MATHEMATISCHE NACHRICHTEN

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REPRINT

Math. Nachr. **284**, No. 5–6, 664 – 680 (2011) / **DOI** 10.1002/mana.200810163

Sequential w-right continuity and summing operators

Joe Diestel[*1], **Antonio M. Peralta**[**1,2], and **Daniele Puglisi**[***1]

[1] Kent State University, Kent, OH 44242, USA

[2] Departamento de Análisis Matemático, Facultad de Ciencias, Universidad de Granada, 18071 Granada, Spain

Received 13 July 2008, revised 30 July 2009, accepted 14 August 2009
Published online 8 March 2011

Key words W-right topology, strong* topology, sequential w-right-norm continuity, weakly compact mappings, summing operators

MSC (2010) Primary: 46G20, 46G25, 46L05, 47B07, 17C65; Secondary: 47A20

We continue the study of the w-right and strong* topologies in general Banach spaces started in [36, 37] and [35]. We show that in $L_1(\mu)$-spaces the w-right convergence of sequences admits a simpler control. Some considerations about these topologies will be contemplated in the particular cases of C*-algebras and JB*-triples in connection with summing operators. We also study (sequential) w-right-norm and strong*-norm continuity for holomorphic mappings.

1 Preliminaries

I. Villanueva, J. D. M. Wright, K. Ylinen and the second author of the present note introduced in [36] two interesting topologies: the strong* and the w-right topology in the following way: let X and Y be two Banach spaces, for every bounded linear operator $T : X \longrightarrow Y$, we can consider a seminorm on X defined by $\|x\|_T := \|T(x)\|$. The *strong*-topology* is the topology generated by the family of seminorms $\|\cdot\|_T$, where $T : X \longrightarrow H$ is a bounded linear operator from X to some Hilbert space H (such a topology is denoted by $S^*(X, X^*)$). Similarly, the *w-right-topology* is the topology generated by the family of seminorms $\|\cdot\|_T$ where T runs in the set of all bounded linear operators from X to a reflexive space [36].

In Section 2, we establish new methods for controlling w-right convergent sequences in $L_1(\mu)$ spaces. Section 3 is devoted to a more detailed study of strong*-norm continuous operators between Banach spaces. In the particular cases of operators whose domain is a C*-algebra or a JB*-triple, we explore the connections with p-C*-summing and p-JB*-triple-summing operators. We prove an extension property for 2-C*-summing and 2-JB*-triple-summing operators (see Theorems 3.6 and 3.9). In this section we shall also introduce and develop p-JB*-triple-summing operators on JB*-triples as suitable generalizations of p-C*-summing operators on C*-algebras in the sense introduced by Pisier in [39].

The last section of the paper is devoted to the study of those holomorphic mappings of bounded type which are sequentially w-right-norm continuous. The main result in [35] establishes that a bounded linear operator $T : X \longrightarrow Y$ is weakly compact if and only if T is w-right-norm continuous. We shall provide examples showing that none of these implications holds for continuous polynomials in general Banach spaces. In the linear case, T is weakly compact if and only if T^{**} is Y-valued. In the setting of multilinear operators, this equivalence has been recently studied in [37]. One of the main results in the just quoted paper proves that when $X_1, \ldots, X_k$ are non zero sequentially right Banach spaces and $T : X_1 \times \cdots \times X_k \longrightarrow X$ is a multilinear operator, then T is RQCC (i.e., T is jointly sequentially w-right-norm continuous) if and only if all of the Aron-Berner extensions of T are X-valued if and only if T has an X-valued Aron-Berner extension. We shall consider here holomorphic mappings of bounded type f between two Banach spaces X and Y with X being a sequentially right Banach

* e-mail: j_diestel@hotmail.com, Phone: +1-330-672-9087, Fax: +1-330-672-2209
** Corresponding author: e-mail: aperalta@ugr.es, Phone: +34-958-246-307, Fax: +34-958-243-272
*** e-mail: puglisi@math.kent.edu, Phone: +39-095-738-3095, Fax: +39-095-330-094

space. We shall prove that such a mapping f is sequentially w-right-norm continuous if and only if its Aron-Berner extension, $AB(f) : X^{**} \longrightarrow Y^{**}$, is Y-valued.

1.1 Notation

Except otherwise stated, all the Banach spaces considered in this paper will be complex. Given a Banach space X, $S(X)$ and $B(X)$ denote, respectively, the unit sphere and the closed unit ball of X. For any pair of Banach spaces X, Y, $\mathcal{L}(X, Y)$ will stand for the space of all bounded linear operators between X and Y, while $X \overset{\vee}{\otimes} Y$ and $X \overset{\wedge}{\otimes} Y$ will denote the injective and projective tensor product of X and Y, respectively.

2 When the w-right-topology and the weak-topology coincide sequentially

In [36, Proposition 2.7] the authors remarked the following.

Proposition 2.1 *Let X be a Banach space. If the w-right-topology coincides with the weak topology on X, then X is finite dimensional.*

Given a set X with two topologies τ_1 and τ_2, we say that τ_1 and τ_2 coincide sequentially if both topologies define the same convergent sequences on X, that is, a sequence $(x_n)_n$ in X is τ_1-convergent to $x \in X$ if and only if $(x_n)_n$ converges to the same x in the τ_2-topology.

We recall that a bounded linear operator $T : X \longrightarrow Y$ is called *completely continuous* if it maps weakly convergent sequences to norm convergent sequences. A Banach space X has the *Dunford-Pettis property* (DPP) if, for every Banach space Y, every weakly compact operator from X to Y is completely continuous. X satisfies the (weaker) *alternative Dunford-Pettis property* (DP1) if every weakly compact operator $T : X \longrightarrow Y$ is a DP1 operator, that is, $T(x_n)$ converges in norm to $T(x)$ whenever $x_n \longrightarrow x$ weakly in X and $\|x_n\| = \|x\| = 1$.

The following result was established in [36, Remark 4.5].

Proposition 2.2 *Let X be a Banach space.*

(a) *The w-right and the weak topologies coincide sequentially if and only if X has the DPP.*

(b) *The w-right and the weak topologies coincide sequentially on the unit sphere of X if and only if X has the DP*1.

To formulate the next result we first recall a deep result due to Rieffel (see [27] for more details).

Theorem 2.3 (Rieffel) *Let (Ω, Σ, μ) be a finite measure space and let X be a Banach space. A vector measure $F : \Sigma \longrightarrow X$ is Bochner-representable with respect to μ (i.e., $F(B) = Bochner - \int_B f\, d\mu$ for all $B \in \Sigma$ and some $f \in L_1(\mu, X)$) if and only if F is μ-continuous, F is of bounded variation, and for each $\epsilon > 0$ there exists $B_\epsilon \in \Sigma$ with $\mu(\Omega \setminus B_\epsilon) < \epsilon$ such that*

$$\left\{ \frac{F(B)}{\mu(B)} \;:\; B \subseteq B_\epsilon,\ B \in \Sigma,\ \mu(B) > 0 \right\}$$

is relatively weakly compact.

