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On the Range of a Vector Measure

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Abstract. Let C be a countable subset of c_0 that lies in the range of an $(c_0)^{**}$-valued measure, then C lies in the range of a c_0-valued measure. We extend this result to $C(K)$, where K is a compact Hausdorff space, i.e., we let C be a countable subset of $C(K)$ that lies in the range of a $C(K)^{**}$-valued measure, then C lies in the range of a $C(K)$-valued measure. We will also see that in any separable Banach space the result still holds.

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1. Introduction

In [10], Professor A. Sofi asks (Problem 6) to which Banach spaces X is it so that if C is a countable subset of X that lies in the range of a countably additive X^{**}-valued measure with a σ-field domain, then there is a countably additive X-valued measure with a σ-field domain, whose range also contains C. Of course, if X is complemented in X^{**}, then the answer is plain and easy. In this note we show that if X is c_0, X is a separable Banach space or X is a $C(K)$-space for a compact Hausdorff space K, then any countable subset C of X that lies in the range of an X^{**}-valued countably additive measure on a σ-field lies in the range of an X-valued countably additive measure on the same σ-field. The "techniques" are all Banach space techniques. We never called on a change of domain.

This problem is related to the following considered by F.J. Freniche (see [3]): given a vector measure μ with values in the bidual X^{**} of the Banach space X, under what conditions can we say μ actually takes its values inside X? The author shows that if X is a Banach space such that its dual closed unit ball is weak-star

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sequentially compact and if the X^{**} measure μ satisfies the Geitz's condition, then its range is contained in X. Therefore, if X is a Banach space whose dual unit ball is weak-star sequentially compact and if C is a countable subset of X that lies in the range of an X^{**}-valued measure that verifies the Geitz's condition, then C lies in the range of an X-valued measure on the same sigma algebra.

The closely related bounded variation version of the problem as treated in this paper is already known to have been resolved in the negative. A counterexample to this effect was constructed by B. Marchena and C. Pineiro (see [6]), and later improved by M.A. Sofi (see [10]).

2. Preliminaries

We will denote the support of a function by supp(f) where

$$\text{supp}(f) = \overline{\{\gamma \in \Gamma : f(\gamma) \neq 0\}},$$

where Γ is any index set. Let X and Y denote Banach spaces, and let $F : \Sigma \to X$ denote the vector measure defined on the σ-field Σ.

Let $\mathcal{K}$ denote a scalar field. Then we let $c_0(\Gamma)$ consist of all the functions $f : \Gamma \to \mathcal{K}$, such that for every $\epsilon > 0$,

$$\{\gamma \in \Gamma : |f(\gamma)| \geq \epsilon\}$$

is finite. Note that each $f \in c_0(\Gamma)$ has countable support and is bounded and that

$$\|f\|_{c_0(\Gamma)} = \|f\|_\infty.$$

If $\Gamma = \mathbb{N}$, we use the usual notation and write c_0, which consists of all sequences converging to zero. Likewise, l_∞ consists of all bounded sequences of scalars.

We say that a set A lies in the range of a Y-valued measure if there is a σ-field Σ and a countably additive measure $F : \Sigma \to Y$ so that $A \subseteq F(\Sigma)$. A set C is *absolutely convex* if and only if for any points x_1 and x_2 in C, and any numbers λ_1 and λ_2 satisfying $|\lambda_1| + |\lambda_2| \leq 1$, the sum $\lambda_1 x_1 + \lambda_2 x_2$ belongs to C. Since the intersection of any collection of absolutely convex sets is a convex set, then for any subset A we denote the *absolutely convex hull* to be the intersection of all absolutely convex sets containing A. A space is *injective* if every isomorphic embedding of it in an arbitrary Banach space Y is the range of a bounded linear projection defined on Y. A topological space is *zero-dimensional*, if its topological dimension is zero, or equivalently, if it has a base consisting of clopen sets. A Banach space is *weakly compactly generated* whenever it is the closed linear span of one of its weakly compact subsets. We use the same notation as in [2].

We will need the following theorems in order to prove the main results. The proof of the first theorem can be found in [2], page 14.

