual of an infinite dimensional von Neumann algebra does not admit a complete q-norm, unless $q = 1$.

Proposition 2.6 *Suppose $1 < q \leq \infty$, and there exists a continuous and complete q-norm on the predual M_* of a von Neumann algebra M. Then M is finite dimensional.*

Proof Suppose, for the sake of contradiction, that M is an infinite dimensional von Neumann algebra, and $\|\cdot\|$ is a continuous and complete q-norm on M_*. Let $\|\cdot\|_0$ and $\|\cdot\|_0^*$ denote the canonical (C*) norm on M, and the canonical norm on M_*, respectively. Arguing as in the proof of above propositions, we deduce the existence of two positive constants m_1, m_2 such that $m_1\|\cdot\|_0^* \le \|\cdot\| \le m_2\|\cdot\|_0^*$.

Since M is infinite-dimensional, we can find a sequence (p_n) of mutually orthogonal projections in M. For each natural n, we define a positive norm-one weak*-continuous functional $\phi_n : \mathbb{C}p_n \to \mathbb{C}$ by the assignment $\phi_n(\lambda\ p_n) := \lambda$. Observing that $\mathbb{C}p_n$ is a von Neumann subalgebra of M, it follows from [28, Proposition 1.24.5] that, for each natural n, there exists a positive norm-one weak*-continuous functional $\varphi_n \in M_*$ satisfying $\varphi_n|_{\mathbb{C}p_n} = \phi_n$. Furthermore, by [31, Lemma III.4.1], $p_n\varphi_n p_n = \varphi_n$ for every n. For $n \ne m$ we have

$$2 = \|\varphi_n\| + \|\varphi_m\| \ge \|\varphi_n - \varphi_m\| \ge \langle \varphi_n - \varphi_m, p_n - p_m\rangle = 2.$$

Therefore, by [31, Theorem III.4.2(ii)], the functionals (φ_n) are mutually orthogonal in M_*. Since $\|\cdot\|$ is an q-norm, the series $\sum_{n=1}^{\infty} \frac{1}{n}\,\varphi_n$ converges with respect to the norm $\|\cdot\|$, and hence with respect to $\|\cdot\|_0^*$, which gives the desired contradiction. □

3 Disjointness preserving maps on Banach lattices

In this section we investigate automatic continuity of maps on Banach lattices. It is known [2] that, if $T : E \to F$ is a linear bijection between two Banach lattices, such that both T and T^{-1} preserve disjointness, then T and T^{-1} are continuous. We consider DSMO preserving maps from a Banach lattice to a Banach space, partially generalizing the result quoted above (note that a disjointness preserving map between Banach lattices is DSMO preserving).

Recall that a Banach lattice E is called *order continuous* if, for any downward directed net $(x_\alpha)_{\alpha\in\mathcal{A}}$ with $\bigwedge_{\alpha\in\mathcal{A}} x_\alpha = 0$, we have $\lim_\alpha \|x\|_\alpha = 0$. We refer the reader to, for instance, [19, Sect. 1.a], [20, Sect. 2.4], or [29, Chap. II] for more information on such lattices. Note that any reflexive Banach lattice is order continuous, as is $L^1(\mu)$. On the other hand, $C(K)$ is not order continuous, unless K is a finite set. The main result of this section is the following:

Theorem 3.1 *Any linear DSMO preserving bijection from an order continuous Banach lattice to a Banach space is continuous.*

DSMO preserving bijections between certain Köthe function spaces must be weighted composition operators. Below, we deal with spaces of (equivalence classes of) functions on measure spaces (Ω, Σ, μ). Throughout, we assume that Ω is Polish, and μ is a σ-finite complete Borel measure. Recall that a measure μ on a Polish space Ω is called *complete* (or *standard*) *Borel* if any Borel set is measurable, any subset of a null set is measurable, and, for any measurable set S, there exists a Borel set S' satisfying $\mu(S \triangle S') = 0$ (here and throughout the section, $\triangle$ stands for the symmetric difference of sets). If all these conditions are satisfied, we say that our measure space is *appropriate*. Examples of appropriate measure spaces include the Lebesgue measure, as well as the counting measure on a countable set.

Now suppose (Ω, Σ, μ) is a complete Borel measure space, and $(E, \|\cdot\|)$ is a Banach space of equivalence classes (modulo equality μ-a.e.) of μ-measurable functions. E is called a *Köthe function space* if the following two conditions hold:

1. If $g \in E$, and $|f| \leq |g|$ μ-a.e., then $f \in E$, and $\|f\| \leq \|g\|$.
2. If $S \subset \Omega$ satisfies $\mu(S) < \infty$, then $\chi_S \in E$.

We refer the reader to [19, Sect. 1.b] for more information on the topic.

Theorem 3.2 *Suppose E_1 and E_2 are Köthe function spaces on appropriate measure spaces $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$, respectively, such that E_1 is order continuous, and μ_1 is finite. Then, for any disjointness preserving linear bijection $T : E_1 \to E_2$, there exists a measurable map $\phi : \Omega_1 \to \Omega_2$, such that $T(f)(t) = F(t) f(\phi(t))$ μ-almost everywhere (here, $F = T(\mathbf{1})$, where $\mathbf{1} = \chi_{\Omega_1}$).*

We can describe all complete p-norms on $L^p(\mu)$, in a manner similar to Kakutani's description of p-additive Banach lattices.

Proposition 3.3 *For any complete p-norm $\|\cdot\|$ on $L^p(\Omega, \mu)$ (where $1 \leq p < \infty$, and μ is σ-finite), there exists a function $\phi \in L^\infty(\Omega, \mu)$, such that $\phi^{-1} \in L^\infty(\Omega, \mu)$, and $\|f\| = \|\phi f\|_{L^p(\Omega,\mu)}$ for every $f \in L^p(\Omega, \mu)$. Conversely, any function ϕ such that both it and its inverse are essentially bounded gives rise to a complete p-norm on $L^p(\Omega, \mu)$.*

Proof Clearly, any function ϕ with the above properties produces a p-norm. To prove the converse, suppose $\|\cdot\|$ is another complete p-norm on $L^p(\mu)$. Denote the space $L^p(\mu)$, equipped with this norm, by X. The formal identity T from $L^p(\Omega, \mu)$ to X is bounded, by Theorem 3.1 (it is well known that $L^p(\mu)$ is order continuous). By Open Mapping Principle, T^{-1} is bounded. Thus, there exists $C \geq 1$ such that

$$C^{-1}\|f\|_{L^p(\Omega,\mu)} \leq \|T(f)\| \leq C\|f\|_{L^p(\Omega,\mu)},$$

for every $f \in L^p(\mu)$.

For any measurable $A \subseteq \Omega$, define $\nu(A) = \|T(\chi_A)\|^p$. The preservation of p-orthogonality ensures that ν is finitely additive. Furthermore, $C^{-p}\mu(A) \leq \nu(A) \leq C^p\mu(A)$, hence ν is countably additive. Finally, ν is absolutely continuous with respect to μ. By Radon-Nikodym Theorem, there exists a measurable function ψ, such that $C^{-p} \leq \psi \leq C^p$ almost everywhere, and $\nu(A) = \int_A \psi \, d\mu$. By the density of simple functions in $L^p(\Omega, \mu)$,

$$\|T(f)\|^p = \int \psi |f|^p \, d\mu = \|\phi f\|^p_{L^p(\Omega,\mu)},$$

where $\phi = \psi^{1/p}$. □

Theorem 3.2 can be viewed as a particular case of [1]. There it is shown that large classes of disjointness preserving mappings between Banach lattices can be represented as weighted composition operators, provided the lattices involved are represented as spaces of continuous extended real valued functions on extremally disconnected compact Hausdorff spaces. For Köthe function spaces, an appropriate representation can be obtained, if one follows [33, Chap. II]. Below we present a self-contained proof, for the benefit of the reader.

Proof of Theorem 3.2 By Theorem 3.1, we can assume that T is continuous. Note first that simple functions are dense in E_1. Indeed, it suffices to show that simple functions are dense in the positive cone. For $f \geq 0$, there exists a sequence of non-negative simple functions (f_n) so that $f_n \nearrow f$ everywhere. By the order continuity, $\lim_n \|f_n - f\| = 0$.

Thus, it suffices to prove the existence of ϕ so that

$$T(\chi_S)(t) = F(t)\chi_{\phi^{-1}(S)}(t)$$

almost everywhere, whenever $S \in \mathcal{B}(\Omega_1)$ (the family of all Borel subsets of Ω_1). To this end, find a Borel set $\Omega \subset \mathrm{supp}(F)$, such that $\mathrm{supp}(F)\triangle\Omega$ has measure 0. Clearly, it suffices to construct ϕ on Ω. On $\Omega_2\backslash\Omega$, we can define ϕ in an arbitrary fashion, since F vanishes there.

For $S \in \mathcal{B}(\Omega_1)$, set $\Phi(S) := \Omega \cap \mathrm{supp}(T(\chi_S))$. Note that Φ maps $\mathcal{B}(\Omega_1)$ into $\mathcal{M}/\mathcal{I}$, where $\mathcal{M}$ is the family of measurable subsets of Ω, and $\mathcal{I}$ is the σ-ideal of null sets. We show that Φ is a σ-homomorphism—that is, it preserves complements and countable unions.

For simplicity of notation, we identify a measurable set with its equivalence class in $\mathcal{M}/\mathcal{I}$ (that is, we write S instead of $[S]$). By definition $\Phi(\Omega_1) = \Omega$. As T is disjointness preserving, $\Phi(S) \cap \Phi(S^c)$ has measure 0. Furthermore, $T(\chi_S) = F \cdot \chi_{\Phi(S)}$. Indeed, $F = T(\chi_S + \chi_{S^c})$, hence

$$F \cdot \chi_{\Phi(S)} = \big(T(\chi_S) + T(\chi_{S^c})\big) \cdot \chi_{\Phi(S)} = T(\chi_S) \cdot \chi_{\Phi(S)}.$$

We claim that $\Phi(S^c) = \Omega\backslash\Phi(S)$. Indeed, as noted above, $\Phi(S^c)$ and $\Phi(S)$ are disjoint (up to a null set). On the other hand,

$$F = T(\mathbf{1}) = T(\chi_S) + T(\chi_{S^c}) = F\chi_{\Phi(S)} + F\chi_{\Phi(S^c)},$$

hence $\Phi(S) \cup \Phi(S^c) = \Omega$ (once again, up to a null set).

In the same fashion, one shows that $\Phi(S_1 \cup S_2) = \Phi(S_1) \cup \Phi(S_2)$ provided S_1 and S_2 are disjoint. Thus, Φ preserves finite unions. To tackle countable unions, consider a sequence $S_1, S_2, \ldots \in \mathcal{B}(\Omega_1)$. Let $G = \bigcup_n S_n$, and $\tilde{G} = \Phi(G)$. For $m \in \mathbb{N}$, let $G_m = \bigcup_{n=1}^m S_n$, and $\tilde{G}_m = \Phi(G_m)$. Clearly, $\tilde{G}_1 \subset \tilde{G}_2 \subset \cdots \subset \tilde{G}$. We have to show that set $H = \tilde{G}\backslash(\bigcup_m \tilde{G}_m)$ has measure 0.

Suppose, for the sake of contradiction, that $\mu(H) > 0$. For $\ell \in \mathbb{N}$ set $H_\ell = \{\omega \in H : |F(\omega)| > 1/\ell\}$. Then $\bigcup_\ell H_\ell = H$ (recall that $H \subset \Omega$, and $|F| > 0$ on Ω), hence there exists $\ell \in \mathbb{N}$ such that $\mu(H_\ell) > 0$. Then

$$\|F\chi_H\| \geq \|F\chi_{H_\ell}\| \geq \ell^{-1}\|\chi_{H_\ell}\| > 0.$$

On the other hand, $\chi_{G\backslash G_m} \searrow 0$, hence, by the order continuity of E_1, $\lim_m \|\chi_{G\backslash G_m}\| = 0$. The continuity of T implies $\lim_m \|T(\chi_{G\backslash G_m})\| = 0$. Furthermore, $\Phi(G\backslash G_m) = \tilde{G}\backslash\tilde{G}_m$, hence, for any $m \in \mathbb{N}$,

$$\|T(\chi_{G\backslash G_m})\| = \|F \cdot \chi_{\tilde{G}\backslash\tilde{G}_m}\| \geq \left\|\left(F \cdot \chi_{\tilde{G}\backslash\tilde{G}_m}\right)\chi_H\right\| = \|F\chi_H\| > 0,$$

leading to a contradiction.

Therefore, Φ is σ-homomorphism. By [18, Theorem 15.9], there exists a measurable $\phi : \Omega \to \Omega_1$, so that $\Phi(S) = \phi^{-1}(S)$, for any $S \in \mathcal{B}(\Omega_1)$. □

Remark 3.4 It is not clear to what extent the order continuity of E_1 is essential. By [13] and [16], any disjointness preserving bijection between $C(K)$ or $C_0(K)$ spaces is a weighted composition operator.

The proof of Theorem 3.1 is more involved. Recall that a subspace F of a Banach lattice E is called an *ideal* if $y \in F$ whenever $x \in F$, and $|y| \leq |x|$. An ideal F is a *band* if $\bigvee_{\alpha\in\mathcal{A}} x_\alpha \in F$ whenever $(x_\alpha)_{\alpha\in\mathcal{A}} \subset F$, and $\bigvee_{\alpha\in\mathcal{A}} x_\alpha$ exists in E.