Remark 2.4 Note that we can reformulate Rieffel‘s theorem in terms of operators as follows: a bounded linear operator $T : L_1(\mu) \longrightarrow X$ is Bochner-representable (i.e., there is a $g \in L_\infty(\mu, X)$ so that $T(f) =$ Bochner $- \int_\Omega f \cdot g\, d\mu$ for every $f \in L_1(\mu)$) if and only if for each $\epsilon > 0$ there exists $\Omega_\epsilon \in \Sigma$ with $\mu(\Omega \setminus \Omega_\epsilon) < \epsilon$ so that $T : L_1\left(\Omega_\epsilon, \Sigma_{\Omega_\epsilon}, \mu|_{\Sigma_{\Omega_\epsilon}}\right) \longrightarrow X$ is weakly compact, where Σ_{Ω_ϵ} denotes the σ-field $\Sigma_{\Omega_\epsilon} = \{F \in \Sigma \;:\; F \subseteq \Omega_\epsilon\}$.

Proposition 2.5 *Let (Ω, Σ, μ) be a finite measure space, and let $(f_n)_n$ be a sequence in $L_1(\mu)$. Then the following are equivalent*:

(0) *$(f_n)_n$ is weakly null.*

(0′) *$(f_n)_n$ is w-right null.*

(1) *For every Banach space X, and every weakly compact operator $T : L_1(\mu) \longrightarrow X$, the sequence $(T(f_n))_n$ is norm null.*

(2) *For each Banach space X, and every representable operator $T : L_1(\mu) \longrightarrow X$, the sequence $(T(f_n))_n$ is norm null.*

(3) *For every Banach space X and every $g \in L_\infty(\mu, X)$, the sequence $(f_n \cdot g)_n$ is weakly null in $L_1(\mu, X)$.*

P r o o f. Since $L_1(\mu)$ has the DPP, then (0) and $(0')$ are equivalent by Proposition 2.2. The equivalence of $(0')$ and (1) follows directly from the definition of the w-right topology.

(1) ⇒ (2) Let X be a Banach space and let $T : L_1(\mu) \longrightarrow X$ be a representable operator. Fixing $\epsilon > 0$, by the previous remark, there exists $\Omega_\epsilon \in \Sigma$ with $\mu(\Omega \setminus \Omega_\epsilon) < \epsilon$ so that

$$T : L_1\big(\Omega, \Sigma_{\Omega_\epsilon}, \mu|_{\Sigma_{\Omega_\epsilon}}\big) \longrightarrow X \quad \text{is weakly compact.}$$

Moreover, since $(f_n)_n$ is weakly null in $L_1(\Omega, \Sigma, \mu)$, by a classical result of Dunford and Pettis (see [15] Chapter IV.2) the sequence $(f_n)_n$ is uniformly integrable. We can then assume that $\big\|f_n\ \chi_{\Omega\setminus\Omega_\epsilon}\big\|_{L_1(\Omega,\Sigma,\mu)} < \frac{\epsilon}{\|T\|}$ uniformly in $n \in \mathbb{N}$.

Now, since $\big\{f_n \chi_{\Omega_\epsilon}\big\}_n$ is weakly null in $L_1\big(\Omega, \Sigma_{\Omega_\epsilon}, \mu|_{\Sigma_{\Omega_\epsilon}}\big)$, then there is an $n_0 \in \mathbb{N}$ so that $\|T(f_n\chi_{\Omega_\epsilon})\| < \epsilon$, for all $n > n_0$. Therefore, for $n > n_0$, we have

$$\|T(f_n)\| \le \|T(f_n\chi_{\Omega_\epsilon})\| + \|T(f_n\chi_{\Omega\setminus\Omega_\epsilon})\| < \epsilon + \|T\|\|f_n\ \chi_{\Omega\setminus\Omega_\epsilon}\|_{L_1(\Omega,\Sigma,\mu)} < 2\epsilon.$$

(2) ⇒ (0) Fix $g \in L_\infty(\mu)$; It's enough to choose $T : L_1(\mu) \longrightarrow \mathbb{C}$ defined by $T(f) = \int_\Omega f\ g\, d\mu$. T is trivially representable and so $\langle f_n\ ,\ g\rangle \overset{n\to\infty}{\longrightarrow} 0$, that is, $(f_n)_n$ is weakly null in $L_1(\mu)$.

(0) ⇒ (3) Fix a Banach space X and an element $g \in L_\infty(\mu, X)$. Let $(f_n)_n$ be a weakly-null sequence in $L_1(\mu)$. Since (Ω, Σ, μ) is a finite measure space we can apply the Diestel-Ruess-Schachermayer Theorem (see [14]) to the weakly relatively compact set $\{f_n\}_{n\in\mathbb{N}}$. Then for each subsequence $(f_{n_k})_k$ of $(f_n)_n$ there exists a sequence $(g_n)_n$ with $g_n \in co\{f_{n_k}\ :\ n_k \ge n\}$ such that $(g_n(\omega))_n$ is a null sequence of scalars for a.e. $\omega \in \Omega$. But the sequence $(g_n \cdot g)_k$ is such that $g_n \cdot g \in co\{f_{n_k} \cdot g\ :\ n_k \ge n\}$ and so $(g_n(\omega) \cdot g(\omega))_k$ is norm null in X for a.e. $\omega \in \Omega$. By the Diestel-Ruess-Schachermayer's Theorem the sequence $(f_n \cdot g)_{n\in\mathbb{N}}$ is weakly null in $L_1(\mu, X)$.

(3) ⇒ (0) It's enough to choose $X = \mathbb{K}$ and $g(\omega) = 1$ for each $\omega \in \Omega$. □

Corollary 2.6 $(0), (0'), (1), (2)$ *of the proposition above are equivalent for any measure space* (Ω, Σ, μ).

P r o o f. Note that $(0), (0')$ and (1) are trivially equivalent because $L_1(\mu)$ has the DPP.

(0) ⇒ (2) Let $(f_n)_n$ be a weakly null sequence in $L_1(\mu)$. It is well known that (see [18], III.8.5) there exists a set Ω_1 in Σ, a sub σ-field of Σ such that the restriction μ_1 of μ to Σ_1 has the properties

(i) the measure space $(\Omega_1, \Sigma_1, \mu_1)$ is σ-finite;

(ii) $\overline{span}\{f_n\ :\ n \ge 1\} \subseteq L_1(\Omega_1, \Sigma_1, \mu_1)$.

Since $L_1(\Omega_1, \Sigma_1, \mu_1)$ is a closed subspace of $L_1(\Omega, \Sigma, \mu)$, we can assume, without loss of generality that (Ω, Σ, μ) is a σ-finite measure space. Thus there exists a sequence (A_n) in Σ of pairwise disjoint sets with finite and positive μ-measure such that $\Omega = \bigcup_{n\in\mathbb{N}} A_n$.

Now, we define $\mu_0 : \Sigma \longrightarrow [0, +\infty)$ as

$$\mu_0(E) = \sum_{n\in\mathbb{N}} \frac{1}{2^n\ \mu(A_n)}\ \mu(A_n \cap E) \quad \text{for every} \quad E \in \Sigma.$$

It is easy to see that μ_0 is a finite measure (it is a probability measure) and, if we consider the function $h = \sum_{n\in\mathbb{N}} 2^n\ \mu(A_n)\ \chi_{A_n}$, the map

$$\begin{aligned} &T : L_1(\mu) \longrightarrow L_1(\mu_0) \\ &f \longmapsto h \cdot f \end{aligned}$$

is a surjective isometry. Since μ and μ_0 have the same null sets, for each Banach space X the identity

$$id : L_\infty(\mu, X) \longrightarrow L_\infty(\mu_0, X)$$

is a surjective isometry.