Theorem 2.1 (Bartle-Dunford-Schwartz). *Let $F : \Sigma \to X$ be a countably additive vector measure on a σ-field Σ. Then the range of F is relatively weakly compact.*

Theorem 2.2 (Rosenthal). *Any weakly compact subset of ℓ_∞ is norm separable.*

Sobczyk (see [9]) proved the following theorem that states that c_0 is complemented in every separable space in which it resides, although the norm of the projection does not need to be equal to one.

Theorem 2.3 (Sobczyk). *If X is a separable Banach space and $Y \subseteq X$ is a closed subspace isometric to c_0, then there is a continuous linear projection P from X onto Y with $\|P\| \leq 2$.*

The proof of the following theorem can be found in [1].

Theorem 2.4 (Amir and Lindenstrauss). *Let X be a weakly compactly generated Banach space. If X_0 is a separable subspace of X and Y_0 is a separable subspace of X^*, then there is a projection $P : X \to X$ whose range is separable such that $X_0 \subseteq P(X)$ and $Y_0 \subseteq P^*(X^*)$.*

A proof of the following two theorems can be found in [8] and [7].

Theorem 2.5 (Miljutin). *If K is an uncountable compact metric space, then the space $C(K)$ is isomorphic to $C(\Delta)$, where Δ denotes the Cantor discontinuum.*

Theorem 2.6 (Pełczynski).

1. *Let K be a zero-dimensional compact metric space. If a separable Banach space X contains a subspace Y that is isometrically isomorphic to $C(K)$, then there are a subspace Z of Y and a projection $P : X \to Z$ (onto Z) such that Z is isometrically isomorphic to $C(K)$ and $\|P\| = 1$.*
2. *Let K be a compact metric space. If a separable Banach space X contains a subspace Y that is isomorphic to $C(K)$, then there is a subspace Z of Y such that Z is isomorphic to $C(K)$ and Z is complemented in X.*

3. Main results

Section I

Theorem 3.1. *If C is a countable subset of c_0 that lies in the range of an c_0^{**}-valued measure, then C lies in the range of a c_0-valued measure.*

Proof. Let $F : \Sigma \to \ell_\infty$ be a countably additive measure defined on the σ-field Σ such that $C \subseteq F(\Sigma)$. By 2.1 we have that $F(\Sigma)$ is a relatively weakly compact subset of ℓ_∞. By 2.2 we have that F has a norm separable range. So $F(\Sigma)$ generates a separable closed linear subspace of ℓ_∞; we can enlarge our set by letting X be the closed linear span (in ℓ_∞) of $F(\Sigma) \cup c_0$. In so doing, we obtain a separable Banach space X that contains C, itself a subset of c_0. Now we are in a position to call on 2.3: The result is a bounded linear projection P from X onto c_0. We also have that $C \subseteq P \circ F(\Sigma)$ and $P \circ F : \Sigma \to c_0$ is a c_0-valued measure whose range contains C. □

Theorem 3.2. *If C is a countable subset of c_0 and $C \subseteq F(\Sigma)$, where $F : \Sigma \to X$, for F a countably additive vector measure mapping from a σ-field Σ to a Banach space*

X containing c_0. Then there exists a countably additive vector measure $G : \Sigma \to c_0$ such that $C \subseteq G(\Sigma)$.

Proof. Let $W = \overline{\text{span}}[F(\Sigma) \cup c_0]$. Then $W \subset X$ and is weakly compactly generated containing c_0. By Theorem 2.4 there exists a continuous projection $P : W \to W$ such that $c_0 \subset P(W) = S$, with S separable. Now Theorem 2.3 yields a projection $Q : S \to S$ with $\|Q\| \le 2$ so that $Q \circ S = c_0$. Finally, the set function

$$G = Q \circ P \circ F : \Sigma \to c_0$$

defines a vector measure such that $C \subset range(G)$. □

Section II

Theorem 3.3. *Let S be a separable Banach space, and suppose C is a countable subset of S that lies in the range of an S^{**}-valued measure. Then C lies in the range of a S-valued measure.*