Suppose E is a Banach lattice. We say that a family of non-trivial mutually disjoint (with respect to the lattice order of E) ideals $(E_\alpha)_{\alpha\in\mathcal{A}}$ forms a *convenient decomposition* of E if any $x \in E$ can has a unique representation as $x = \sum x_\alpha$, with $x_\alpha \in E_\alpha$, $\{\alpha : x_\alpha \neq 0\}$ is at most countable, and the series for x converges unconditionally, with $\|\sum_{\alpha\in\mathcal{A}'} x_\alpha\| \leq \|x\|$ for any set $\mathcal{A}' \subset \mathcal{A}$.

Lemma 3.5 *Suppose $(E_\alpha)_{\alpha\in\mathcal{A}}$ is a convenient decomposition of a Banach lattice E, X is a Banach space, and $T : E \to X$ is a DSMO preserving linear bijection. Then, for any α, $T(E_\alpha)$ is closed.*

Proof Suppose (e_n) is sequence in E_α, such that $(T(e_n))$ converges to $x \in X$. By the surjectivity of T, there exists $e \in E$ for which $T(e) = x = \lim_k T(e_k)$. Denote by P_α the canonical projection on E_α. More precisely, any $f \in E$ has a unique representation $f = \sum_{\beta\in\mathcal{A}} f_\beta$, with $f_\beta \in E_\beta$. We define $P_\alpha(f) = f_\alpha$. Let $P_\alpha^\perp = I - P_\alpha$. Clearly, $P_\alpha(f)$ is disjoint from $P_\alpha^\perp(f)$. We shall show that $P_\alpha^\perp(e) = 0$. To this end, note that, due to the DSMO preservation,

$$\|TP_\alpha^\perp(e)\| = \|TP_\alpha^\perp(e - e_k)\| \leq \|T(e) - T(e_k)\|,$$

for every k. Passing to the limit in the right hand side, we obtain $\|TP_\alpha^\perp(e)\| = 0$. To complete the proof, we invoke the injectivity of T. □

Lemma 3.6 *Suppose $(E_\alpha)_{\alpha\in\mathcal{A}}$ is a convenient decomposition of a Banach lattice E, X is a Banach space, and $T : E \to X$ is a DSMO preserving linear bijection. Then there exists $\mathcal{A}' \subset \mathcal{A}$, such that $\mathcal{A}\backslash\mathcal{A}'$ is finite, $T|_{E_\alpha}$ is continuous for every α in $\mathcal{A}'$ and $\sup_{\alpha\in\mathcal{A}'} \|T|_{E_\alpha}\|$ is finite.*

Proof Suppose otherwise. Then there exists a sequence (α_n) of distinct elements of $\mathcal{A}$, and $e_n \in E_{\alpha_n}$, such that $\|e_n\| < 2^{-n}$, yet $\|T(e_n)\| > 2^n$ for every n. Let $e = \sum_n e_n$. Then, for every j, $\|T(e)\| \geq \|TP_{\alpha_j}(e)\| \geq 2^j$, a contradiction. □

Lemma 3.7 *Suppose $(E_\alpha)_{\alpha\in\mathcal{A}}$ is a convenient decomposition of a Banach lattice E, X is a Banach space, and $T: E \to X$ is a DSMO preserving linear bijection. Suppose, furthermore, that $\mathcal{A}'$ is a subset of $\mathcal{A}$, such that $T|_{E_\alpha}$ is bounded for any $\alpha \in \mathcal{A}'$. Then*

$$\sup_{F\subset\mathcal{A}',\,|F|<\infty} \|T|_{\mathrm{span}[E_\alpha:\alpha\in F]}\|$$

is finite.

Proof Suppose otherwise. Then there exists a sequence (F_n) of finite subsets of $\mathcal{A}'$, such that $\|T|_{\mathrm{span}[E_\alpha:\alpha\in F_1]}\| > 5$, and

$$\|T|_{\mathrm{span}[E_\alpha:\alpha\in F_n]}\| > 10\|T|_{\mathrm{span}[E_\alpha:\alpha\in F_{n-1}]}\|,$$

for $n > 1$. Let $G_1 = F_1$, and $G_n = F_n\backslash(\bigcup_{j<n} F_j)$. Then the sets (G_n) are finite and disjoint. By induction, we show that

$$\|T|_{\mathrm{span}[E_\alpha:\alpha\in G_n]}\| > 5^n \tag{3.1}$$

for any $n \in \mathbb{N}$. The base ($n = 1$) is clear from the definition. To deal with the inductive step, suppose (3.1) holds for $n = k - 1$, and prove it for $n = k$. Pick a norm 1 $f \in \mathrm{span}[E_\alpha : \alpha \in F_k]$, so that $\|T(f)\| > 10\|T|_{\mathrm{span}[E_\alpha:\alpha\in F_{k-1}]}\| > 5\cdot 10^{k-1}$. Write f can be represented, in a unique way, as $f = \sum_{j=1}^k g_j$, with $g_j \in \mathrm{span}[E_\alpha : \alpha \in G_j]$. Then $\|g_j\| \leq 1$ for every j. Moreover, $G_j \subset F_j$, hence, for $j < k$,

$$\|T(g_j)\| \leq \|T|_{\mathrm{span}[E_\alpha:\alpha\in F_j]}\| \leq 5^{-(k-j-1)}\|T|_{\mathrm{span}[E_\alpha:\alpha\in F_{k-1}]}\| < \frac{5^{-(k-j-1)}}{10}\|T(f)\|.$$

Then

$$\|T(g_k)\| \geq \|Tf\| - \sum_{j=1}^{k-1}\|T(g_j)\| > \left(1 - \frac{1}{10}\sum_{j=1}^{k-1}5^{-(k-j-1)}\right)\|Tf\|$$
$$> \frac{1}{2}\cdot 5\cdot 10^{k-1} \geq 5^k,$$

yielding (3.1) holds for $n = k$.

Thus, we can construct a sequence (e_n) such that, for any n, $e_n \in \mathrm{span}[E_\alpha : \alpha \in G_n]$, $\|e_n\| < 2^{-n}$, and $\|Te_n\| > 2^n$.

To complete the proof, suppose F is a subset of $\mathcal{A}$, not necessarily finite. Any $f \in E$ has a unique representation $f = \sum_{\beta\in\mathcal{A}} f_\beta$, with $f_\beta \in E_\beta$. We define $P_F(f) = \sum_{\beta\in F} f_\beta$ (the right hand side makes sense, due to the unconditional convergence of the series $\sum_\beta f_\beta$). Set $P_F^\perp = I - P_F = P_{\mathcal{A}\backslash F}$. Clearly, $P_F(f)$ and $P_F^\perp(f)$ have disjoint supports.

In our case, taking $e = \sum_{n=1}^\infty e_n$, we have $\|T(e)\| \geq \|TP_{G_j}(e)\| \geq 2^j$, for every j, yielding a contradiction. □

Lemma 3.8 *In the notation of Lemma 3.7, T is bounded on*

$$\mathrm{span}[E_\alpha : \alpha \in \mathcal{A}'].$$

Proof Clearly, $E_0 = \mathrm{span}[E_\alpha : \alpha \in \mathcal{A}']$ is an ideal in E. Together with $(E_\alpha)_{\alpha \in \mathcal{A} \setminus \mathcal{A}'}$, it forms a convenient decomposition of E. Thus, $T(E_0)$ is closed.

Denote by $\mathcal{F}$ the set of all finite subsets of $\mathcal{A}'$. For any $F \subset \mathcal{A}'$, we denote by E_F the closed linear span of the ideals E_α, for $\alpha \in F$. Applying Lemma 3.7, and scaling T if necessary, we can assume that $\|T|_{E_F}\| < 1$ for any $F \in \mathcal{F}$. We shall show that $\|Te\| \leq \|e\|$ for any $e \in E_0$. To this end, we view $\mathcal{F}$ as a net (ordered by inclusion). Then $(P_F(e))_{F \in \mathcal{F}}$ is a Cauchy net, hence so is $(TP_F(e))_{F \in \mathcal{F}}$. As $T(E_0)$ is closed, there exists $f \in E_0$ such that $\lim_F \|T(f) - TP_F(e)\| = 0$. More explicitly, for any $\varepsilon > 0$ there exists a set $G \in \mathcal{F}$ so that $\|T(f) - TP_F(e)\| < \varepsilon$, whenever $G \subset F$.

It suffices to show that $f = e$. Once this is accomplished, we are done, since $\|T(f)\| = \lim_F \|TP_F(e)\| \leq \sup_F \|P_F(e)\| \leq \|e\|$. Furthermore, to show $f = e$, it suffices to prove that $P_F(f) = P_F(e)$ for any $F \in \mathcal{F}$. To this end, fix $\varepsilon > 0$, and pick a set $G \in \mathcal{F}$, such that $F \subset G$, and $\|T(f - P_G(e))\| < \varepsilon$. Since $P_F(f - P_G(e)) = P_F(f) - P_F(e)$ and $P_F^\perp(f - P_G(e))$ are disjointly supported, we have $\|T(P_F(f) - P_F(e))\| \leq \|T(f - P_G(e))\| < \varepsilon$. As ε is arbitrary, we conclude that $\|T(P_F(f) - P_F(e))\| = 0$, and complete the proof using the injectivity of T. ☐

Next we prove Theorem 3.1 in a particular setting.

Lemma 3.9 *Suppose E is an order continuous Köthe function space on (Ω, Σ, μ), where μ is a σ-finite measure. Then any DSMO preserving linear bijection T from E to a Banach space X is continuous.*

Proof For any $S \in \Sigma$, denote by E_S the set of all $f \in E$, vanishing outside of S. The "canonical" projection, P_S, of E onto E_S is defined by setting $P_S(f) := f\chi_S$. Note that, if the sets $S_\alpha \in \Sigma$ $(\alpha \in \mathcal{A})$ are disjoint, then at most countably many of them have positive measure. Furthermore, if $\Omega = \bigcup_{\alpha \in \mathcal{A}} S_\alpha$, then the ideals (in fact, bands) E_{S_α} form a convenient decomposition of E. For convenience of notation, we denote by Σ^+ the set of all $S \in \Sigma$ with $\mu(S) > 0$.

First show that

$$\forall S \in \Sigma^+, \exists S' \subseteq S \text{ such that } \mu(S') > 0 \text{ and } T \text{ is bounded on } E_{S'}. \tag{3.2}$$

Indeed, if S has an atom, we can take S' to be this atom (then $E_{S'}$ is 1-dimensional). Otherwise, write S as an infinite disjoint union of sets $S_k \in \Sigma^+$. By Lemma 3.6, T is bounded on E_{S_k}, for all but finitely many values of k.

Next observe that, if the sets $S_k \in \Sigma^+$ $(k \in \mathbb{N})$ are such that T is bounded on E_{S_k} for every k, then T is bounded on $E_{\bigcup_k S_k}$. Indeed, by passing from S_k to $S_k \setminus \bigcup_{j<k} S_j$ if necessary, we can assume that the sets S_k are disjoint. Then apply Lemma 3.8. This proves (3.2).

Denote by Σ' the set of equivalence classes of sets from $S \in \Sigma^+$ (modulo sets of measure 0). Denote by $\mathcal{S}$ the set of all equivalence classes $[S] \in \Sigma'$ for which T is bounded on E_S (clearly, this definition does not depend on the choice of a representative of an equivalence class). We shall show that $[\Omega] \in \mathcal{S}$.

Define the relation $\prec$ on $\mathcal{S}$ by writing $[S_1] \prec [S_2]$ if $\mu(S_2 \backslash S_1) \geq 0$, and $\mu(S_1 \backslash S_2) = 0$. Note that $\prec$ is a partial order. Furthermore, any chain in $\mathcal{S}$ has an upper bound. Indeed, any such chain can have at most countably many distinct elements, due to the σ-finiteness of μ. We have observed above that $[\bigcup_k S_k] \in \mathcal{S}$ whenever $[S_k] \in \mathcal{S}$ for every k (cf. Lemmas 3.6, 3.7 and 3.8). Thus, by Zorn's Lemma, $\mathcal{S}$ has at least one maximal element. But, by (3.2), $[\Omega]$ is the only possible maximal element. □

Proof of Theorem 3.1 Suppose E is an order continuous Banach lattice. By [19, Proposition 1.a.9], E admits a convenient decomposition into a direct sum of mutually disjoint ideals E_α, each one having its own weak order unit. By [19, Proposition 1.b.14 and p. 29], each E_α is order isometric to an order continuous Köthe function space. Combining Lemmas 3.5 and 3.9, we conclude that T is bounded on each of the ideals E_α. Finally, Lemma 3.8 implies that T is bounded on E. □

Remark 3.10 We do not know whether a disjointness preserving linear bijection between general Banach lattices must be continuous. It is known that any band preserving linear map is continuous [20, Theorem 3.1.12].

We shall conclude this section exploring the automatic continuity of every L-norm on the predual of a commutative von Neumann algebra. We have already commented that $L_1(\mu)$ is an order continuous Banach lattice. Since every orthogonality-to-1-orthogonality preserving linear mapping T from $L_1(\mu)$ to a Banach space is DSMO the following corollary derives from Theorem 3.1, or directly from Proposition 3.3.