Fix a Banach space X and a representable operator $S : L_1(\mu) \longrightarrow X$. Then there is a $g \in L_\infty(\mu, X)$ so that $S(f) = \int_\Omega f \cdot g\, d\mu$. By the last paragraph above, we can consider g as element of $L_\infty(\mu_0, X)$. Then the operator

$$\widetilde{S} : L_1(\mu_0) \longrightarrow X$$
$$\widetilde{S}(f) = \int_\Omega f \cdot g\, d\mu_0 \quad \text{for each} \quad f \in L_1(\mu_0)$$

is trivially representable. Since $(T(f_n))_n$ is w-right null in $L_1(\mu_0)$ (actually, every bounded linear operator maps w-right null sequences into w-right null sequences), by the previous proposition we have

$$\|S(f_n)\| = \|\widetilde{S}(f_n \cdot h)\| \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty$$

because for each $f \in L_1(\mu, X)$ we have $\int_\Omega f\, d\mu = \int_\Omega f \cdot h\, d\mu_0$, which gives (2).

The implication (2) ⇒ (0) follows similarly. □

3 Strong*-norm continuous operators

We recall a result established in [35]. We should note here that after the publication of the just quoted paper, we were told about the significative papers [40] and [42], which are directly connected with the results obtained in [35]. In fact, the main result in [35] follows as a consequence of [40, Proposition 2.6 and Theorems 3.1 and 3.2], proved there in a more general setting. The equivalence of (i) and (iii) in [35, Corollary 5] can be also obtained from [42, Lemmas 2.1 and 3.2].

Theorem 3.1 *Let X and Y be two Banach spaces, and let $T : X \longrightarrow Y$ be a bounded linear operator. Then the following are equivalent*

(a) *T is w-right-norm continuous.*

(b) *T is w-right-norm continuous on the closed unit ball of X.*

(c) *T is weakly compact.*

Similarly we have:

Theorem 3.2 *Let X and Y be two Banach spaces, and let $T : X \longrightarrow Y$ be a bounded linear operator. Then T is strong*-norm continuous if and only if T factors through a Hilbert space.*

P r o o f. Let $T : X \longrightarrow Y$ be a strong*-norm linear operator. The set

$$U := \{x \in X : \|T(x)\| \le 1\}$$

is a strong*-neighborhood of zero in X. Then there exist Hilbert spaces $H_1, \dots, H_n$, and operators $G_i : X \longrightarrow H_i$, $i = 1, \dots, n$, satisfying that $\bigcap_{i=1}^n \{x \in X : \|G_i(x)\| \le 1\} \subseteq U$. Consider $H := \big(\bigoplus_{i=1}^n H_i\big)_{\ell_2}$, and $G : X \longrightarrow H$ defined by $G(x) = (G_i(x))_{i=1}^n$. The inclusion

$$\{x \in X : \|G(x)\| \le 1\} \subseteq \bigcap_{i=1}^n \{x \in X : \|G_i(x)\| \le 1\}$$

implies that

$$\|T(x)\| \le \|G(x)\| \quad \text{for each} \quad x \in X. \tag{3.1}$$

The kernel of G is a closed subspace of X and the mapping $x + \ker(G) \longmapsto |||x|||_G = \|G(x)\|$ is a prehilbertian norm on the quotient $X/\ker(G)$. The inequality (3.1) guarantees that the law

$$R : X/\ker(G) \longrightarrow Y$$
$$x + \ker(G) \longmapsto T(x)$$

is a well-defined continuous operator on $X/\ker(G)$ with $\|R\| \le \|T\|$. If H_G denotes the completion of $X/\ker(G)$, then H_G is a Hilbert space and R admits an extension $\widehat{R} : H_G \longrightarrow Y$. If π denotes the canonical projection of

X onto $X/\ker(G)$ and j_G the inclusion of $X/\ker(G)$ into H_G, then we have $T = \widehat{R}\, j_G\, \pi$, which shows that T factors through a Hilbert space (i.e., $T \in \Gamma_2(X,Y)$). Clearly every operator in $\Gamma_2(X,Y)$ is strong*-norm continuous. □

Corollary 3.3 *Every strong*-norm continuous operator between two Banach spaces is uniformly convexifying in the sense of Beauzamy* [7].

Corollary 3.4 *Every strong*-norm continuous operator between two Banach spaces is a Banach-Saks operator* [8].

Corollary 3.5 *The class of all strong*-norm continuous operators between Banach spaces is an injective* (*closed*) *operator ideal in the Pietsch sense* [38].

When the domain space is a C*-algebra (respectively, a JB*-triple) then strong*-norm continuous operators coincide with 2-C*-summing (respectively, 2-JB*-triple-summing) operators. p-C*-summing operators on C*-algebras were introduced by Pisier in [39]. We recall that an operator T from a C*-algebra A to a Banach space Y is said to be *p-C*-summing* ($p > 0$) if there exists a constant C such that for every finite sequence of self-adjoint elements $(a_1, \dots, a_n)$ in A the next inequality holds

$$\left(\sum_1^n \|T(a_i)\|^p\right)^{\frac{1}{p}} \leq C \left\| \left(\sum_1^n |a_i|^p\right)^{\frac{1}{p}} \right\|, \tag{3.2}$$

where, for each $x \in A$, we denote $|x| = \left(\frac{xx^*+x^*x}{2}\right)^{\frac{1}{2}}$. The smallest constant C verifying the above inequality is denoted by $C_p(T)$.

The following Pietsch's factorization theorem for p-C*-summing operators was established by Pisier in [39]: if $T : A \longrightarrow Y$ is a bounded linear operator from a C*-algebra to a complex Banach space, then T is a p-C*-summing operator if and only if there is a norm-one positive linear functional φ in A^* and a positive constant $K_p(T)$ such that

$$\|T(x)\| \leq K_p(T)(\varphi(|x|^p))^{\frac{1}{p}}$$

for every x in A. Every p-summing operator from a C*-algebra to a Banach space is p-C*-summing but the converse is false in general (compare [39, Remark 1.2]). It follows from the little Grothendieck's inequality for C*-algebras (see [23, 39]) that an operator $T : A \longrightarrow Y$ is 2-C*-summing if and only if it is strong*-norm continuous.

The following result shows that 2-C*-summing operators enjoy an extension property which is the appropriate version of [13, Theorem 4.15].