Proof. Let $F : \Sigma \longrightarrow S^{**}$ be a countable additive measure on the σ-field Σ such that $C \subseteq F(\Sigma)$. By Theorem 2.1 $F(\Sigma)$ is weakly compact. Let K denote the absolutely convex hull of $F(\Sigma)$ in S^{**}; the set K is the unit ball of its linear span Y_K, where Y_K is a Banach space, contained (as linear subspace) in S^{**}. By Theorem 2.4 (see [1]) we can assume that Y_K is separable in the S^{**} norm topology. Moreover, the inclusion map

$$Y_K \hookrightarrow S^{**}$$

is weakly compact and has separable range. A close look at the still marvelous factorization scheme of Davis, Figiel, Johnson and Pelczynski tell us that there is a separable reflexive Banach space R and bounded linear operators (with $\|a\| \le 1$)

$$a : Y_K \longrightarrow R \quad , \quad b : R \longrightarrow S^{**}$$

so that $b \circ a$ <u>is</u> the inclusion $Y_K \hookrightarrow S^{**}$ of Y_K into S^{**}.

Since X is S is separable, let $\mathcal{D} = \{x_n^* \ , \ n \in \mathbb{N}\}$ be a weak*-dense sequence consisting of linearly independent elements of S^*. Let $\mathcal{F}$ be a countable $\|\cdot\|$ dense subset of Y_K ($\|\cdot\|$ is the norm in S^{**}) and look at $\mathcal{F} \cup \mathcal{C} = \{z_n^{**} \ , \ n \in \mathbb{N}\}$, which lies inside Y_K. Let $r_n = a(z_n^{**})$ for each $n \in \mathbb{N}$. Since $a : Y_K \longrightarrow R$ is a bounded linear operator, a is also $\|\cdot\|$-continuous and so $\{r_n \ , \ n \in \mathbb{N}\}$ is norm dense in R. Let

$$F_n = \text{span}\{z_1^{**}, \ldots, z_n^{**}\} \subseteq S^{**}$$

and

$$\widetilde{F_n} = \text{span}\{x_1^*, \ldots, x_n^*\} \subseteq S^*$$

By the principle of Local reflexivity we can find for each $n \in \mathbb{N}$ an injective linear map

$$T_n : F_n \longrightarrow X$$

such that

(A) $T_n|_{F_n \cap X} = id|_{F_n \cap S}$,

(B) $\|T_n\| \le 1$, $\|T_n^{-1}\| \le 1 + \frac{1}{n}$,

(C) $x^*(\ T_n\ (b(r)\ )\ ) = x^*(\ b(r)\ )$, $x^* \in \widetilde{F_n}$ and for $r \in span\{a(z_1^{**}), \ldots, a(z_n^{**})\}$ or, what is the same,

(C)′ $x^* T_n z^{**} = z^{**}(x^*)$, $x^* \in \widetilde{F_n}$, $z_n^{**} \in F_n$.

Let $R_n = a(F_n)$.

Define $U_n : R_n \longrightarrow S$ by

$$\underbrace{R_n \xrightarrow{b} F_n \xrightarrow{T_n} X.}_{U_n}$$

Let

$$\widetilde{U}_n \ : \ S^* \longrightarrow R^*$$

be given by

$$\widetilde{U}_n x^*(r) = \begin{cases} x^* U_n(r), & \text{if } r \in R_n; \\ 0, & \text{otherwise.} \end{cases}$$

$\widetilde{U}_n$ is homogeneous and it can be viewed as a point of $(B_{R^*}, \text{weak}^*)^{\mathcal{D}}$, a compact metrizable space. (To be more precise, $\widetilde{U_n}|_D \in (B_{R^*}, \text{weak}^*)^{\mathcal{D}}$).
Viewing $(\widetilde{U}_n)_n$ in this way we see that $(\widetilde{U_n})_n$ has a limit point $\widetilde{U} \in (B_{R^*}, \text{weak}^*)^{\mathcal{D}}$ that can be extended to all S^* with values still in B_{R^*}, by standard way (using the fact that $\mathcal{D}$ is linearly independent). The compact metrizable nature on $(B_{R^*}, \text{weak}^*)^{\mathcal{D}}$ tells us that $\widetilde{U}$ is actually a pointwise limit (over $\mathcal{D}$) of a subsequence $(\widetilde{U}_{n_k})_k$ of $(\widetilde{U}_n)_n$; density of $\mathcal{D}$ in S^* lets us extend $\widetilde{U}$ to a map of S^* to R^*.

Step 1. Now we note that $\widetilde{U}$ is actually a bounded linear operator.