Corollary 3.11 *Let (Ω, Σ, μ) be a measure space where μ is finite and positive, and let X be a Banach space. Then every O-1-O preserving linear bijection $T : L_1(\mu) \to X$ is automatically continuous.*

Theorem 3.12 *Let M be a commutative von Neumann algebra and let X be a Banach space. Then, every O-1-O preserving linear bijection $T : M_* \to X$ is continuous. Equivalently, every complete L-norm on the predual space of a commutative von Neumann algebra is equivalent to the original norm.*

Proof By [28, Proposition 1.18.1] we have

$$M = \bigoplus_{\alpha \in I}^{\ell_\infty} L_\infty(\Omega_\alpha, \Sigma_\alpha, \mu_\alpha)$$

for a family $(\mu_\alpha)_{\alpha \in I}$ of positive finite measures. Therefore,

$$M_* = \bigoplus_{\alpha \in I}^{\ell_1} L_1(\Omega_\alpha, \Sigma_\alpha, \mu_\alpha).$$

Proposition 2.3 implies that, for each $\alpha \in I$, $T\left(L_1(\Omega_\alpha, \Sigma_\alpha, \mu_\alpha)\right)$ is norm closed and the restriction

$$T|_{L_1(\Omega_\alpha, \Sigma_\alpha, \mu_\alpha)} : L_1(\Omega_\alpha, \Sigma_\alpha, \mu_\alpha) \to T\left(L_1(\Omega_\alpha, \Sigma_\alpha, \mu_\alpha)\right)$$

is an orthogonality-to-1-orthogonality preserving linear bijection. Corollary 3.11 assures that the mapping $T_\alpha := T|_{L_1(\Omega_\alpha, \Sigma_\alpha, \mu_\alpha)}$ is continuous.

The argument given in the proof of Proposition 2.1, shows that $K := \sup\{\|T_\alpha\| : \alpha \in I\} < \infty$, and since for each $\varphi = (\varphi_\alpha)_\alpha$ in M_*, there exists a countable subset $I_0 \subset I$ such that $\varphi_\alpha = 0$, for every $\alpha \in I \backslash I_0$ and $\|\varphi\| = \sum_{\alpha \in I_0} \|\varphi_\alpha\|$. Propositions 2.3 and 2.1 show that $T|_{\bigoplus_{\alpha \in I_0}^{\ell_1} L_1(\Omega_\alpha, \Sigma_\alpha, \mu_\alpha)}$, is continuous and

$$\|T((\varphi_\alpha)_\alpha)\| = \left\| \sum_{\alpha \in I_0} T(\varphi_\alpha) \right\| = \sum_{\alpha \in I_0} \|T(\varphi_\alpha)\| \leq K \sum_{\alpha \in I_0} \|\varphi_\alpha\| = K \|\varphi\|,$$

which proves that T is continuous with $\|T\| \leq K$. □

Corollary 3.13 *Every complete L-norm on the dual space of an abelian C^*-algebra is equivalent to the original norm.*

Proof If A is a commutative C*-algebra, then A^{**} is a commutative von Neumann algebra. By Theorem 3.12, $A^* = (A^{**})_*$ has a unique (up to equivalence) complete L-norm. □

Corollary 3.14 *Let M and N be von Neumann algebras with N abelian. Then every L-orthogonality preserving linear bijection $T : N_* \to M_*$ is continuous.*

4 *p*-norms on Schatten spaces

Throughout this section, H denotes a complex Hilbert space, equipped with the inner product $(\cdot|\cdot)$. For a fixed $p \in [1, \infty)$, we consider the Schatten space $\mathcal{S}^p(H)$ (defined in Sect. 1), with its norm $\|\cdot\|_p$. Our main result is:

Theorem 4.1 *For $1 \leq p < \infty$, any O-p-O preserving linear surjection from $\mathcal{S}^p(H)$ to a Banach space is continuous.*

Now consider a family of Hilbert spaces $(H_i)_{i \in I}$. The von Neumann algebra $M = (\bigoplus_{i \in I} B(H_i))_{\ell_\infty}$ can be equipped with the faithful normal semi-finite trace $\tau = \bigoplus_{i \in I} \mathrm{tr}_i$, where tr_i is the canonical trace on $B(H_i)$. Then $L^p(\tau)$ can be identified with $(\bigoplus_{i \in I} \mathcal{S}^p(H_i))_{\ell_p}$, equipped with the norm

$$\|(\phi_i)_{i \in I}\|_p = \left(\sum_{i \in I} \|\phi_i\|_{\mathcal{S}^p(H_i)}^p \right)^{1/p}.$$

The elements ϕ and ψ of $L^p(\tau)$ are orthogonal if and only if $\phi_i \perp \psi_i$ for every $i \in I$.

It turns out that any *atomic* or *discrete* von Neumann algebra (that is, an algebra where every projection has an atomic abelian subprojection) is of the form described above. Moreover, these von Neumann algebras are the second duals of compact C*-algebras. Recall that a Banach algebra A is called *compact* (or *dual*) if, for any $a \in A$, the map $b \mapsto aba$ is compact. By [3], a C*-algebra is compact if and only if it is of the form $(\bigoplus_{i \in I} K(H_i))_{c_0}$.

Corollary 4.2 *Suppose τ is the canonical trace on the discrete von Neumann algebra $(\bigoplus_{i \in I} B(H_i))_{\ell_\infty}$, X is a Banach space, and $1 \le p < \infty$. Then any O-p-O preserving linear bijection $T : L^p(\tau) \to X$ is continuous.*

Proof As explained above, we have $L^p(\tau) = (\bigoplus_{i \in I} \mathcal{S}^p(H_i))_{\ell_p}$. By Proposition 2.3, for each $i \in I$, $T(\mathcal{S}^p(H_i))$ is norm closed and the restriction

$$T|_{\mathcal{S}^p(H_i)} : \mathcal{S}^p(H_i) \to T\left(\mathcal{S}^p(H_i)\right)$$

is an O-p-O preserving linear bijection. Theorem 4.1 implies that the mapping $T_i := T|_{\mathcal{S}^p(H_i)}$ is continuous.

It remains to show that $\sup\{\|T_i\| : i \in I\} < \infty$. Otherwise there exists a sequence of distinct indices $i_n \in I$, and elements $\phi_n \in \mathcal{S}^p(H_{i_n})$, so that $\|\phi_n\| < 2^{-n}$, and $\|T\phi_n\| > 2^n$. Define $\psi = (\psi_j)_{j \in I} \in L^p(A)$ by setting $\psi_j = \phi_n$ if $j = i_n$ for some n, and $\psi_j = 0$ otherwise. Note that $\psi - \phi_n$ is orthogonal to ϕ_n for every n, hence $\|T\psi\| \ge \|T\phi_n\| > 2^n$, which is impossible. □

Theorem 4.1 will be proved following a series of auxiliary lemmas. Finite rank operators play an important role. Note that any rank 1 operator on H has the form $\omega_{\xi,\eta} = \xi \otimes \eta$, defined via $\xi \otimes \eta(h) = (h|\eta)\xi$. Clearly, $\|\omega_{\xi,\eta}\| = \|\omega_{\xi,\eta}\|_p = \|\xi\|\|\eta\|$, where $\|\omega_{\xi,\eta}\|$ denotes the norm of $\omega_{\xi,\eta}$ in $B(H)$. Moreover, $\omega_{\xi_1,\eta_1} \perp \omega_{\xi_2,\eta_2}$ if, and only if, $(\xi_1|\xi_2) = (\eta_1|\eta_2) = 0$. For the sake of brevity, we write $\omega_\xi = \omega_{\xi,\xi}$.

Our first lemma is a simple algebraic exercise.

Lemma 4.3 *Suppose ξ and η are mutually orthogonal norm-one elements of H. Then*:

(1) $\omega_{\xi+\eta} \perp \omega_{\xi-\eta}$ *and* $\omega_{\xi,\eta} \perp \omega_{\eta,\xi}$.
(2) $\omega_{\xi+\eta} + \omega_{\xi-\eta} = 2(\omega_\xi + \omega_\eta)$, *and* $\omega_{\xi+\eta} - \omega_{\xi-\eta} = 2(\omega_{\xi,\eta} + \omega_{\eta,\xi})$.

Corollary 4.4 *Suppose ξ and η are mutually orthogonal norm-one elements of H, and a linear map $T : \mathcal{S}^p(H) \to X$ is O-p-O preserving. Then*

$$\|T(\omega_{\xi,\eta})\|^p + \|T(\omega_{\eta,\xi})\|^p = \|T(\omega_\xi)\|^p + \|T(\omega_\eta)\|^p.$$

Proof By Lemma 4.3(2),

$$2^p\left(\|T(\omega_\xi)\|^p + \|T(\omega_\eta)\|^p\right) = \|2T(\omega_\xi + \omega_\eta)\|^p = \|T(\omega_{\xi+\eta} + \omega_{\xi-\eta})\|^p,$$

and

$$2^p\left(\|T(\omega_{\xi,\eta})\|^p + \|T(\omega_{\eta,\xi})\|^p\right) = \|2T(\omega_{\xi,\eta} + \omega_{\eta,\xi})\|^p = \|T(\omega_{\xi+\eta} - \omega_{\xi-\eta})\|^p.$$

By Lemma 4.3(*1*), $\|T(\omega_{\xi+\eta} + \omega_{\xi-\eta})\| = \|T(\omega_{\xi+\eta} - \omega_{\xi-\eta})\|$, hence

$$2^p\big(\|T(\omega_\xi)\|^p + \|T(\omega_\eta)\|^p\big) = \|T(\omega_{\xi+\eta} - \omega_{\xi-\eta})\|^p = \|2T(\omega_{\xi,\eta} + \omega_{\eta,\xi})\|^p$$
$$= 2^p\big(\|T(\omega_{\xi,\eta})\|^p + \|T(\omega_{\eta,\xi})\|^p\big). \qquad \square$$

Lemma 4.5 *Suppose the linear map $T : \mathcal{S}^p(H) \to X$ is O-p-O preserving. Then*

$$\sup\big\{\|T(\omega_\xi)\| \ : \ \xi \in H, \ \|\xi\| = 1\big\} < \infty.$$

Proof Suppose, for the sake of contradiction, that the supremum above is infinite. Construct recursively an orthonormal sequence (ξ_n), such that $\|T(\omega_{\xi_n})\| > 4^n$. Start by selecting $\xi_1 \in H$ with $\|\xi_1\| = 1$, and $\|T(\omega_{\xi_1})\| > 4$.

Now suppose the orthonormal vectors $\xi_1, \ldots, \xi_n$ have already been chosen, so that $\|T(\omega_{\xi_k})\| > 4^k$ for $1 \le k \le n$. Let $H_n = \mathrm{span}[\xi_1, \ldots, \xi_n]$, and $C = \sup\{\|T(\omega_\xi)\| : \|\xi\| = 1, \xi \in H_n\}$. Pick $\eta \in H$ so that $\|\eta\| = 1$, and $\|T(\omega_\eta)\| > 3(4^{n+1} + C)$. Clearly, $\eta \notin H_n$. Write $\eta = \alpha\xi + \beta\xi_{n+1}$, where $\|\xi\| = \|\xi_{n+1}\| = 1$, ξ_{n+1} is orthogonal to H_n, ξ belongs to H_n, and $|\alpha|^2 + |\beta|^2 = 1$. We claim that $\|T(\omega_{\xi_{n+1}})\| > 4^{n+1}$. Indeed,

$$\omega_\eta = |\alpha|^2\omega_\xi + |\beta|^2\omega_{\xi_{n+1}} + \alpha\overline{\beta}\omega_{\xi,\xi_{n+1}} + \overline{\alpha}\beta\omega_{\xi_{n+1},\xi},$$

hence

$$\|T(\omega_\eta)\| \le \|T(\omega_\xi)\| + \|T(\omega_{\xi_{n+1}})\| + \|T(\omega_{\xi,\xi_{n+1}})\| + \|T(\omega_{\xi_{n+1},\xi})\|.$$

However, by Corollary 4.4, $\|T(\omega_{\xi,\xi_{n+1}})\|$ and $\|T(\omega_{\xi_{n+1},\xi})\|$ do not exceed $\|T(\omega_\xi)\| + \|T(\omega_{\xi_{n+1}})\|$. Thus, $\|T(\omega_\eta)\| \le 3(C + \|T(\omega_{\xi_{n+1}})\|)$, and therefore, $\|T(\omega_{\xi_{n+1}})\| \ge \|T(\omega_\eta)\|/3 - C > 4^{n+1}$.