Theorem 3.6 *Let A and B be two C*-algebras with B a C*-subalgebra of A and let Y be a Banach space. Then every 2-C*-summing operator $T : B \longrightarrow Y$ admits a norm preserving 2-C*-summing extension $\widetilde{T} : A \longrightarrow Y$.*

Proof. Let $T : B \longrightarrow Y$ be a 2-C*-summing operator. According to the Pietsch factorisation theorem, there is a norm-one positive linear functional φ in B^* and a positive constant $K_p(T)$ such that

$$\|T(x)\| \leq K_2(T)(\varphi(|x|^2))^{\frac{1}{2}} \tag{3.3}$$

for every x in B. Proposition 3.1.6 in [29] implies the existence of a positive functional $\phi \in A^*$ satisfying that $\|\phi\| = \|\varphi\|$ and $\phi|_B = \varphi$. The set $N_\phi = \{x \in A : \phi(xx^* + x^*x) = 0\}$ is a closed subspace of A and the sesquilinear form

$$(x + N_\phi, y + N_\phi) \longmapsto \frac{1}{2}\phi(xy^* + y^*x)$$

defines a pre-inner product on the preHilbert space A/N_ϕ. The completion of the latter space is a Hilbert space that will be denoted by H_ϕ. Let $j_\phi : A \longrightarrow H_\phi$ denote the composition of the canonical projection and inclusion.

The norm-closure of $j_\phi(B) = B/(N_\phi \cap B)$ is a closed subspace of H_ϕ which is denoted by K. Let π be the orthogonal projection of H_ϕ onto K.

The operator $B/(N_\phi \cap B) \longrightarrow Y$, $x + (N_\phi \cap B) \longmapsto T(x)$ is well-defined by (3.3). Therefore there exists a unique operator $R : K \longrightarrow Y$ satisfying $R(x + (N_\phi \cap B)) = T(x)$. Finally, $\widetilde{T} : A \longrightarrow Y$, $\widetilde{T} = R \circ \pi \circ j_\phi$ is a norm preserving 2-C*-summing extension of T. □

Proposition 3.7 *Let X be a Banach space. Suppose that the w-right topology and the $S^*(X, X^*)$-topology coincide on bounded subsets of X. Then the following statements holds*:

(a) *X satisfies the DPP if and only if every strong*-norm continuous operator from X to a Banach space is completely continuous.*

(b) *X satisfies the DP*1 *if and only if every strong*-norm continuous operator from X to a Banach space is a DP*1 *operator.*

P r o o f. (a) We prove only the if-implication, because the other implication follows easily. Suppose that every strong*-norm continuous operator from X to a Banach space is completely continuous. Let (x_n) be a weakly-null sequence in X and let $T : X \longrightarrow Y$ be a weakly compact operator. Since the w-right topology and the $S^*(X, X^*)$-topology coincide on bounded subsets of X, then there exist a bounded linear operator G from X to a Hilbert space and a mapping $N : (0, \infty) \longrightarrow (0, \infty)$ satisfying

$$\|T(x)\| \le N(\varepsilon)\|G(x)\| + \varepsilon\|x\|,$$

for all $x \in X$ and $\varepsilon > 0$ (compare [36, Proposition 5.1]).

Let us fix $\delta > 0$. Since (x_n) is bounded, we can find an appropriate $\varepsilon_0 > 0$ satisfying that $\varepsilon_0 \, \|x_n\| < \frac{\delta}{2}$, for every natural n. By hypothesis, $G(x_n) \longrightarrow 0$ in norm. So there exists a natural m satisfying that

$$N(\varepsilon_0)\|G(x_n)\| < \frac{\delta}{2}, \quad \text{for all} \quad n \ge m,$$

which gives that $\|T(x_n)\| < \delta$, for all $n \ge m$.

The proof of statement (b) follows similarly. □

For each C*-algebra A, the w-right and the strong* topologies coincide on bounded sets of A (compare [2, Theorem II.7]). The extension property of 2-C*-summing operators proved in Theorem 3.6 together with the above Proposition 3.7 give an alternative proof to [11, Corollary 2] and [19, Corollary 3.2].

Corollary 3.8 *Every C*-subalgebra of a C*-algebra satisfying the DPP* (*respectively, the DP*1) *also satisfies the same property.*

Let u be a norm-one element in a Banach space X. The set of states of X relative to u, $D(X, u)$, is defined as the non empty, convex, and weak*-compact subset of X^* given by

$$D(X, u) := \{\Phi \in X^* : \Phi(u) = 1 = \|\Phi\|\}.$$

For $x \in X$, the *numerical range* of x relative to u, $V(X, u, x)$, is given by $V(X, u, x) := \{\Phi(x) : \Phi \in D(X, u)\}$. The *numerical radius* of x relative to u, $v(X, u, x)$, is given by

$$v(X, u, x) := \max\{|\lambda| : \lambda \in V(X, u, x)\}.$$

It is well-known that a bounded linear operator T on a complex Banach space X is hermitian if and only if $V(\mathcal{L}(X), I_X, T) \subseteq \mathbb{R}$ (compare [9, Section 5, Lemma 2]). If T is a bounded linear operator on X, then we have $V(\mathcal{L}(X), I_X, T) = \overline{co}\, W(T)$ where

$$W(T) = \{x^*(T(x)) : (x, x^*) \in \Gamma\},$$

and $\Gamma \subseteq \{(x, x^*) : x \in S_X, x^* \in S_{X^*}, x^*(x) = 1\}$ verifies that its projection onto the first coordinate is norm dense in the unit sphere of X [9, Section 9]. Moreover, the numerical radius of T can be calculated as follows

$$v(\mathcal{L}(X), I_X, T) = \sup\{|x^*(T(x))| : (x, x^*) \in \Gamma\}.$$

In particular if $X = Y^*$, then by the Bishop-Phelps-Bollobás theorem, it follows that

$$v(\mathcal{L}(X), I_X, T) = \sup\{|x^*(T(x))| : x \in S_X, \quad x^* \in S_Y, \quad x^*(x) = 1\}.$$

Originally introduced by Kaup in [26], the class of complex Banach spaces called JB*-triples includes all C*-algebras, Hilbert spaces, spin factors and operators between complex Hilbert spaces. A JB*-triple is a complex Banach space E with a continuous triple product $\{.,.,.\} : E \times E \times E \longrightarrow E$ which is bilinear and symmetric in the outer variables and conjugate linear in the middle one, and satisfies:

(JB1) (Jordan Identity) $L(a,b)\{x,y,z\} = \{L(a,b)x,y,z\} - \{x, L(b,a)y, z\} + \{x,y,L(a,b)z\}$, for all a, b, c, x, y, z in E, where $L(a,b)x := \{a,b,x\}$;

(JB2) The map $L(a,a) : E \longrightarrow E$ is an hermitian operator with non negative spectrum for all a in E;

(JB3) $\|\{a,a,a\}\| = \|a\|^3$, for all a in E.

For each element x in a JB*-triple E, we shall denote $x^{[1]} := x$, $x^{[3]} := \{x,x,x\}$, and $x^{[2n+1]} := \left\{x, x, x^{[2n-1]}\right\}$, $(n \in \mathbb{N})$. Given a subset $F \subset E$, the symbol $F \,\square\, F$ will denote the set

$$\{L(x,y) : x, y \in F\} \subset \mathcal{L}(E).$$

Examples: every C*-algebra is a JB*-triple with respect to the product $\{a,b,c\} := \frac{1}{2}(ab^*c + cb^*a)$. The above product remains valid for the space $\mathcal{L}(H,K)$ of all continuous operators between two complex Hilbert spaces H, K.