- Linearity: Indeed, let $x^*, y^* \in S^*$, $\lambda, \mu \in \mathbb{K}$ and $r \in R$. Let us consider $\overline{n} \in \mathbb{N}$ such that $r \in R_n$ for each $n \geq \overline{n}$. Then

$$\begin{aligned} \langle \widetilde{U}(\lambda x^* + \mu y^*), r\rangle &= \lim_{k\to\infty} \langle \widetilde{U}_{n_k}(\lambda x^* + \mu y^*), r\rangle \\ &= \lim_{k\to\infty} \langle \lambda x^* + \mu y^*, U_{n_k}(r)\rangle \\ &= \lambda \lim_{k\to\infty} \langle x^*, U_{n_k}(r)\rangle + \mu \lim_{k\to\infty} \langle y^*, U_{n_k}(r)\rangle \\ &= \lambda \lim_{k\to\infty} \langle \widetilde{U}_{n_k}(x^*), r\rangle + \mu \lim_{k\to\infty} \langle \widetilde{U}_{n_k} y^*, r\rangle \\ &= \lambda \langle \widetilde{U}(x^*), r\rangle + \mu \langle \widetilde{U}(y^*), r\rangle. \end{aligned}$$

- Continuity: Indeed, let $x^* \in S^*$, and $r \in B_R$. Let us consider $\overline{n} \in \mathbb{N}$ such that $r \in B_n$ for each $n \geq \overline{n}$. Then

$$\begin{aligned} \langle \widetilde{U}(x^*), r\rangle &= \lim_{k\to\infty} \langle \widetilde{U}_{n_k}(x^*), r\rangle \\ &= \lim_{k\to\infty} \langle x^*, U_{n_k}(r)\rangle \\ &\leq \lim_{k\to\infty} \|x^*\| \ \|U_{n_k}(r)\| \\ \text{(since } \|U_n\| \leq 1 \ \forall n \in \mathbb{N}) \quad &\leq \|x^*\| \end{aligned}$$

Step 2. For all $r \in \bigcup_{n\in\mathbb{N}} R_n$, $\{U_n(r)\}_{n\in\mathbb{N}}$ is relative weakly compact and from this we see that $\{U_n(r)\}_{n\in\mathbb{N}}$ is relative weakly compact for each $r \in R$ – simple epsĭlonuš argument.
This follow directly from (C) above and

$$x^* U_n(r) = T_n(b(r))x^* = b(r)x^*$$

for any $r \in \operatorname{span}\{a(z_1^{**}), \dots, a(z_n^{**})\}$ and $x^* \in \widetilde{F_n}$.

Step 3. $\widetilde{U}$ is weak*-weak* continuous.
Indeed, let $(s_n^*)_n$ be a sequence in S^* such that $s_n^* \xrightarrow{\text{weak}^*} s^*$ and let $r \in R$.
Then

$$\begin{aligned}
\lim_{n\to\infty} \langle \widetilde{U}(s_n^*), r\rangle &= \lim_{n\to\infty} \langle \lim_{k\to\infty} \widetilde{U}_{n_k}(s_n^*), r\rangle \\
&= \lim_{n\to\infty} \lim_{k\to\infty} \langle \widetilde{U}_{n_k}(s_n^*), r\rangle \\
\text{(since we are considering } n_k \text{ big enough)} \quad &= \lim_{n\to\infty} \lim_{k\to\infty} \langle s_n^*, U_{n_k}(r)\rangle \\
(*) \quad &= \lim_{k\to\infty} \lim_{n\to\infty} \langle s_n^*, U_{n_k}(r)\rangle \\
&= \lim_{k\to\infty} \lim_{n\to\infty} \langle \widetilde{U}_{n_k}(s_n^*), r\rangle \\
&= \lim_{k\to\infty} \langle \widetilde{U}_{n_k}(s^*), r\rangle \\
&= \langle \widetilde{U}(s^*), r\rangle
\end{aligned}$$

where in $(*)$ we used the double limit Grothendieck theorem (see [5] *Corollaire* 1 to the *Théorème* 7) and Step 2.