Now consider $\phi = \sum_{n=1}^{\infty} 2^{-n}\omega_{\xi_n} \in \mathcal{S}^p(H)$. Then, for any natural n, $\omega_{\xi_n} \perp (\phi - 2^{-n}\omega_{\xi_n})$, hence

$$\|T(\phi)\| \ge 2^{-n}\|T(\omega_{\xi_n})\| > 2^{-n} \cdot 4^n = 2^n,$$

which is impossible. $\square$

Corollary 4.6 *Suppose the linear map $T : \mathcal{S}^p(H) \to X$ is O-p-O preserving. Then the set $\{\|T(\omega_{\xi,\eta})\| : \xi, \eta \in H, \|\xi\| = \|\eta\| = 1\}$ is bounded.*

Proof By Lemma 4.5, $K = \sup_{\|\xi\|=1} \|T(\omega_\xi)\|$ is finite. We show that $\|T(\omega_{\xi,\eta})\| \le 3K$ whenever $\|\xi\| = \|\eta\| = 1$. If $\eta \in \mathrm{span}[\xi]$, we are done. Otherwise, write $\eta = \alpha\xi + \beta\zeta$, where $\|\zeta\| = 1$, $(\xi|\zeta) = 0$, and $|\alpha|^2 + |\beta|^2 = 1$. Then $\omega_{\xi,\eta} = \overline{\alpha}\omega_\xi + \overline{\beta}\omega_{\xi,\zeta}$. But $\|T(\omega_\xi)\| \le K$. Furthermore, by Corollary 4.4,

$$\|T(\omega_{\xi,\zeta})\| \le \big(\|T(\omega_\xi)\|^p + \|T(\omega_\zeta)\|^p\big)^{1/p} \le 2K.$$

Therefore, $\|T(\omega_{\xi,\eta})\| \le |\alpha|\|T(\omega_\xi)\| + |\beta|\|T(\omega_{\xi,\zeta})\| \le \sqrt{5}K < 3K$. $\square$

Remark 4.7 Using similar methods, one can prove the following statement: suppose $\xi_1, \dots, \xi_n$ is an orthonormal family in H, and $\lambda_1, \dots, \lambda_n$ a complex numbers. Then, for any O-1-O preserving linear map $T : \mathcal{S}^1(H) \to X$, we have

$$\left\| T(\omega_{\lambda_1\xi_1+\cdots+\lambda_n\xi_n}) \right\| \leq \frac{1}{2} \sum_{i,j=1}^{n} |\lambda_i|\, |\lambda_j| \, \left(\|T(\omega_{\xi_i})\| + \|T(\omega_{\xi_j})\| \right).$$

We denote by $\mathcal{F}(H)$ the space of finite rank operators on H. When equipped with the norm inherited from $\mathcal{S}^p(H)$, this space is denoted by $\mathcal{F}^p(H)$.

Corollary 4.8 *Suppose $T : \mathcal{S}^p(H) \to X$ is an O-p-O preserving linear map. Then T is bounded on $\mathcal{F}^p(H)$.*

Proof By Corollary 4.6,

$$K = \sup_{\|\xi\| \leq 1, \|\eta\| \leq 1} \|T(\omega_{\xi,\eta})\|$$

is finite. Any $\phi \in \mathcal{F}^p(H)$ admits a polar decomposition $\phi = \sum_{k=1}^n \alpha_k \omega_{\xi_k,\eta_k}$, where the systems (ξ_k) and (η_k) are orthonormal, and $(\alpha_k) \subset \mathbb{R}^+$ are the singular numbers of ϕ (with $\|\phi\|_p^p = \sum_k \alpha_k^p$). Then, since T is O-p-O preserving,

$$\|T(\phi)\|^p = \sum_{k=1}^{n} |\alpha_k|^p \|T(\omega_{\xi_k,\eta_k})\|^p \leq K^p \sum_{k=1}^{n} |\alpha_k|^p = K^p \|\phi\|_p^p,$$

and hence $\|T|_{\mathcal{F}^p(H)}\| \leq K$. □

It is easy to see that, for any $\phi \in \mathcal{S}^p(H)$, there exists a projection p_0 in $B(H)$ with separable range, such that $p_0 \phi p_0 = \phi$. Here and below, the word "projection" refers to a self-adjoint idempotent on a Hilbert space.

Proposition 4.9 *Suppose ϕ is an element of $\mathcal{S}^p(H)$, and p_0 is a projection with separable range, such that $p_0 \phi p_0 = \phi$. Suppose, furthermore, that X is a Banach space, and the linear map $T : \mathcal{S}^p(H) \to X$ is O-p-O preserving. Then there exists $x \in X$ such that*

$$\lim_i \|x - T(l_i \phi r_i)\| = 0$$

whenever (l_i) and (r_i) are increasing sequences of projections, converging strongly to p_0.

The proof relies on an easy

Lemma 4.10 *Suppose ϕ, p_0, (l_i), and (r_i) are as in the previous proposition. Then*

$$\lim \|\phi - l_i \phi\|_p = \lim \|\phi - \phi r_i\|_p = \lim \|\phi - l_i \phi r_i\|_p = 0.$$

Proof Finite rank operators are dense in $\mathcal{S}^p(H)$, hence it suffices to consider the case of $\phi = \omega_{\xi,\eta}$. We shall show that $\lim_i \|\phi - l_i \phi r_i\|_p = 0$ (the other equalities are handled similarly). An easy computation shows that

$$\phi - l_i \phi r_i = \omega_{\xi,\,(p_0 - r_i)\eta} + \omega_{(p_0 - l_i)\xi,\, r_i \eta}.$$

Therefore, $\|\phi - l_i \phi r_i\|_p \leq \|\xi\| \|(p_0 - r_i)\eta\| + \|(p_0 - l_i)\xi\| \|\eta\|$. To complete the proof, recall that $\lim_i \|(p_0 - r_i)\eta\| = \lim_i \|(p_0 - l_i)\xi\| = 0$. □

Proof of Proposition 4.9 Let $K = \|T|_{\mathcal{F}^p(H)}\|$ (cf. Corollary 4.8). Suppose first that (l_i) and (r_i) are sequences of projections as above. Let $\phi_i = l_i \phi r_i$. Then (ϕ_i) is a Cauchy sequence in $\mathcal{F}^p(H)$. Indeed, for $j > i$, $\phi_j - \phi_i = l_j(\phi - \phi_i) r_j$, hence $\|\phi_j - \phi_i\|_p \leq \|\phi - \phi_i\|_p$, and the right hand side tends to 0, by Lemma 4.10. Thus, $\lim_{i,j} \|\phi_j - \phi_i\|_p = 0$. The operator T is bounded on $\mathcal{F}^p(H)$, hence the image of the Cauchy sequence $(\phi_i)_{i=1}^\infty$ is again Cauchy. Since X is complete, the sequence $(T(\phi_i))_{i=1}^\infty$ converges to some $x \in X$.

Now suppose (l'_j) and (r'_j) are two other sequences of projections, increasing to p_0. As above, we see that $\phi'_i = l'_i \phi r'_i$ form a Cauchy sequence in $\mathcal{F}^p(H)$. For any $\varepsilon > 0$, we can find $N \in \mathbb{N}$ such that $\|\phi - \phi_i\|_p < \varepsilon/2$ and $\|\phi - \phi'_i\|_p < \varepsilon/2$ whenever $i > N$. By the triangle inequality, $\|\phi_j - \phi'_i\|_p < \varepsilon$ whenever $i, j > N$. Moreover, ϕ_j and ϕ'_i have finite rank, hence $\|T(\phi_j) - T(\phi'_i)\| < K\varepsilon$. Letting j grow without a bound, we conclude that $\|x - T(\phi'_i)\| \leq K\varepsilon$ for $i > N$. As ε is arbitrary, we conclude that $\lim_i \|x - T(\phi'_i)\| = 0$. □

Proof of Theorem 4.1 Suppose $T : \mathcal{S}^p(H) \to X$ is linear bijection (X is a Banach space). By Corollary 4.8 and scaling if necessary, we can assume $\|T|_{\mathcal{F}^p(H)}\| = 1$. Suppose ϕ is a norm-one element of $\mathcal{S}^p(H)$. We can always find two increasing sequences of finite rank projections (p_i) and (q_i), such that their strong limits satisfy $\lim_i p_i = \lim_i q_i = p_0$, where p_0 is a projection with separable range, and $p_0 \phi p_0 = \phi$. By Proposition 4.9, there exists $x \in X$ such that $x = \lim_i T(e_i \phi f_i)$ whenever (e_i) and (f_i) are increasing sequences of finite rank projections, such that their strong limits satisfy $\lim_i e_i = \lim_i f_i = p_0$, p_0 is a projection with separable range, and $p_0 \phi p_0 = \phi$. We shall show that $T(\phi) = x$. Once this is established, observe that

$$\|x\| = \|T(\phi)\| = \lim_i \|T(e_i \phi f_i)\| \leq \|T|_{\mathcal{F}^p(H)}\| \lim_i \|e_i \phi f_i\|_p = 1,$$

which implies $\|T\| \leq 1$.

By the bijectivity of T, there exists $\psi \in \mathcal{S}^p(H)$ such that $T(\psi) = x$. We have to show that $\psi - \phi = 0$. To this end, use polar decomposition to write $\psi - \phi = \sum_{i=1}^\infty c_i \omega_{\xi_i, \eta_i}$, where $c_1 \geq c_2 \geq \cdots \geq 0$ are the singular values of $\psi - \phi$, while $(\xi_i)_{i \in \mathbb{N}}$ and $(\eta_i)_{i \in \mathbb{N}}$ are orthonormal systems in H. Find a projection p_0 with separable range such that $p_0 \phi p_0 = \phi$, and $p_0 \psi p_0 = \psi$. Pick the vectors $(\xi'_i)_{i \in \mathbb{N}}$ and $(\eta'_i)_{i \in \mathbb{N}}$ in $p_0(H)$, such that:

1. $p_0(H) = \mathrm{span}[\xi_1, \xi'_1, \xi_2, \xi'_2, \ldots] = \mathrm{span}[\eta_1, \eta'_1, \eta_2, \eta'_2, \ldots]$.
2. For every i, either $\xi'_i = 0$, or $\|\xi'_i\| = 1$. Similarly, either $\eta'_i = 0$, or $\|\eta'_i\| = 1$.
3. For $i \neq j$, $(\xi'_i | \xi'_j) = (\eta'_i | \eta'_j) = 0$. For any i and j, $(\xi'_i | \xi_j) = (\eta'_i | \eta_j) = 0$.

Let r_m and l_m be the projections onto span$[\eta_i, \eta_i' : 1 \le i \le m]$ and span$[\xi_i, \xi_i' : 1 \le i \le m]$, respectively. Then (l_m) and (r_m) form sequences of finite rank projections, increasing toward p_0. Moreover, $l_m(\psi - \phi)r_m = \sum_{i=1}^m c_i \omega_{\xi_i,\eta_i}$, and

$$l_m(\psi - \phi)r_m = l_s(\psi - \phi)r_s + (l_m - l_s)(\psi - \phi)(r_m - r_s) \tag{4.1}$$

for $s < m$ (the terms in the right hand side are orthogonal).

We have to show that, for any $\varepsilon > 0$, there exists $M \in \mathbb{N}$ such that $\|T(l_m(\psi - \phi)r_m)\| \le \varepsilon$ for any $m \ge M$. Once this is accomplished, (4.1) establishes that, for any s,

$$\|T(l_s(\psi - \phi)r_s)\| \le \lim_m \|T(l_m(\psi - \phi)r_m)\| = 0.$$

The injectivity of T implies $l_s(\psi - \phi)r_s = 0$. As s is arbitrary, we conclude that $\psi - \phi = 0$.

For the sake of brevity, we use the notation $q^\perp = p_0 - q$ whenever q is a subprojection of p_0. Note that, for $n > m$, $l_m l_n = l_m$, and $r_n r_m = r_m$, and therefore,

$$\begin{aligned} l_n\phi r_n - \psi &= (l_m^\perp + l_m)(l_n\phi r_n - \psi)(r_m^\perp + r_m) \\ &= l_m^\perp(l_n\phi r_n - \psi)r_m^\perp + l_m^\perp(l_n\phi r_n - \psi)r_m \\ &\quad + l_m(l_n\phi r_n - \psi)r_m^\perp + l_m(l_n\phi r_n - \psi)r_m. \end{aligned} \tag{4.2}$$

Note that $l_m(l_n\phi r_n - \psi)r_m = l_m(\phi - \psi)r_m$, and furthermore, $l_m(\phi - \psi)r_m$ is orthogonal to $l_m^\perp(l_n\phi r_n - \psi)r_m^\perp$. As T is O-p-O preserving, we have

$$\|T(l_m(\phi - \psi)r_m)\| \le \|T(l_m(\phi - \psi)r_m + l_m^\perp(l_n\phi r_n - \psi)r_m^\perp)\|.$$

By (4.2),

$$\begin{aligned} &l_m(\phi - \psi)r_m + l_m^\perp(l_n\phi r_n - \psi)r_m^\perp \\ &\quad = (l_n\phi r_n - \psi) - l_m^\perp(l_n\phi r_n - \psi)r_m - l_m(l_n\phi r_n - \psi)r_m^\perp, \end{aligned}$$

hence, by the triangle inequality,

$$\begin{aligned} \|T(l_m(\phi - \psi)r_m)\| &\le \|T(l_m(\phi - \psi)r_m + l_m^\perp(l_n\phi r_n - \psi)r_m^\perp)\| \\ &\le \|T(l_n\phi r_n - \psi)\| + \|T(l_m^\perp(l_n\phi r_n - \psi)r_m)\| \\ &\quad + \|T(l_m(l_n\phi r_n - \psi)r_m^\perp)\|. \end{aligned} \tag{4.3}$$

Recall that $\lim_n T(l_n\phi r_n) = T(\psi)$, hence there exists $N_0 \in \mathbb{N}$ such that $\|T(l_n\phi r_n - \psi)\| < \varepsilon/3$ for $n > N_0$. Note also that

$$l_m^\perp(l_n\phi r_n - \psi)r_m = l_n l_m^\perp \phi r_m - l_m^\perp \psi r_m,$$

hence

$$\|l_m^\perp(l_n\phi r_n - \psi)r_m\|_p \le \|l_m^\perp \phi\|_p + \|l_m^\perp \psi\|_p.$$

By Lemma 4.10, there exists $N_1 \in \mathbb{N}$ such that $\|l_m^\perp \phi\|_p + \|l_m^\perp \psi\|_p < \varepsilon/3$, for $m > N_1$. Moreover,

$$\text{rank}\left(l_m^\perp (l_n \phi r_n - \psi) r_m\right) \leq \text{rank}\, r_m < \infty,$$

hence

$$\|T(l_m^\perp (l_n \phi r_n - \psi) r_m)\| \leq \|l_m^\perp (l_n \phi r_n - \psi) r_m\|_p < \frac{\varepsilon}{3}$$

for $n > m \geq N_1$. Similarly, there exists $N_2 \in \mathbb{N}$ such that

$$\|T(l_m (l_n \phi r_n - \psi) r_m^\perp)\| < \varepsilon/3,$$

for $n > m > N_2$. By (4.3), we conclude that $\|T(l_m(\phi - \psi) r_m)\| < \varepsilon$ for $m > \max\{N_0, N_1, N_2\}$. This completes the proof. □

5 Orthogonality preserving maps between non-commutative L^p spaces

This section is devoted to the structure of orthogonality (equivalently, O-p-O) preserving maps between non-commutative L^p spaces. We begin by describing orthogonality preserving bijections between Schatten spaces. Throughout this section, H and K stand for Hilbert spaces.