For each JB*-triple E and every state $\Phi \in D(\mathcal{L}(E), I_E)$, the assignment $x \longmapsto \|x\|_\Phi := \Phi(L(x,x))^{\frac{1}{2}}$ defines a prehilbertian seminorm on E. Further, whenever φ is a norm-one element in E^* and $z \in S_{E^*}$ with $\varphi(z) = 1$, the mapping $x \longmapsto \|z\|_\varphi = \varphi\{x,x,z\}^{\frac{1}{2}}$ does not depend on the point of support z, and defines also a prehilbertian seminorm on E (compare [34, Section 1]).

In the more general setting of JB*-triples, only the notion of 2-C*-summing operator has been generalized in [32]. An operator T from a JB*-triple E to a Banach space Y is said to be *2-JB*-triple-summing* if there exists a positive constant C such that for every finite sequence $(x_1, \ldots, x_n)$ of elements in E we have

$$\sum_{i=1}^{n} \|T(x_i)\|^2 \le C \left\| \sum_{i=1}^{n} L(x_i, x_i) \right\|. \tag{3.4}$$

The smallest constant C for which (3.4) holds is denoted by $C_2(T)$.

The corresponding Pietsch factorization theorem for 2-JB*-triple-summing operators was established in [32, Theorem 3.6]. Indeed: if $T : E \longrightarrow Y$ is a 2-JB*-triple-summing operator then there are norm-one functionals φ_1, φ_2 in E^* and a positive constant $C(T)$ such that

$$\|T(x)\| \le C(T)\|x\|_{\varphi_1,\varphi_2}$$

for all $x \in E$. This result together with the little Grothendieck inequality for JB*-triples and the Hahn-Banach theorem allow us to prove the following result with a verbatim adaptation of the proof of Theorem 3.6.

Theorem 3.9 *Let E and F be two JB*-triples with F a JB*-subtriple of E and let Y be a Banach space. Then every 2-JB*-triple-summing operator $T : F \longrightarrow Y$ admits a norm preserving 2-JB*-triple-summing extension $\widetilde{T} : E \longrightarrow Y$.*

Having in mind that for every JB*-triple E, the w-right and strong* topologies coincide on bounded subsets of E (compare [33, p. 621]), the results [12, Corollary 6] and [1, Corollary 1] follow now as a direct consequence.

Corollary 3.10 *Every JB*-subtriple of a JB*-triple satisfying the DPP (respectively, the DP1) also satisfies the same property.*

Our next goal is to introduce a suitable variation of p-summing operators in JB*-triples.

Let x be an element in a (general) JB*-triple E and let E_x denote the JB*-subtriple generated by x. It is known that E_x is a commutative JB*-triple. Therefore, the closed linear span of $E_x \,\square\, E_x|_{E_x} \subset \mathcal{L}(E_x)$ is an abelian C*-algebra (compare [26, Proposition 1.5]). This structure allows to define $L(x,x)^{\frac{p}{2}}|_{E_x}$ as an element in $\mathcal{L}(E_x)$.

However, this definition, based on “local theory,” does not satisfy our needs because $L(x,x)^{\frac{p}{2}}$ should be an element in $\mathcal{L}(E)$.

We shall see how local theory, wisely applied, can help us to avoid this obstacle. Let x be an element in a JB*-triple E. It is known that E_x is JB*-triple isomorphic (and hence isometric) to $C_0(\Omega)$ for some locally compact Hausdorff space Ω contained in $(0, \|x\|]$, such that $\Omega \cup \{0\}$ is compact and $C_0(\Omega)$ denotes the Banach space of all complex-valued continuous functions vanishing at 0. It is also known that there exists a triple isomorphism Ψ from E_x onto $C_0(\Omega)$, $\Psi(x)(t) = t$ $(t \in \Omega)$ (cf. [25, 4.8], [26, 1.15] and [20]). The set $\overline{\Omega} = \mathrm{Sp}(x)$ is called the *triple spectrum* of x. We should note that $C_0(\mathrm{Sp}(x)) = C(\mathrm{Sp}(x))$, whenever $0 \notin \mathrm{Sp}(x)$.

Local theory in JB*-triples gave rise to the so-called *triple functional calculus*. To avoid possible confusion with the classical continuous functional calculus in C*-algebras, given a function $f \in C_0(\mathrm{Sp}(x))$, $f(x)$ shall have its usual meaning when E_x is regarded as an abelian C*-algebra and $f_t(x)$ shall denote the same element of E_x when the latter is regarded as a JB*-subtriple of E. Thus, for any odd polynomial, $P(\lambda) = \sum_{k=0}^{n} \alpha_k \lambda^{2k+1}$, we have $P_t(x) = \sum_{k=0}^{n} \alpha_k x^{[2k+1]}$. The symbol $x^{[p]}$ will stand for $f_t(x)$, where $f(\lambda) := \lambda^p$, $(\lambda \in \mathrm{Sp}(x))$.

The general lack of order in JB*-triples of the same kind that exists for C*-algebras prevents us to affirm any property on a finite sum of the form $\sum_j x_j^{\left[\frac{p}{2}\right]}$, where $x_1, \ldots, x_k$ are arbitrary elements in a JB*-triple E. In order to have a common order, not depending on the local structure, we make use of the space $\mathcal{L}(E)$. The following definition does not require the existence of an order.

Definition 3.11 Let E be a JB*-triple, let Y be a Banach space and let $p > 0$. An operator $T : E \longrightarrow Y$ is said to be p-JB*-triple-summing if there exists a positive constant C such that for every finite sequence $(x_1, \ldots, x_n)$ of elements in E we have

$$\sum_{i=1}^{n} \|T(x_i)\|^p \leq C \left\| \sum_{i=1}^{n} L\left(x_i^{\left[\frac{p}{2}\right]}, x_i^{\left[\frac{p}{2}\right]}\right) \right\|. \tag{3.5}$$

The smallest constant C for which (3.5) holds is denoted by $C_p(T)$.

Let A be a C*-algebra. We recall that two elements a and b in A are said to be *orthogonal* if $ab^* = b^*a = 0$, equivalently, $L(a,b) = 0$. When a and b belong to a JB*-triple E, we say that a and b are orthogonal whenever $L(a,b) = 0$. When a C*-algebra is regarded as a JB*-triple, these two notions of orthogonality coincide on A. We refer to [10, Lemma 1] for several reformulations of orthogonality in C*-algebras and JB*-triples.

C*-algebras have a dual structure as JB*-triples and C*-algebras. Our next result shows that, in the setting of C*-algebras, p-C*-summing operators and p-JB*-triple-summing operators coincide.

Lemma 3.12 *Let $(x_1, \ldots, x_n)$ be a finite sequence of elements in the C*-algebra A and let X be a Banach space. The following statements hold*:

(a) $\left\| \sum_{i=1}^{n} |x_i|^2 \right\| \leq \left\| \sum_{i=1}^{n} L(x_i, x_i) \right\| \leq 2 \left\| \sum_{i=1}^{n} |x_i|^2 \right\|$.

(b) *When $x_1 \ldots, x_n$ are assumed to be hermitian we have*

$$\left\| \sum_{i=1}^{n} L\left(x_i^{\left[\frac{p}{2}\right]}, x_i^{\left[\frac{p}{2}\right]}\right) \right\| = \left\| \sum_{i=1}^{n} |x_i|^p \right\|,$$

for every $p > 0$.