So $\widetilde{S}$ is a weak*-weak* bounded linear operator. That means there is a bounded linear operator

$$V : R \longrightarrow X$$

such that

$$V^* = \widetilde{U}$$

Now consider the new σ-additive measure

$$\begin{array}{llll}
\widetilde{F} : \Sigma & \xrightarrow{F} F_C & & \\
& & {}_a\searrow & \\
& & & R \xrightarrow{V} \quad X
\end{array}$$

The last effort is to show that $\mathcal{C} \subseteq \widetilde{F}(\Sigma)$.

Indeed, for all $x \in \mathcal{C}$

$$\begin{aligned}\langle V(a(x)), x^*\rangle &= \langle a(x), \widetilde{U}(x^*)\rangle \\ &= \lim_{k\to\infty} \langle a(x), \widetilde{U}_{n_k}(x^*)\rangle \\ &= \lim_{k\to\infty} \langle U_{n_k}(a(x)), x^*\rangle \\ &= \lim_{k\to\infty} \langle T_{n_k}(b \circ a(x)), x^*\rangle \\ &= \lim_{k\to\infty} \langle T_{n_k}(x), x^*\rangle \\ (\text{ by (A) above}) &= \langle x, x^*\rangle \qquad \square\end{aligned}$$

Comment 3.4. *The second author believes that, assuming that Martin's axiom* (*see* [4]) *is true and a transfinite induction instead of a natural induction, the separability hypothesis of the previous theorem can be avoided.*

Section III

The reader might realize that when a Banach space X is complemented in X^{**}, then the problem has a positive solution. In fact, consider the projection $P : X^{**} \to X$ and compose it with the vector measure $F : \Sigma \to X^{**}$ to obtain the countably additive X-valued measure

$$P \circ F : \Sigma \to X.$$

Theorem 3.5. *Let C be a countable subset of $C(K)$, where K is a compact Hausdorff space. If C lies in the range of a $C(K)^{**}$-valued measure, then C lies in the range of a $C(K)$-valued measure.*

Proof. First, we recall a construction due to Eilenberg: If $k_1, k_2 \in K$, then we say $k_1 \sim k_2$ if $f(k_1) = f(k_2)$ for each $f \in C$; of course $\sim$ is an equivalence relation on K and between equivalence classes $[k_1]$ and $[k_2]$, we can define a metric $d([k_1], [k_2])$ by

$$d([k_1],[k_2]) = \sum_{f_n \in C} \frac{|f_n(k_1) - f_n(k_2)|}{(\|f_n\| + 1)2^n}$$

remembering that $C = \{f_n : n \in \mathbb{N}\}$ is countable. Each $f \in C$ "lifts" to an $\tilde{f} \in C(K_0)$, K_0 the metric space of equivalence classes; the map $q : K \to K_0$ that takes k to $[k]$ is a continuous surjection. So K_0 is a compact metric space and $q : K \twoheadrightarrow K_0$ is a surjective continuous map; q induces an isometric linear embedding $q^\circ : C(K_0) \to C(K)$, where $q^\circ(\tilde{f})(k) = \tilde{f}([k])$, for any $\tilde{f} \in C(K_0)$. It is important to realize that if $\tilde{C}$ is the result of lifting members of C in $C(K)$ to members of $C(K_0)$, then $q^\circ(\tilde{C}) = C$. Here is the setup.

$$\tilde{C} \subseteq C(K_0) \subseteq C(K) \qquad \text{and} \qquad C \subseteq F(\Sigma) \subseteq C(K)^{**}$$

for some countably additive $F : \Sigma \to C(K)^{**}$ with a σ-field domain Σ. Because $C(K_0)^{**}$ is isometrically isomorphic to a subspace of $C(K)^{**}$; but $C(K_0)^{**}$ is an injective Banach space so there is a bounded linear projection $P : C(K)^{**} \to$

$C(K)^{**}$ on $C(K)^{**}$ with range $P(C(K)^{**}) = C(K_0)^{**}$. So we have a $C(K_0)^{**}$ such that the countable set $\tilde{C}$ lies in $P \circ F(\Sigma)$. The previous theorem tells us there is a $C(K_0)$-valued countably additive measure G on Σ so that $\tilde{C} \subseteq G(\Sigma)$. Look at:

$$C = q^\circ(\tilde{C}) \subseteq q^\circ(G(\Sigma)) \subseteq q^\circ(C(K_0)) \subseteq C(K). \qquad \square$$

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