Theorem 5.1 *Suppose* $1 \leq p < \infty$, *and* $T : S^p(H) \to S^p(K)$ *is an orthogonality preserving linear bijection. Then there exists a scalar* γ, *and unitary operators* $U \in B(H, K)$, $V \in B(K, H)$, $\widetilde{U} \in B(H^*, K)$ *and* $\widetilde{V} \in B(K, H^*)$, *such that either* $T(\phi) = \gamma U \phi V$, *or* $T(\phi) = \gamma\, \widetilde{U} \phi^t \widetilde{V}$, *for every* ϕ *in* $\mathcal{S}^p(H)$ (*where* ϕ^t *stands for the transpose of* ϕ). *In particular,* T *is a scalar multiple of an isometry.*

The proof relies on several auxiliary results. As in Sect. 4, $\omega_{h,k}$ ($h, k \in H$) stands for the rank 1 operator, defined by $\omega_{h,k}(\xi) = (\xi|k)h$. First, we express the rank of an operator a (denoted by rank(a)) in terms of mutually orthogonal summands. Here, rank(a) is defined as dim(ran(a)), if ran(a) is finite dimensional, and rank(a) $= \infty$ otherwise. Thus, rank(a) takes values in the set $\mathbb{Z}_+ \cup \{\infty\}$, where $\mathbb{Z}_+ = \{0, 1, \ldots\}$. For $S \subset \mathbb{Z}_+$, $\sup S$ is defined (in the standard manner) to be either a non-negative integer, or ∞.

Note that our definition doesn't distinguish between different types of "infinite cardinals": rank(a) $= \infty$ when ran(a) is either separable infinite dimensional, or non-separable. However, in this paper, we mostly work with operators in $\mathcal{S}^p(H)$, which are compact. Employing a standard singular value decomposition technique (as used in the proof below), one can show that any compact operator $a \in B(H)$ has separable range.

Lemma 5.2 *Suppose* $\mathcal{A}$ *denotes either* $B(H)$, *or* $\mathcal{S}^p(H)$ $(1 \leq p < \infty)$. *Let a be an element in* $\mathcal{A}$. *When a has finite rank, then* rank(a) *equals the supremum of all* $n \in \mathbb{Z}_+$ *for which a can be represented as a sum of n non-zero mutually orthogonal elements of* $\mathcal{A}$. *Moreover,* rank(a) $= \infty$ *if, and only if, a can be represented as a sum of n non-zero mutually orthogonal elements of* $\mathcal{A}$ *for any* $n \geq 1$.

Proof First, we show that $\mathrm{rank}(a) \geq n$ if there exist mutually orthogonal non-zero operators $(a_i)_{i=1}^n$, satisfying $a = \sum_{i=1}^n a_i$. Indeed, there exist non-zero projections (p_i) and (q_i), so that $p_i p_j = q_i q_j = 0$ if $i \neq j$, and $p_i a_i q_i = a_i$ for any i. For $1 \leq i \leq n$, pick a norm 1 vector $\xi_i \in q_i(H)$, so that $\eta_i = a_i \xi_i \neq 0$. It is easy to see that the range of $\sum_i a_i$ contains the linear span of the vectors η_i. Moreover, the vectors η_i are linearly independent. Therefore, $\mathrm{rank}(a) \geq n$. Taking the supremum, we obtain $\mathrm{rank}(a) \geq n$.

Now suppose $a \in \mathcal{A}$ has rank $n \in \mathbb{Z}_+$. Note that $n = 0$ if and only if $a = 0$. If $n \geq 1$, then, by [30, Theorem 1.4] or [14, Sect. II.2], a admits a singular value decomposition $a = \sum_{i=1}^n \mathbf{s}_i(a)\omega_{\xi_i,\eta_i}$, where (ξ_i) and (η_i) are orthonormal systems in H, and the singular numbers $\mathbf{s}_i(a)$ are positive. It is easy to see that the operators $a_i = \mathbf{s}_i(a)\omega_{\xi_i,\eta_i}$ $(1 \leq i \leq n)$ are mutually orthogonal, and therefore, a can be represented as a sum of n mutually orthogonal non-zero elements of $\mathcal{A}$. This establishes the statement of the lemma for finite rank a.

It remains to show that, whenever $a \in \mathcal{A}$ has infinite rank, then, for any $n \in \mathbb{N}$, there exist mutually orthogonal, non-zero members of to $\mathcal{A}$, satisfying $a = \sum_{i=1}^n a_i$.

If a is compact, the singular value decomposition yields $a = \sum_{i=1}^\infty \mathbf{s}_i(a)\omega_{\xi_i,\eta_i}$. Set $a_i = \mathbf{s}_i(a)\omega_{\xi_i,\eta_i}$ $(1 \leq i \leq n-1)$, and $a_n = \sum_{i=n}^\infty \mathbf{s}_i(a)\omega_{\xi_i,\eta_i}$. The operators $(a_i)_{i=1}^n$ are mutually orthogonal, non-zero, belong to $\mathcal{A}$, and satisfy $a = \sum_{i=1}^n a_i$.

Now suppose a is not compact (this can only happen if $\mathcal{A} = B(H)$). Write $a = ub$, where $b = (a^*a)^{1/2}$ is positive, u is an isometry from the range of b to the range of a (see e.g. [30, Sect. 1.1]). If $\sigma(b)$ (the spectrum of b) is infinite, we can represent $\sigma(b)$ as a disjoint union of non-empty Borel subsets $(S_i)_{i=1}^n$. Then the operators $a_i = u\chi_{S_i}(b)b$ are mutually orthogonal, non-zero, and satisfy $\sum_{i=1}^n a_i = a$.

If $\sigma(b)$ is finite, then it contains an eigenvalue $\beta > 0$ with infinite multiplicity. Let $p = \chi_{\{\beta\}}(b)$ be the corresponding spectral projection. For $n \geq 2$, we can write $p = p_1 + \cdots + p_n$, where the non-zero projections $p_1, \ldots, p_n$ are mutually orthogonal. Let $a_i = \beta u p_i$ $(1 \leq i \leq n-1)$, and $a_n = a - \sum_{i=1}^{n-1} a_i = u(\mathbf{1} - \beta p)b + \beta u p_n$. Once again, the operators $(a_i)_{i=1}^n$ have the required properties. □

Remark 5.3 For $a \in \mathcal{A}$, denote by $\mathrm{rank}_{\mathcal{A},\mathrm{orth}}(a)$ the supremum of all $n \in \mathbb{Z}_+$ for which a can be represented as a sum of n non-zero mutually orthogonal elements of $\mathcal{A}$ (we set $\mathrm{rank}_{\mathcal{A},\mathrm{orth}}(a) = 0$ if no such representation exists). As we do not distinguish between different types of "infinite cardinals" when computing $\mathrm{rank}(a)$, we conclude that the proof above shows that $\mathrm{rank}(a) = \mathrm{rank}_{\mathcal{A},\mathrm{orth}}(a)$ for any $a \in \mathcal{A}$.

From Lemma 5.2 we obtain:

Corollary 5.4 *For $1 \leq p < \infty$, any orthogonality preserving T linear bijection from $\mathcal{S}^p(H)$ to $\mathcal{S}^p(K)$ is rank-nondecreasing. That is,* $\mathrm{rank}(T(\phi)) \geq \mathrm{rank}(\phi)$, *for any $\phi \in \mathcal{S}^p(H)$. In particular, the preimage of any rank 1 operator also has rank 1.* □

The following result may be of independent interest. Here and below, we use the notation $\mathcal{S}^p(H_1, H_2)$ (H_1 and H_2 are Hilbert spaces) for the set of all compact operators ϕ from H_1 to H_2, whose sequence of singular values $(\mathbf{s}_i(\phi))_{i\in\mathbb{N}}$ is p-summable. Equipping it with the norm $\|\phi\|_p = (\sum_i \mathbf{s}_i(\phi)^p)^{1/p}$, we turn $\mathcal{S}^p(H_2, H_1)$ into a Banach space. One easily observes that $\mathcal{S}^p(H, H) = \mathcal{S}^p(H)$. Furthermore, if H_1 and H_2

are subspaces of K_1 and K_2, we can view $\mathcal{S}^p(H_1, H_2)$ as a subspace of $\mathcal{S}^p(K_1, K_2)$. More precisely, the canonical isometric embedding $I : \mathcal{S}^p(H_1, H_2) \to \mathcal{S}^p(K_1, K_2)$ is defined via $I\phi = J\phi p$, where p and J denote the orthogonal projection from K_1 onto H_1, and the canonical injection of H_2 into K_2, respectively.

Proposition 5.5 *Suppose $1 \le p < \infty$, and S is an isomorphism of $\mathcal{S}^p(K)$ onto a subspace $\mathcal{A}$ of $\mathcal{S}^p(H)$, which takes rank 1 maps to rank 1 maps. Then one of the following statements is true*:

(a) *There exists $\zeta_0 \in H$, and a subspace H_0 of H, so that $\mathcal{A} = \zeta_0 \otimes H_0$.*
(b) *There exists $\zeta_0 \in H$, and a subspace H_0 of H, so that $\mathcal{A} = H_0 \otimes \zeta_0$.*
(c) *$\mathcal{A} = \mathcal{S}^p(H_1, H_2)$, where H_1 and H_2 are subspaces of H, isomorphic to K. There exist invertible operators $U \in B(H_1, K)$ and $V \in B(K, H_2)$, such that $S(\phi) = J_2 V \phi U p_1$ for every ϕ in $\mathcal{S}^p(K)$.*
(d) *$\mathcal{A} = \mathcal{S}^p(H_1, H_2)$, where H_1 and H_2 are subspaces of H, isomorphic to K. There exist invertible operators $\widetilde{U} \in B(H_1, K^*)$ and $\widetilde{V} \in B(K^*, H_2)$, such that $S(\phi) = J_2 \widetilde{V} \phi^t \widetilde{U} p_1$ for every ϕ in $\mathcal{S}^p(K)$, where ϕ^t denotes the transpose of ϕ.*

In (c) *and* (d), *p_1 and J_2 denote the orthogonal projection from H onto H_1, and the canonical injection of H_2 into H, respectively.*

Clearly, (a) or (b) in the above Proposition can occur if and only if K is finite dimensional.

The proof depends on several lemmas, and will be completed once Lemma 5.9 is established. Throughout our reasoning, the letters H, K, $\mathcal{A}$, and S have the same meaning as in Proposition 5.5. We assume (without loss of generality) that S^{-1} is a contraction. We let $C = \|S\|$.

First, introduce some notation. As before, we denote by $\omega_{h,k}$ $(h, k \in H)$ the rank 1 operator, defined by $\omega_{h,k}(\xi) = (\xi|k)h$. As S takes rank 1 map to rank 1 maps, for every $\xi, \eta \in K\backslash\{0\}$ there exist $h, k \in H\backslash\{0\}$ so that $S(\omega_{\xi,\eta}) = \omega_{h,k}$. Let

$$\Gamma(\xi, \eta) := \left\{(h, k) \in H \times H : S(\omega_{\xi,\eta}) = \omega_{h,k}\right\}. \tag{5.1}$$

Note that, if both (h, k) and (h', k') belong to $\Gamma(\xi, \eta)$, then $\omega_{h,k} = \omega_{h',k'}$, and therefore, there exists $\lambda \in \mathbb{C}\backslash\{0\}$ so that $h' = \lambda h$ and $k' = \lambda^{-1} k$. Conversely, if $(h, k) \in \Gamma(\xi, \eta)$, then also $(\lambda h, \lambda^{-1} k) \in \Gamma(\xi, \eta)$ for any $\lambda \in \mathbb{C}\backslash\{0\}$. Thus, $\Phi(\xi, \eta) = \text{span}[h : (h, k) \in \Gamma(\xi, \eta)]$ and $\Psi(\xi, \eta) = \text{span}[k : (h, k) \in \Gamma(\xi, \eta)]$ are 1-dimensional subspaces of H.