(c) *If $T \in \mathcal{L}(A, X)$, then T is p-C*-summing whenever it is p-JB*-triple-summing. Moreover, if T is p-C*-summing then there exists $C > 0$ satisfying*

$$\sum_{i=1}^{n} \|T(x_i)\|^p \leq C \left\| \sum_{i=1}^{n} L\left(x_i^{\left[\frac{p}{2}\right]}, x_i^{\left[\frac{p}{2}\right]}\right) \right\|,$$

for every finite sequence $(x_1, \ldots, x_n)$ of hermitian elements in A.

P r o o f. (a) Let 1 denote the unit element in A^{**}. For every finite sequence $(x_1, \ldots, x_n)$ of elements in A we have

$$\left\| \sum_{i=1}^{n} L(x_i, x_i) \right\| \geq \left\| \sum_{i=1}^{n} L(x_i, x_i)(1) \right\| = \left\| \sum_{i=1}^{n} |x_i|^p \right\|.$$

To see the other inequality let us denote $S := \sum_{i=1}^{n} L(x_i, x_i)$. Clearly S is a hermitian operator on A, Sinclair's theorem (compare [9, Remark in p. 54]) assures that

$$\|S\| = \sup\{|\phi(S(z))| : z \in S_A, \phi \in S_{A^*}, \phi(z) = 1\}.$$

It is worth mentioning that $\phi(S(z)) \geq 0$ for any ϕ and z in the above setting. Let $z \in S_A$ and $\phi \in S_{A^*}$ with $\phi(z) = 1$. We define $\psi(x) := \phi(x \circ z)$, where the symbol $\circ$ denotes the natural Jordan product in A. It can be easily seen that $\psi \in S_{A^*}$, $\psi(1) = \phi(z) = 1$. Moreover,

$$\psi(L(x,x)(1)) = \phi(L(x,x)(1) \circ z) = \frac{1}{2}\phi(\{x,x,z\} + \{x^*,x^*,z\}) \geq \frac{1}{2}\phi(L(x,x)(z)),$$

for all $x \in A$. Furthermore, $\phi(L(x,x)(z)) = \psi(L(x,x)(1))$, whenever $x = x^*$. In particular $\phi(S(z)) \leq 2\psi(S(1))$, and hence

$$\begin{aligned} \|S\| &\leq 2\sup\{\psi(S(1)) : \psi \in S_{A^*}, \psi(1) = 1\} \\ &= 2\sup\left\{\psi\left(\sum_{i=1}^{n} |x_i|^2\right) : \psi \in S_{A^*}, \psi(1) = 1\right\} = 2\left\|\sum_{i=1}^{n} |x_i|^2\right\|. \end{aligned}$$

When $x_1, \ldots, x_n$ are hermitian elements, the constant 2 in the above inequality can be omitted, which in particular gives: $\left\| \sum_{i=1}^{n} |x_i|^2 \right\| = \left\| \sum_{i=1}^{n} L(x_i, x_i) \right\|$.

(b) Every self-adjoint element $x \in A$ admits a decomposition in the form $x = x^+ - x^-$, where x^+ and x^- are orthogonal positive elements in A. It is not hard to see that $x^{\left[\frac{p}{2}\right]} = (x^+)^{\left[\frac{p}{2}\right]} - (x^-)^{\left[\frac{p}{2}\right]}$. Since $(x^+)^{\left[\frac{p}{2}\right]}$ and $(x^-)^{\left[\frac{p}{2}\right]}$ are orthogonal, we have $\left|x^{\left[\frac{p}{2}\right]}\right|^2 = (x^+)^p + (x^-)^p = |x|^p$. Let $x_1, \ldots, x_n$ be self-adjoint elements in A. The last paragraph in the proof of the previous statement shows that

$$\left\| \sum_{i=1}^{n} L\left(x_i^{\left[\frac{p}{2}\right]}, x_i^{\left[\frac{p}{2}\right]}\right) \right\| = \left\| \sum_{i=1}^{n} \left|x_i^{\left[\frac{p}{2}\right]}\right|^2 \right\| = \left\| \sum_{i=1}^{n} |x_i|^p \right\|.$$

(c) The formula stated in (b) proves the required statements. □

Let A be a C*-algebra and let X be a Banach space. The question is clearly whether p-JB*-triple-summing and p-C*-summing operators coincide in $\mathcal{L}(A, X)$. A strengthening of the inequality in Lemma 3.12, (b) seems to be necessary.

Proposition 3.13 *Let A be a C*-algebra and let $p \geq 2$. Then the formula*

$$\sum_{i=1}^{n} L\left(x_i^{\left[\frac{p}{2}\right]}, x_i^{\left[\frac{p}{2}\right]}\right)(1) \geq \sum_{i=1}^{n} |x_i|^p,$$

holds for every finite sequence of elements $x_1, \ldots, x_n$ in A. In particular p-JB-triple-summing and p-C*-summing operators on A coincide.*

P r o o f. Considering A^{**} instead of A, we may assume that A is a von Neumann algebra.

Let e be a partial isometry in A. It is easy to check that $e^{\left[\frac{p}{2}\right]} = e$. Since $\frac{ee^* + e^*e}{2}$ is a positive element in the closed unit ball of A and $p/2 \geq 1$ we have

$$L\left(e^{\left[\frac{p}{2}\right]}, e^{\left[\frac{p}{2}\right]}\right)(1) = L(e,e)(1) = \frac{ee^* + e^*e}{2} \geq |e|^p = \left(\frac{ee^* + e^*e}{2}\right)^{\frac{p}{2}}.$$

Let a be an algebraic element in A when the latter is regarded as a JBW*-triple. That is, $a = \sum_i^k \alpha_i e_i$, where $\alpha_i \in \mathbb{R}^+$ and (e_i) are mutually orthogonal partial isometries (tripotents) in A. Since the e_i's are mutually orthogonal, we have $a^{[\frac{p}{2}]} = \sum_i^k \alpha_i^{\frac{p}{2}} e_i$. Thus,

$$L\left(a^{[\frac{p}{2}]}, a^{[\frac{p}{2}]}\right)(1) = \sum_i^k \alpha_i^{\frac{p}{2}} L(e_i, e_i)(1) \geq \sum_i^k \alpha_i^{\frac{p}{2}} |e_i|^p = \left(\sum_i^k \alpha_i^2 |e_i|^2\right)^{\frac{p}{2}} = |a|^p. \tag{3.6}$$

It is known that the set of tripotents is norm-total in every JBW*-triple, i.e., for every element x in A there exists a sequence (a_k) of algebraic elements in A converging in norm to x (compare [24, Lemma 3.11]). Since $\left(a_k^{[\frac{p}{2}]}\right)$ and $(|a_k|^p)$ converge in norm to $x^{[\frac{p}{2}]}$ and $|x|^p$, respectively, inequality (3.6) proves the statement. □

From now on, given an element a in a C*-algebra A, $\sigma_A(a)$ will stand for the *spectrum* of a in A.

Remark 3.14 The inequality established in the above Proposition 3.13 does not hold for $0 < p < 2$. Indeed, let us consider $A = C([0,1], M_2(\mathbb{C}))$ the C*-algebra of all continuous functions on $[0,1]$ with values in $M_2(\mathbb{C})$. We define $e \equiv e(t) := \begin{pmatrix} \sqrt{t} & \sqrt{1-t} \\ 0 & 0 \end{pmatrix} \in A$. In this case, we have

$$(ee^* + e^*e)(t) = \begin{pmatrix} 1+t & \sqrt{t(1-t)} \\ \sqrt{t(1-t)} & 1-t \end{pmatrix}.$$

Since for each $t \in [0,1]$, the spectrum $\sigma_{M_2(\mathbb{C})}(ee^* + e^*e(t)) = \left\{1+\sqrt{t}, 1-\sqrt{t}\right\}$, it can be easily seen that $\sigma_A\left(\frac{ee^*+e^*e}{2}\right) = [0,1]$. We claim that, for $0 < p < 2$, there is no positive constant $C > 0$ satisfying

$$C\, L\left(e^{[\frac{p}{2}]}, e^{[\frac{p}{2}]}\right)(1) \geq |e|^p.$$

Otherwise, we have

$$C\frac{ee^* + e^*e}{2} = C\, L(e,e)(1) = C\, L\left(e^{[\frac{p}{2}]}, e^{[\frac{p}{2}]}\right)(1) \geq |e|^p = \left(\frac{ee^* + e^*e}{2}\right)^{\frac{p}{2}},$$

which is impossible, since $Ct \ngeq t^{\frac{p}{2}}$ in $C[0,1]$.

However, for each $0 < p < 2$, the question whether every p-C*-summing operator on a C*-algebra is automatically p-JB*-triple-summing remains open.

Following standard arguments, a Pietsch factorisation theorem for p-JB*-triple- summing operators on JB*-triples can be established now.

Theorem 3.15 *Let T be a bounded operator from a JB*-triple E to a Banach space X. For each $p > 0$, the following assertions are equivalent.*

(a) *T is p-JB*-triple-summing.*

(b) *There is a state $\Psi \in D(\mathcal{L}(E), I_E)$ and a positive constant $C(T)$ such that*

$$\|T(x)\|^p \leq C(T)\Psi\Big(L\Big(x^{[\frac{p}{2}]}, x^{[\frac{p}{2}]}\Big)\Big),$$

for every $x \in E$.

(c) *There exist two norm-one functionals $\varphi_1, \varphi_2 \in E^*$ and a positive constant $K(T)$ such that*

$$\|T(x)\|^p \leq K(T)\left(\left\|x^{[\frac{p}{2}]}\right\|_{\varphi_1}^2 + \left\|x^{[\frac{p}{2}]}\right\|_{\varphi_2}^2\right),$$

for every $x \in E$.

Proof. (a) $\Rightarrow$ (b) Let us denote $K := D(\mathcal{L}(E), Id_E)$. Clearly, K is a weak*-compact subset in $\mathcal{L}(E)^*$. For every finite sequence $x_1, \ldots, x_k \in E$, we define the convex function $f_{x_1,\ldots,x_k} : K \longrightarrow \mathbb{R}$ by

$$f_{x_1,\ldots,x_k}(\Phi) := \sum_{i=1}^{k} \|T(x_i)\|^p - C_p(T)\, \Phi \left(\sum_{i=1}^{k} L\left(x_i^{\left[\frac{p}{2}\right]}, x_i^{\left[\frac{p}{2}\right]} \right) \right).$$

The set $\Gamma := \left\{ f_{x_1,\ldots,x_k} : x_1, \ldots, x_k \in E \right\} \subset C(K, \mathbb{R})$ is convex and hence concave in the terminology of [38, E.4]. Since for each $x_1, \ldots, x_k \in E$, the operator $S = \sum_{i=1}^{k} L\left(x_i^{\left[\frac{p}{2}\right]}, x_i^{\left[\frac{p}{2}\right]} \right)$ is hermitian, Sinclair's Theorem (compare [9, Theorem 11.17]) assures that

$$\|S\| = \sup_{\Phi \in K} |\Phi(S)| = \max_{\Phi \in K} |\Phi(S)|. \tag{3.7}$$

Thus, there exists $\Phi_S \in K$ satisfying that $\Phi_S(S) = \|S\|$, and hence

$$f_{x_1,\ldots,x_k}(\Phi_S) = \sum_{i=1}^{k} \|T(x_i)\|^p - C_p(T) \left\| \sum_{i=1}^{k} L\left(x_i^{\left[\frac{p}{2}\right]}, x_i^{\left[\frac{p}{2}\right]} \right) \right\| \leq 0.$$

By the Ky Fan lemma (see [38, E.4]) there exists an element $\Psi \in K$ such that $f_{x_1,\ldots,x_k}(\Psi) \leq 0$ for every $f_{x_1,\ldots,x_k} \in \Gamma$, which in particular implies that

$$\|T(x)\|^p \leq C(T) \Psi \left(L\left(x^{\left[\frac{p}{2}\right]}, x^{\left[\frac{p}{2}\right]} \right) \right),$$

for every $x \in E$.

(b) $\Rightarrow$ (c) Let $\Psi \in D(\mathcal{L}(E), I_E)$, satisfying the assumption (b). The map $\|.\|_\Psi$ is a prehilbertian seminorm on E. Denoting $N := \{x \in E : \|x\|_\Psi = 0\}$, then the quotient E/N can be completed to a Hilbert space H. Let us denote by Q the natural quotient map from E to H. By [33, Corollary 1] (see also [34, Corollary 1.11]) there exist two norm-one functionals $\varphi_1, \varphi_2 \in E^*$ such that the inequality

$$\|Q(x)\|^2 = \|x\|_\Psi^2 = \Psi(L(x,x)) \leq 4 \left(\|x\|_{\varphi_1}^2 + \|x\|_{\varphi_2}^2 \right)$$

holds for every $x \in E$. We therefore have:

$$\|T(x)\|^p \leq 4C(T) \left(\left\| x^{\left[\frac{p}{2}\right]} \right\|_{\varphi_1}^2 + \left\| x^{\left[\frac{p}{2}\right]} \right\|_{\varphi_2}^2 \right),$$

for every $x \in E$.

(c) $\Rightarrow$ (a) Let $\varphi \in S_{E^*}$ and $z \in S_{E^{**}}$ with $\varphi(z) = 1$. Since for every finite sequence $(x_1, \ldots, x_n)$ in E we have

$$\sum_i \left\| x_i^{\left[\frac{p}{2}\right]} \right\|_\varphi^2 = \sum_i \varphi \left\{ x_i^{\left[\frac{p}{2}\right]}, x_i^{\left[\frac{p}{2}\right]}, z \right\} = \varphi \sum_i L\left(x_i^{\left[\frac{p}{2}\right]}, x_i^{\left[\frac{p}{2}\right]} \right)(z) \leq \left\| \sum_{i=1}^{n} L\left(x_i^{\left[\frac{p}{2}\right]}, x_i^{\left[\frac{p}{2}\right]} \right) \right\|,$$

and hence $a)$ follows from $c)$. □

4 w-right-norm continuous holomorphic mappings

Given Banach spaces X and Y, letting $m = 1, 2, \ldots$, we shall denote by $\mathcal{L}(^m X, Y)$ the Banach space of all continuous m-linear mappings from $X^m = X \times \ldots \times X$ (m times) to Y, with respect to the pointwise vector operations and the norm defined by

$$\|A\| = \sup_{x_1 \neq 0, \ldots, x_m \neq 0} \frac{\|A(x_1, \ldots, x_m)\|}{\|x_1\| \cdots \|x_m\|}$$

where $A \in \mathcal{L}(^m X, Y)$ and $x_1, \ldots, x_m \in X$.