As an illustration, consider the case of $S : \mathcal{S}^p(K) \to \mathcal{S}^p(H) : \phi \mapsto U\phi V$, with $U \in B(K, H)$ and $V \in B(H, K)$. Then $S(\omega_{\xi,\eta}) = \omega_{U(\xi), V^*(\eta)}$, hence $\Gamma(\xi, \eta) = \{(\lambda U(\xi), \lambda^{-1} V^*(\eta)) : \lambda \in \mathbb{C}\backslash\{0\}\}$, $\Phi(\xi, \eta) = \text{span}[U(\xi)]$, and $\Psi(\xi, \eta) = \text{span}[V^*(\eta)]$.

Below, we use the symbol $\sim$ to indicate the collinearity of non-zero vectors: we write $\xi \sim \eta$ if $\text{span}[\xi] = \text{span}[\eta]$, and $\xi \nsim \eta$ otherwise.

The following basic observation will be used several times.

Lemma 5.6 *The following statements hold*:

(i) *Suppose* ξ, η_1, *and* η_2 *are non-zero vectors in* K, *with* $\eta_1 \nsim \eta_2$. *Then exactly one of the two following statements is true*:
   (1) $\Phi(\xi, \eta_1) = \Phi(\xi, \eta_2)$, *and* $\Psi(\xi, \eta_1) \neq \Psi(\xi, \eta_2)$.
   (2) $\Phi(\xi, \eta_1) \neq \Phi(\xi, \eta_2)$, *and* $\Psi(\xi, \eta_1) = \Psi(\xi, \eta_2)$.

(ii) *Suppose* ξ_1, ξ_2 *and* η *are non-zero vectors in* K, *with* $\xi_1 \nsim \xi_2$. *Then exactly one of the two following statements is true*:
   (1) $\Phi(\xi_1, \eta) = \Phi(\xi_2, \eta)$, *and* $\Psi(\xi_1, \eta) \neq \Psi(\xi_2, \eta)$.
   (2) $\Phi(\xi_1, \eta) \neq \Phi(\xi_2, \eta)$, *and* $\Psi(\xi_1, \eta) = \Psi(\xi_2, \eta)$.

Proof (i) Note that, for any $(\alpha_1, \alpha_2) \in \mathbb{C}^2 \backslash \{(0,0)\}$,

$$S(\omega_{\xi, \alpha_1\eta_1 + \alpha_2\eta_2}) = \overline{\alpha_1} S(\omega_{\xi,\eta_1}) + \overline{\alpha_2} S(\omega_{\xi,\eta_2})$$

must be a rank 1 operator.

Suppose first $\Phi(\xi, \eta_1) = \Phi(\xi, \eta_2)$ and $\Psi(\xi, \eta_1) = \Psi(\xi, \eta_2)$. Then there exists $h, k_1, k_2 \in H$ so that $(h, k_1) \in \Gamma(\xi, \eta_1)$, $(h, k_2) \in \Gamma(\xi, \eta_2)$, and $\mathrm{span}[k_1] = \mathrm{span}[k_2]$. Find non-zero α_1 and α_2 so that $\overline{\alpha_1} k_1 + \overline{\alpha_2} k_2 = 0$, hence $S(\omega_{\xi, \alpha_1\eta_1 + \alpha_2\eta_2}) = 0$, which is impossible.

If, on the other hand, $\Phi(\xi, \eta_1) \neq \Phi(\xi, \eta_2)$ and $\Psi(\xi, \eta_1) \neq \Psi(\xi, \eta_2)$, then $u = S(\omega_{\xi, \eta_1+\eta_2})$ has rank 2, which contradicts our hypothesis. To verify the last statement, pick $h_1, h_2, k_1, k_2 \in H$, with the property that $(h_i, k_i) \in \Gamma(\xi, \eta_i)$ $(i = 1, 2)$. Find $\zeta \in H$, which is orthogonal to k_2, but not to k_1. Note that $u = \omega_{h_1,k_1} + \omega_{h_2,k_2}$, hence h_1 belongs to the range of u. Similarly, this range contains h_2.

Statement (ii) follows similarly. □

Lemma 5.7 *Suppose* ξ, η_1, *and* η_2 *are non-zero vectors in* K, *with* $\eta_1 \nsim \eta_2$.

(1) *If* $\Phi(\xi, \eta_1) = \Phi(\xi, \eta_2)$, *and* $\Psi(\xi, \eta_1) \neq \Psi(\xi, \eta_2)$, *then, for any* $\eta \in K$, $\Phi(\xi, \eta_1) = \Phi(\xi, \eta)$.
(2) *If* $\Phi(\xi, \eta_1) \neq \Phi(\xi, \eta_2)$, *and* $\Psi(\xi, \eta_1) = \Psi(\xi, \eta_2)$, *then, for any* $\eta \in K$, $\Psi(\xi, \eta_1) = \Psi(\xi, \eta)$.

Proof We prove (1), since (2) is established in a similar manner. It suffices to consider the case when η is not a scalar multiple of either η_1 or η_2. By Lemma 5.6(i), one of the following holds:

$$\begin{aligned} &(1) \quad \Phi(\xi, \eta_1) = \Phi(\xi, \eta), \quad \text{and} \quad \Psi(\xi, \eta_1) \neq \Psi(\xi, \eta), \\ &(2) \quad \Phi(\xi, \eta_1) \neq \Phi(\xi, \eta), \quad \text{and} \quad \Psi(\xi, \eta_1) = \Psi(\xi, \eta). \end{aligned} \tag{5.2}$$

Similarly, one of the statements below holds:

$$\begin{aligned} &(1) \quad \Phi(\xi, \eta_2) = \Phi(\xi, \eta), \quad \text{and} \quad \Psi(\xi, \eta_2) \neq \Psi(\xi, \eta), \\ &(2) \quad \Phi(\xi, \eta_2) \neq \Phi(\xi, \eta), \quad \text{and} \quad \Psi(\xi, \eta_2) = \Psi(\xi, \eta). \end{aligned} \tag{5.3}$$

We show that, if (5.2)(i) and (5.3)(j) hold at the same time, then $i = j = 1$. Suppose first that (5.2)(1) and (5.3)(2) hold. Then $\Phi(\xi, \eta_1) = \Phi(\xi, \eta) \neq \Phi(\xi, \eta_2)$, which contradicts $\Phi(\xi, \eta_1) = \Phi(\xi, \eta_2)$. The combination of (5.2)(2) and (5.3)(1) is ruled out similarly. Furthermore, if (5.2)(2) and (5.3)(2) hold at the same time, then $\Psi(\xi,\eta_1)=\Psi(\xi,\eta)=\Psi(\xi,\eta_2)$, which contradicts $\Psi(\xi,\eta_1)\neq\Psi(\xi,\eta_2)$. □

Now fix $\xi_0,\eta_1,\eta_2\in K\backslash\{0\}$, such that $\|\xi_0\|=1$, and $\eta_1\nsim\eta_2$. Suppose, without loss of generality, that $\Phi(\xi_0,\eta_1)=\Phi(\xi_0,\eta_2)$. By Lemma 5.7, for every $\eta\in K\backslash\{0\}$, $\Phi(\xi_0,\eta_1)=\Phi(\xi_0,\eta)$, and $\Psi(\xi_0,\eta_1)\neq\Psi(\xi_0,\eta)$ whenever $\eta\nsim\eta_1$. Find a norm 1 vector $\zeta_0\in H\in\Phi(\xi_0,\eta_1)$. By (5.1) and the discussion following it, for every $\eta\in K\backslash\{0\}$ there exists a unique $\boldsymbol{\psi}(\eta)\in H$, so that $(\zeta_0,\boldsymbol{\psi}(\eta))\in\Gamma(\xi_0,\eta)$ (hence $\Phi(\xi_0,\eta)=\text{span}[\zeta_0]$). Define $\boldsymbol{\psi}(0)=0$. Then, for any $\eta\in K$, $\boldsymbol{\psi}(\eta)$ is the unique element $k\in H$, satisfying $\omega_{\zeta_0,k}=S(\omega_{\xi_0,\eta})$. Note that the maps $\eta\mapsto S(\omega_{\xi_0,\eta})$ and $k\mapsto\omega_{h,k}$ (defined on K and H, respectively) are anti-linear, hence the map $\boldsymbol{\psi}:K\to H$ is linear. Furthermore, for $\eta\neq 0$,

$$\|\boldsymbol{\psi}(\eta)\|=\|\omega_{\zeta_0,\boldsymbol{\psi}(\eta)}\|_p=\|S(\omega_{\xi_0,\eta})\|_p\in\big[\|\eta\|,C\|\eta\|\big]$$

(recall that S^{-1} is a contraction, and $C=\|S\|$). In particular, $\boldsymbol{\psi}$ is an isomorphism from K to a closed subspace of H. Note also that, for $\eta\neq 0$, $\Psi(\xi_0,\eta)=\text{span}[\boldsymbol{\psi}(\eta)]$.

Lemma 5.8 *In the above notation, suppose $\xi\in K$ is not collinear to ξ_0. Then exactly one of the two following statements is true*:

(1) $\Phi(\xi,\eta)=\Phi(\xi_0,\eta)$, *and* $\Psi(\xi,\eta)\neq\Psi(\xi_0,\eta)$ *for any* $\eta\in K\backslash\{0\}$.
(2) $\Phi(\xi,\eta)\neq\Phi(\xi_0,\eta)$, *and* $\Psi(\xi,\eta)=\Psi(\xi_0,\eta)$ *for any* $\eta\in K\backslash\{0\}$.

Proof First fix $\eta_0\in K\backslash\{0\}$. Applying Lemma 5.6(ii) to ξ_0,ξ and η_0, we see that exactly one of the two following statements holds.

$$\begin{aligned}
&(1)\quad \Phi(\xi,\eta_0)=\Phi(\xi_0,\eta_0)=\text{span}[\zeta_0],\quad\text{and}\\
&\qquad\ \Psi(\xi,\eta_0)\neq\Psi(\xi_0,\eta_0)=\text{span}[\boldsymbol{\psi}(\eta_0)],\\
&(2)\quad \Phi(\xi,\eta_0)\neq\Phi(\xi_0,\eta_0)=\text{span}[\zeta_0],\quad\text{and}\\
&\qquad\ \Psi(\xi,\eta_0)=\Psi(\xi_0,\eta_0)=\text{span}[\boldsymbol{\psi}(\eta_0)].
\end{aligned}\tag{5.4}$$

Similarly, for any $\eta\in K\backslash\{0\}$, we have one of the two:

$$\begin{aligned}
&(1)\quad \Phi(\xi,\eta)=\Phi(\xi_0,\eta)=\text{span}[\zeta_0],\quad\text{and}\\
&\qquad\ \Psi(\xi,\eta)\neq\Psi(\xi_0,\eta)=\text{span}[\boldsymbol{\psi}(\eta)],\\
&(2)\quad \Phi(\xi,\eta)\neq\Phi(\xi_0,\eta)=\text{span}[\zeta_0],\quad\text{and}\\
&\qquad\ \Psi(\xi,\eta)=\Psi(\xi_0,\eta)=\text{span}[\boldsymbol{\psi}(\eta)].
\end{aligned}\tag{5.5}$$

We show that (5.4)(1) ⇔ (5.5)(1), and (5.4)(2) ⇔ (5.5)(2). Indeed, suppose (5.4)(1) and (5.5)(2) hold. Then $\Phi(\xi,\eta_0)\neq\Phi(\xi,\eta)$, and $\Psi(\xi,\eta_0)\neq\Psi(\xi,\eta)$, which contradicts Lemma 5.6. Similarly, (5.4)(2) is incompatible with (5.5)(1). □

Lemma 5.9 *Suppose there exist norm-one vectors $\xi_0\in K$, $\zeta_0\in H$, and a linear isomorphism $\boldsymbol{\psi}$ from K into H, so that, for every $\eta\in K$, $S(\omega_{\xi_0,\eta})=\omega_{\zeta_0,\boldsymbol{\psi}(\eta)}$. Then exactly one of the two statements holds*:

(A) *For any* $\xi, \eta \in K$, *with* $\xi_0 \nsim \xi$, $\Phi(\xi, \eta) = \mathrm{span}[\zeta_0]$, *and* $\Psi(\xi, \eta) \neq \mathrm{span}[\boldsymbol{\psi}(\eta)]$.
(B) *For any* $\xi, \eta \in K$, *with* $\xi_0 \nsim \xi$, $\Phi(\xi, \eta) \neq \mathrm{span}[\zeta_0]$, *and* $\Psi(\xi, \eta) = \mathrm{span}[\boldsymbol{\psi}(\eta)]$.

Proof Pick $\xi_1 \in K$, not collinear to ξ_0. By the dichotomy from Lemma 5.8, one of the following two statements holds:

(A′) For any $\eta \in K$, $\Phi(\xi_1, \eta) = \mathrm{span}[\zeta_0]$, and $\Psi(\xi_1, \eta) \neq \mathrm{span}[\boldsymbol{\psi}(\eta)]$.
(B′) For any $\eta \in K$, $\Phi(\xi_1, \eta) \neq \mathrm{span}[\zeta_0]$, and $\Psi(\xi_1, \eta) = \mathrm{span}[\boldsymbol{\psi}(\eta)]$.

We shall prove first that (A′) implies (A). Suppose, for the sake of contradiction, that the statement of (A) is false for some non-zero $\overline{\xi}$ and $\overline{\eta}$ in K, with $\xi_0 \nsim \overline{\xi}$. That is, $\Phi(\overline{\xi}, \overline{\eta}) \neq \mathrm{span}[\zeta_0]$, or $\Psi(\overline{\xi}, \overline{\eta}) = \mathrm{span}[\boldsymbol{\psi}(\eta)]$. Under these conditions, we may assume that $\overline{\xi} \nsim \xi_1$ and $\overline{\xi} \nsim \xi_0$.

Having in mind our assumptions and applying Lemma 5.6(ii) to ξ_0, $\overline{\xi}$, and $\overline{\eta}$, we obtain

$$\Phi(\overline{\xi}, \overline{\eta}) \neq \mathrm{span}[\zeta_0], \quad \text{and} \quad \Psi(\overline{\xi}, \overline{\eta}) = \mathrm{span}[\boldsymbol{\psi}(\overline{\eta})].$$

Doing the same for ξ_1, $\overline{\xi}$, and $\overline{\eta}$, we see that

$$\Phi(\overline{\xi}, \overline{\eta}) \neq \Phi(\xi_1, \overline{\eta}) = \mathrm{span}[\zeta_0], \quad \text{and} \quad \Psi(\overline{\xi}, \overline{\eta}) = \Psi(\xi_1, \overline{\eta}).$$

Thus, $\Psi(\xi_1, \overline{\eta}) = \mathrm{span}[\boldsymbol{\psi}(\overline{\eta})]$, which contradicts (A′).

Similarly, (B') implies (B). If, for some $\overline{\xi}$ and $\overline{\eta}$ in K, (B) is false, then

$$\Phi(\overline{\xi}, \overline{\eta}) = \mathrm{span}[\zeta_0], \quad \text{and} \quad \Psi(\overline{\xi}, \overline{\eta}) \neq \mathrm{span}[\boldsymbol{\psi}(\overline{\eta})].$$

Therefore $\Phi(\overline{\xi}, \overline{\eta}) \neq \Phi(\xi_1, \overline{\eta})$, and $\Psi(\overline{\xi}, \eta) \neq \Psi(\xi_1, \overline{\eta})$. This is ruled out by Lemma 5.6(i). $\square$

Below, we denote by $\mathcal{F}_{(k)}(H_1, H_2)$ the set of operators from H_1 to H_2, of rank not exceeding k.

Proof of Proposition 5.5 Pick norm 1 vectors $\xi_0, \eta_0 \in K$. By Lemma 5.6, Lemma 5.7 and the comments following the latter, there exists a norm one $\zeta_0 \in H$ for which one of the following two statement holds:

(1) $\Phi(\xi_0, \eta) = \mathrm{span}[\zeta_0]$ for any $\eta \in K$, and $\Psi(\xi_0, \eta_1) \neq \Psi(\xi_0, \eta_2)$ whenever $\eta_1 \nsim \eta_2$.
(2) $\Psi(\xi_0, \eta) = \mathrm{span}[\zeta_0]$ for any $\eta \in K$, and $\Phi(\xi_0, \eta_1) \neq \Phi(\xi_0, \eta_2)$ whenever $\eta_1 \nsim \eta_2$.

We consider (1), as (2) is dealt with in the same manner. We shall show that, in this case, either (a) or (c) of Proposition 5.5 holds. Note that $S(\omega_{\xi_0,\eta}) = \omega_{\zeta_0,\boldsymbol{\psi}(\eta)}$, where $\boldsymbol{\psi} : K \to H$ is a (well defined) linear map, with

$$\|\eta\| \leq \|\boldsymbol{\psi}(\eta)\| \leq C\|\eta\|.$$

Lemma 5.9 provides us with the following dichotomy:

(1.1) For any $\xi, \eta \in K$, $\Phi(\xi, \eta) = \text{span}[\zeta_0]$.
(1.2) For any $\xi, \eta \in K$, $\Phi(\xi, \eta) \neq \text{span}[\zeta_0]$ unless $\xi \sim \xi_0$, and $\Psi(\xi, \eta) = \text{span}[\boldsymbol{\psi}(\eta)]$.

If (1.1) holds, clearly S maps $\mathcal{F}_{(1)}(K)$ into $\zeta_0 \otimes H_0$, where $H_0 = \boldsymbol{\psi}(K)$ is a closed subspace of H. By the density of the linear span of $\mathcal{F}_{(1)}(K)$ in $\mathcal{S}^p(K)$, $\mathcal{A} = S(\mathcal{S}^p(K)) = \zeta_0 \otimes H_0$, and Statement (a) of the proposition is true.

Now suppose (1.2) holds. In this case, write $S(\omega_{\xi_0,\eta}) = \omega_{\zeta_0,\boldsymbol{\psi}(\eta)}$. Combining Lemma 5.6 and Lemma 5.7 (applied to $\xi \nsim \xi_0$, η_1, and η_2 with $\eta_1 \nsim \eta_2$), we have

$$\Phi(\xi, \eta_1) = \Phi(\xi, \eta), \quad \text{for every } \eta \in K$$

or

$$\Psi(\xi, \eta_1) = \Psi(\xi, \eta), \quad \text{for every } \eta \in K.$$

When the second statement holds, it follows, by (1.2), that $\boldsymbol{\psi}(\eta_1) \sim \boldsymbol{\psi}(\eta)$ for every $\eta \in K$. We deduce that $\boldsymbol{\psi}$ is a rank-one operator, and hence K is one dimensional, this is covered by case (1.1). Therefore, we may assume that

$$\Phi(\xi, \eta_1) = \Phi(\xi, \eta_2), \quad \text{for every } \eta_1, \eta_2 \text{ and } \xi \text{ in } K\backslash\{0\}. \tag{5.6}$$

We have to show that, in this situation, (c) is true.

First, we define a linear isomorphism $\boldsymbol{\phi} : K \to H_2$ (where H_2 is a closed subspace of H) with the property that

$$\text{span}[S(\omega_{\xi,\eta})] = \text{span}[\omega_{\boldsymbol{\phi}(\xi),\boldsymbol{\psi}(\eta)}] \quad \text{for any } \xi, \eta \in K. \tag{5.7}$$

Set $\boldsymbol{\phi}(0) = 0$. For $\xi \in K\backslash\{0\}$, there exists a unique $\boldsymbol{\phi}(\xi) \in H$ so that $(\boldsymbol{\phi}(\xi), \boldsymbol{\psi}(\eta_0)) \in \Gamma(\xi, \eta_0)$ (see (5.1) and the remarks following it). In other words, $\boldsymbol{\phi}(\xi)$ is the unique element $h \in H$ with the property that $S(\omega_{\xi,\eta_0}) = \omega_{h,\boldsymbol{\psi}(\eta_0)}$. Note that the map $\xi \mapsto S(\omega_{\xi,\eta_0})$ is linear, hence $\boldsymbol{\phi}$ is also linear. Furthermore, $\|\boldsymbol{\psi}(\eta_0)\| \in [1, C]$, and

$$\|\boldsymbol{\phi}(\xi)\| \|\boldsymbol{\psi}(\eta_0)\| = \|S(\omega_{\xi,\eta_0})\| \in \big[\|\xi\|, C\|\xi\|\big],$$

hence $C^{-1}\|\xi\| \leq \|\boldsymbol{\phi}(\xi)\| \leq C\|\xi\|$. In particular, $H_2 = \boldsymbol{\phi}(K)$ is a closed subspace of H, and $\boldsymbol{\phi} : K \to H_2$ is a linear isomorphism.

Now consider arbitrary non-zero $\xi, \eta \in K$. By (5.6) and Condition (1.2),

$$\Phi(\xi, \eta) = \Phi(\xi, \eta_0) = \text{span}[\boldsymbol{\phi}(\xi)], \quad \text{and}$$
$$\Psi(\xi, \eta) = \text{span}[\boldsymbol{\psi}(\eta)] \quad \text{for every } \eta, \xi \in K.$$

Then, for every $(h, k) \in \Gamma(\xi, \eta)$, there exist non-zero $\lambda, \mu \in \mathbb{C}$ so that $h = \lambda\boldsymbol{\phi}(\xi)$, and $k = \mu\boldsymbol{\psi}(\eta)$. Therefore,

$$S(\omega_{\xi,\eta}) = \omega_{h,k} = \lambda\overline{\mu}\omega_{\boldsymbol{\phi}(\xi),\boldsymbol{\phi}(\eta)}.$$

This establishes (5.7). Thus, for any $\xi, \eta \in K\backslash\{0\}$, there exists a unique $\gamma(\xi, \eta) \in \mathbb{C}\backslash\{0\}$ satisfying

$$S(\omega_{\xi,\eta}) = \gamma(\xi, \eta)\omega_{\boldsymbol{\phi}(\xi),\boldsymbol{\psi}(\eta)}.$$

We define $\gamma(0,\eta) = \gamma(\xi,0) = 0$. For $\xi,\eta \in K\backslash\{0\}$, we have

$$|\gamma(\xi,\eta)| = \frac{\|S(\omega_{\xi,\eta})\|_p}{\|\boldsymbol{\phi}(\xi)\|\|\boldsymbol{\psi}(\eta)\|}.$$

As noted above, $\|\boldsymbol{\phi}(\xi)\| \in [C^{-1}\|\xi\|, C\|\xi\|]$, and $\|\boldsymbol{\phi}(\xi)\| \in [\|\eta\|, C\|\eta\|]$, while $\|S(\omega_{\xi,\eta})\|_p \in [\|\xi\|\|\eta\|, C\|\xi\|\|\eta\|]$. Therefore,

$$C^{-2}\|\xi\|\|\eta\| \leq |\gamma(\xi,\eta)|\|\boldsymbol{\phi}(\xi)\|\|\boldsymbol{\psi}(\eta)\| \leq C^2\|\xi\|\|\eta\|.$$

Thus, S maps $\mathcal{F}_{(1)}(K)$ onto $\mathcal{F}_{(1)}(H_1, H_2)$, where $H_1 = \boldsymbol{\psi}(K)$ and $H_2 = \boldsymbol{\phi}(K)$. Thus, $\mathcal{A} = \mathcal{S}^p(H_1, H_2)$. By [23, Theorem 3.3], there exist invertible operators $U \in B(H_2, K)$, $V \in B(K, H_1)$, $\widetilde{U} \in B(H_2, K^*)$ and $\widetilde{V} \in B(K^*, H_1)$, such that either $S(\phi) = J_1 V \phi U p_2$ for every ϕ in $\mathcal{S}^p(K)$, or $S(\phi) = J_1 \widetilde{V} \phi^t \widetilde{U} p_2$ for every $\phi \in \mathcal{S}^p(H)$. Here, ϕ^t denotes the transpose of ϕ, p_2 is the orthogonal projection of H onto H_2, and J_1 denotes the canonical injection of H_1 into H. □

Proof of Theorem 5.1 Suppose $T : \mathcal{S}^p(H) \to \mathcal{S}^p(K)$ is an orthogonality preserving linear bijection. By Theorem 4.1, T is an isomorphism. By Corollary 5.4, $T^{-1} : \mathcal{S}^p(K) \to \mathcal{S}^p(H)$ takes rank 1 operators to rank 1 operators. By Proposition 5.5, there exist invertible $S, R \in B(H, K)$, $\widetilde{S}, \widetilde{R} \in B(H^*, K)$, such that either $T(\phi) = R\phi S^*$ for every $\phi \in \mathcal{S}^p(H)$, or $T(\phi) = \widetilde{R}\phi^t \widetilde{S}^*$ for every $\phi \in \mathcal{S}^p(H)$.

We have to show that R and S (resp., $\widetilde{S}, \widetilde{R}$) are multiples of unitaries. By the famous theorem of U. Uhlhorn (see [6] for a recent generalization), it suffices to show that $(R(\xi)|R(\eta)) = (S^*(\xi)|S^*(\eta)) = 0$ whenever $(\xi|\eta) = 0$. We consider the case of $T(\phi) = R\phi S^*$ (the other possibility is handled in a similar manner). If $\xi \perp \eta$, then $\phi = \omega_\xi$ and $\psi = \omega_\eta$ are orthogonal in $\mathcal{S}^p(H)$. Then $T(\phi) = \omega_{R(\xi), S^*(\xi)}$ and $T(\psi) = \omega_{R(\eta), S^*(\eta)}$ are also orthogonal, which leads to the desired conclusion concerning R and S. □

We have already seen that the notion of orthogonality can be also considered in the predual of a general (non necessarily tracial) von Neumann algebra. We shall explore now the automatic continuity of those linear bijections between von Neumann algebra preduals which are orthogonality preserving.

Theorem 5.10 *Let A be a compact C^*-algebra, and N a von Neumann algebra. Then every orthogonality preserving linear bijection $T : A^* \to N_*$ is continuous. In this situation, N is isometric to the second dual of A (as a von Neumann algebra).*

We start by showing that orthogonality preserving maps between preduals of von Neumann algebras "respect" central projections. For the sake of brevity, we use the notation $e^\perp$ for $\mathbf{1} - e$.

Lemma 5.11 *Suppose T is a orthogonality preserving linear bijection from N_* to M_*, where N and M are von Neumann algebras. Then, for any central projection e in N, there exists a central projection f in M, so that T maps N_*e and $N_*e^\perp$ onto M_*f and $M_*f^\perp$, respectively.*

Proof Let $X_1 = T(N_* e)$ and $X_2 = T(N_* e^\perp)$. As T preserves orthogonality, $x_1 \perp x_2$ whenever $x_1 \in X_1$ and $x_2 \in X_2$. For $k = 1, 2$, let $l_k = \bigvee_{x_k \in X_k} l(x_k)$ and $r_k = \bigvee_{x_k \in X_k} r(x_k)$. By the above, $l_1 \perp l_2$, and $r_1 \perp r_2$. Furthermore, $X_k = l_k X_k r_k$. By the bijectivity of T, $M_* = X_1 \oplus_1 X_2$. Therefore, $l_1 + l_2 = \mathbf{1}$. Indeed, if $l_1 + l_2 < \mathbf{1}$, then $(l_1 + l_2) M_* \neq M_* = X_1 \oplus_1 X_2$, a contradiction. Similarly, $r_1 + r_2 = \mathbf{1}$.

We shall show that $l_1 = r_1$, $l_2 = r_2$, and that these projections are central. Note that we have $l_1 M_* r_2 = l_2 M_* r_1 = 0$, hence, by duality, $l_1 M r_2 = l_2 M r_1 = 0$. In particular, no subprojection of l_1 (resp., l_2) is equivalent to a subprojection of r_2 (resp., r_1). By [31, Lemma V.1.7], $\mathbf{c}(l_1) \perp \mathbf{c}(r_2)$, and $\mathbf{c}(l_2) \perp \mathbf{c}(r_1)$ (here and below, $\mathbf{c}(p)$ denotes the central cover of a projection p). Thus,

$$2 \cdot \mathbf{1} = \big(l_1 + l_2\big) + \big(r_1 + r_2\big) \leq \big(\mathbf{c}(l_1) + \mathbf{c}(r_2)\big) + \big(\mathbf{c}(l_2) + \mathbf{c}(r_1)\big) \leq 2 \cdot \mathbf{1}.$$

Thus, we have equality in the centered expression, which can only happen if $l_k = \mathbf{c}(l_k)$ and $r_k = \mathbf{c}(r_k)$ for $k = 1, 2$. We prove next that $r_1 \leq l_1$. Indeed, by [31, Theorem III.4.2(i)], $l(x) \sim r(x)$ for any x. Therefore, for any $x \in X_1$, $r(x) \leq \mathbf{c}(r(x)) = \mathbf{c}(l(x)) \leq l_1$. Thus, $r_1 \leq l_1$. But the converse inequality is also true, hence $l_1 = r_1$. Similarly, $l_2 = r_2$. □

Corollary 5.12 *Suppose T is a orthogonality preserving continuous linear bijection from $N_* = (\bigoplus_{i \in I} \mathcal{S}^1(H_i))_{\ell_1}$ to $M_* = (\bigoplus_{j \in J} \mathcal{S}^1(K_j))_{\ell_1}$. Then, for any $i \in I$, there exists a set $J(i) \subset J$ so that T maps $\mathcal{S}^1(H_i)$ onto $(\bigoplus_{j \in J(i)} \mathcal{S}^1(K_j))_{\ell_1}$.*

Proof By Lemma 5.11, for every $i \in I$ there exists a non-zero central projection $f_i \in M = (\bigoplus_{j \in J(i)} B(K_j))_{\ell_\infty}$ so that $T(\mathcal{S}^1(H_i)) = f_i M_*$. Furthermore, the projections $\{f_i\}_{i \in I}$ are mutually orthogonal. We complete the proof by observing that the central projections in M are precisely the "coordinate" projections. □

We need one more technical lemma.

Lemma 5.13 *If $1 \leq p < \infty$, and $\phi, \psi \in \mathcal{S}^p(K)$ are such that, for every $\alpha > 0$, $\alpha\phi + \psi$ is orthogonal to $\phi - \alpha\psi$, then $\phi = \psi = 0$.*

Proof Suppose ϕ and ψ are as in the statement of the lemma—that is, for any $\alpha > 0$,

$$\big(\alpha\phi + \psi\big)^*\big(\phi - \alpha\psi\big) = \psi^*\phi + \alpha\big(\phi^*\phi - \psi^*\psi\big) - \alpha^2\phi^*\psi = 0,$$
$$\big(\alpha\phi + \psi\big)\big(\phi - \alpha\psi\big)^* = \psi\phi^* + \alpha\big(\phi\phi^* - \psi\psi^*\big) - \alpha^2\phi\psi^* = 0.$$

Comparing the coefficients of different powers of α, we conclude that $\psi^*\phi = \psi\phi^* = 0$ (that is, $\phi \perp \psi$), and $\phi^*\phi = \psi^*\psi$. Multiplying both sides of the last equality by ϕ on the left, we obtain $\phi\phi^*\phi = (\phi\psi^*)\psi = 0$. However, $\|\phi\phi^*\phi\|_\infty = \|\phi\|_\infty^3$ (here, $\|\cdot\|_\infty$ denotes the usual operator norm), hence $\phi = 0$. Similarly, $\psi = 0$. □

Proof of Theorem 5.10 As discussed above, T is O-1-O preserving. An application of Corollary 4.2 shows that T is continuous. Thus, T is an isomorphism. If N is not discrete, then, by [27, Lemma], it contains a subalgebra, isometric to $L_\infty(0, 1)$, which is the range of a weak* continuous conditional expectation. Thus, N_* contains an isometric copy of $L_1(0,1)$. However, since $A^* = (\bigoplus_{i\in I} \mathcal{S}^1(H_i))_{\ell_1}$ is isomorphic to a complemented subspace of some $\mathcal{S}^1(K)$, for a suitable Hilbert space K, by the comments preceding Theorem 6 in [4] (or by [15, Lemma 6.4]), A^* does not contain $L_1(0,1)$ isomorphically.

Thus, $A^* = (\bigoplus_{i\in I} \mathcal{S}^1(H_i))_{\ell_1}$, and $N_* = (\bigoplus_{j\in J} \mathcal{S}^1(K_j))_{\ell_1}$. By Corollary 5.12, for any $i \in I$ there exists $J(i) \subset J$ so that T maps $\mathcal{S}^1(H_i)$ onto $(\bigoplus_{j\in J(i)} \mathcal{S}^1(K_j))_1$. It remains to show that, for any i, $J(i)$ is a singleton. Once this is established, we conclude that $\dim H_i = \dim K_{j(i)}$, where $\{j(i)\} = J(i)$. Therefore, A^{**} and N are isomorphic as von Neumann algebras.

For the sake of brevity, write H instead of H_i, and $\mathcal{U}$ instead of $J(i)$. Suppose, for the sake of contradiction, that $\mathcal{U}$ is not a singleton. For $j \in \mathcal{U}$, let $X_j = T^{-1}(\mathcal{S}^1(K_j)) \subset \mathcal{S}^1(H)$. For a set $\mathcal{V} \subset \mathcal{U}$, denote by $Q_{\mathcal{V}}$ the "coordinate" projection from $(\bigoplus_{j\in\mathcal{U}} \mathcal{S}^1(K_j))_{\ell_1}$ onto $(\bigoplus_{j\in\mathcal{V}} \mathcal{S}^1(K_j))_{\ell_1}$. Then $P_{\mathcal{V}} = T^{-1} Q_{\mathcal{V}} T$ is a projection from $\mathcal{S}^1(H)$ onto $\mathrm{span}[X_j : j \in \mathcal{V}]$. For singletons, we use the notation Q_j and P_j instead of $Q_{\{j\}}$ and $P_{\{j\}}$, respectively.

Now fix $j \in \mathcal{U}$, and let $\mathcal{V} = \mathcal{U}\backslash\{j\}$. By Corollary 5.4, T is rank-nondecreasing, hence $T^{-1}|_{\mathcal{S}^1(K_j)}$ preserves rank 1 elements. Applying Proposition 5.5 to $T^{-1}|_{\mathcal{S}^1(K_j)}$, we see that $X_j = T^{-1}(\mathcal{S}^1(K_j)) = \mathcal{S}^1(H_2, H_1)$, where H_1 and H_2 are closed subspaces of H, at least one of them proper. Without loss of generality, assume that H_1 is a proper subspace of H. Now, note that $P_{\mathcal{V}} + P_j = I$ (the identity on $\mathcal{S}^1(H)$), hence $\ker P_{\mathcal{V}} = X_1$.

Pick norm one vectors $\xi_2 \in H_1$ and $\eta_2 \in H_2$. Furthermore, pick norm 1 $\xi_1 \in H\backslash H_1$, and $\eta_1 \in H$, so that $(\xi_1|\xi_2) = (\eta_1|\eta_2) = 0$. Let

$$\phi_1 = \omega_{\xi_1,\eta_1}, \qquad \phi_2 = \omega_{\xi_2,\eta_2}, \qquad \text{and} \qquad \phi_0 = \omega_{\xi_1,\eta_2} + \omega_{\xi_2,\eta_1}.$$

A direct calculation shows that, for any $\alpha > 0$,

$$\alpha^2\phi_1 + \alpha\phi_0 + \phi_2 = \omega_{\alpha\xi_1+\xi_2,\alpha\eta_1+\eta_2}$$

and

$$\phi_1 - \alpha\phi_0 + \alpha^2\phi_2 = \omega_{\xi_1-\alpha\xi_2,\eta_1-\alpha\eta_2}.$$

Note that $(\alpha\xi_1 + \xi_2|\xi_1 - \alpha\xi_2) = (\alpha\eta_1 + \eta_2|\eta_1 - \alpha\eta_2) = 0$, hence

$$\left(\alpha^2\phi_1 + \alpha\phi_0 + \phi_2\right) \perp \left(\phi_1 - \alpha\phi_0 + \alpha^2\phi_2\right).$$

Now note that $Q_{\mathcal{V}}T = TP_{\mathcal{V}}$ preserves orthogonality. Furthermore, ϕ_2 lies in $X_j = \ker P_{\mathcal{V}}$, while $\phi_1 \notin \ker P_{\mathcal{V}}$. Therefore,

$$\left(\alpha^2 Q_{\mathcal{V}}T(\phi_1) + \alpha Q_{\mathcal{V}}T(\phi_0)\right) \perp \left(Q_{\mathcal{V}}T(\phi_1) - \alpha Q_{\mathcal{V}}T(\phi_0)\right)$$

for any $\alpha > 0$. Lemma 5.13 implies that $Q_{\mathcal{V}}T(\phi_1) = Q_{\mathcal{V}}T(\phi_0) = 0$. In particular, $TP_{\mathcal{V}}(\phi_1) = Q_{\mathcal{V}}T(\phi_1) = 0$, which, by the injectivity of T, gives $P_{\mathcal{V}}(\phi_1) = 0$ and we get the desired contradiction. □

In this paper, we do not consider L^p spaces arising from non-tracial von Neumann algebras. In the tracial case, a version of Theorem 5.10 holds for $p \in [1, \infty)$.

Theorem 5.14 *Suppose M_1 and M_2 are von Neumann algebras, equipped with normal faithful semi-finite traces τ_1 and τ_2, respectively. Suppose, furthermore, that M_1 is discrete, $p \in [1, \infty)$, and $T : L^p(\tau_1) \to L^p(\tau_2)$ is an orthogonality preserving linear bijection. Then T is continuous, and M_1 is isomorphic to M_2 as a von Neumann algebra.*

Proof (Sketch) We proceed as in Theorem 5.10. The only difference is that, for $p \neq 1$, we use a different method of proving that M_2 is discrete. If M_2 is not discrete, we can use e.g. [31, Proposition V.1.35] to show that $L^p(\tau_2)$ contains a copy of $L^p(0, 1)$. The latter space is not contained in $L^p(\tau_1)$, by [4, Theorem 6]. □

Remark 5.15 Equip the von Neumann algebra $M = (\bigoplus_{i\in I} B(H_i))_{\ell_\infty}$ with its canonical trace $\tau = \bigoplus_i \mathrm{tr}_i$, where tr_i is the usual trace on $B(H_i)$. Consider $p \in [1, \infty)$, and a von Neumann algebra M'. For $p \neq 1$, we assume that M' is equipped with a normal faithful semi-finite trace τ'. Suppose T is an orthogonality preserving bijection from $L^p(\tau)$ to $L^p(\tau')$ (to M'_*, if $p = 1$). By Theorems 5.10 and 5.14, T is an isomorphism. Moreover, M' can be identified with $(\bigoplus_{i\in I} B(H_i))_{\ell_\infty}$, and $\tau' = \bigoplus_{i\in I} c_i \mathrm{tr}_i$, with $c_i > 0$. Furthermore, there exist $\gamma_i \in [c_i^{-1/p}\|T^{-1}\|^{-1}, c_i^{-1/p}\|T\|]$, and unitaries $U_i, V_i \in B(H_i)$ $(i \in I)$, and $J \subset I$, so that, for $\phi = (\phi_i)_{i\in I})$, we have $T(\phi) = (\gamma_i U_i \tilde{\phi}_i V_i)_{i\in I}$. Here, $\tilde{\phi}_i = \phi_i$ for $i \in J$, and $\tilde{\phi}_i = \phi_i^t$ for $i \in I \backslash J$. Thus, $T(\phi) = \gamma \cdot \alpha_p(\phi)$, where $\gamma = (\gamma_i \mathbf{1}_{H_i})_{i\in I}$ is an invertible element of the center of M', and α_p arises from the triple isomorphism $\alpha : M \to M'$, $(\phi_i)_{i\in I} \mapsto (U_i \tilde{\phi}_i V_i)_{i\in I}$. Similar results for orthogonality preservers on C*-algebras were obtained by the second author and his co-authors in [7, 8].

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