An element $A \in \mathcal{L}(^m X, Y)$ is said to be symmetric if

$$A(x_1, \ldots, x_m) = A(x_{\sigma(1)}, \ldots, x_{\sigma(m)})$$

for any permutation $\sigma : \{1, \ldots, m\} \longrightarrow \{1, \ldots, m\}$. The symbol $\mathcal{L}_s(^m X, Y)$ will denote the closed subspace of $\mathcal{L}(^m X, Y)$ of all symmetric continuous m-linear mappings.

A continuous m-homogeneous polynomial P from X to Y is a mapping $P : X \longrightarrow Y$ for which there is a unique $A \in \mathcal{L}_s(^m X, Y)$ such that

$$P(x) = A(x, \ldots, x) \quad \text{for any} \quad x \in X.$$

The m-linear operator A is called the *generating operator* for P and in the sequel will be denoted by $\widehat{P}$. By a 0-homogeneous polynomial we mean a constant function. $\mathcal{P}(^m X, Y)$ will denote the Banach space of all continuous m-homogeneous polynomials from X to Y, with respect the pointwise vector operations and the norm defined by

$$\|P\| = \sup_{x \neq 0} \frac{\|P(x)\|}{\|x\|}.$$

Every m-homogeneous polynomial $P : X \longrightarrow Y$ satisfies the following polarization formula:

$$\widehat{P}(x_1, \ldots, x_m) = \frac{1}{2^m m!} \sum_{\varepsilon_i = \pm 1} \varepsilon_1 \cdot \ldots \cdot \varepsilon_m \; P\left(\sum_{i=1}^{m} \varepsilon_i x_i\right), \tag{4.1}$$

Jointly w-right-norm continuous multilinear operators have been studied in [22, 30] and [37]. A multilinear operator $T : X_1 \times \ldots \times X_m \longrightarrow X$ is jointly w-right-to-norm continuous if and only if it is jointly w-right-to-norm continuous at 0 if and only if there exist reflexive Banach spaces $R_1, \ldots, R_m$ and bounded linear operators $T_i : X_i \longrightarrow R_i$ satisfying, for each x_i in X_i,

$$\|T(x_1, \ldots, x_m)\| \leq |||x_1|||_{T_1} \;\cdots\; |||x_m|||_{T_m}.$$

(compare [22, Theorem 4] and [37, Proposition 3.11] or [30, Theorem 1]).

The polarization formula (4.1) guarantees that an m-homogeneous polynomial P is w-right-norm continuous if and only if its generating multilinear operator is jointly w-right-norm continuous (at 0) if and only if P is w-right-norm continuous at 0. The corresponding affirmation for the strong* topology is also true.

Arens [3, 4] was the first author in considering extensions of bilinear operators to the product of the biduals. For multilinear operators, Aron and Berner introduced, in [5], a method to extend k-linear mappings to the product of the biduals that can be described as follows: Let $X_1, \ldots, X_k$, and X be Banach spaces and $T : X_1 \times \cdots \times X_k \longrightarrow X$ a k-linear operator. Let $\pi : \{1, \ldots, k\} \longrightarrow \{1, \ldots, k\}$ (denoted $i \longmapsto \pi_i$) be a permutation. We define the Aron-Berner extension of T associated to π

$$AB(T)_\pi : X_1^{**} \times \cdots \times X_k^{**} \longrightarrow X^{**}$$

by

$$AB(T)_\pi(z_1, \ldots, z_k) = \text{weak}^* - \lim_{\alpha_{\pi_1}} \cdots \text{weak}^* - \lim_{\alpha_{\pi_k}} T\big(x_1^{\alpha_1}, \ldots, x_k^{\alpha_k}\big),$$

where $(z_1, \ldots, z_k) \in X_1^{**} \times \cdots \times X_k^{**}$ and, for $1 \leq i \leq k$, $\big(x_i^{\alpha_i}\big)_{\alpha_i} \subset X_i$ is a net weak* convergent to z_i. $AB(T)_\pi$ is bounded and has the same norm as T. For each k-linear operator there are $k!$ possibly different extensions. However, for each symmetric k-linear operator T the restriction of $AB(T)_\pi$ to the diagonal does not depend on the permutation π.

Given an m-homogeneous polynomial $P : X \longrightarrow Y$, the m-homogeneous polynomial $AB(P) : X \longrightarrow Y$, $AB(P)(x) := AB(\widehat{P})_\pi(x, \ldots, x)$ (where π is any permutation of the set $\{1, \ldots, k\}$), will be called the Aron-Berner extension of P.

A *continuous polynomial* P from X to Y is a finite sum of continuous homogeneous polynomials. We shall denote by $\mathcal{P}(X,Y)$ the space of all continuous polynomials from X to Y with respect to pointwise vector operations. Following [31], a polynomial $P : X \longrightarrow Y$ is said to be *weakly compact* if P maps bounded sets in X into relatively weakly compact sets in Y.

We have already noticed that a bounded linear operator $T : X \longrightarrow Y$ is weakly compact if and only if T is w-right-norm continuous. The following examples show that none of these implications holds for continuous polynomials in general Banach spaces.

Example 4.1 Let $P : \ell_2 \longrightarrow \ell_1$ be the 2-homogeneous polynomial whose generating operator is defined by

$$\widehat{P} : \ell_2 \times \ell_2 \longrightarrow \ell_1,$$
$$\widehat{P}(x,y) = x \cdot y,$$

where $x \cdot y$ denotes the pointwise multiplication. It follows by Hölder's inequality that $\widehat{P}$ is well defined with $\|\widehat{P}\| \leq 1$. Since ℓ_2 is a reflexive Banach space, and for any reflexive Banach space the w-right topology coincides with the norm topology, we trivially have that P is w-right-norm continuous. However, P cannot be weakly compact because P maps the canonical basis of ℓ_2 to the canonical basis of ℓ_1 and the latter admits no weakly convergent subsequences.

A weakly compact polynomial on a Banach space X need not be w-right-norm continuous, even when X satisfies the Dunford-Pettis property.

Example 4.2 Since the interval $\left[\frac{1}{2},1\right]$ is not scattered, there is a continuous surjective linear map $q : C\left(\left[\frac{1}{2},1\right]\right) \longrightarrow \ell_2$ (compare [13, Corollary 4.16]). By the open mapping theorem, we can pick $f_n \in C\left(\left[\frac{1}{2},1\right]\right)$ with $\|f_n\| = 1$ such that $q(f_n) = e_n$, for every $n \in \mathbb{N}$, where (e_n) denotes the canonical basis of ℓ_2. We can define a sequence (g_n) in $C([0,1])$ satisfying that $g_n|_{[\frac{1}{2},1]} = f_n$ and $g_n|_{[0,\frac{1}{4}]} = 0$.

On the other hand, the assignment $f \longmapsto \left(f\left(\frac{1}{6n}\right) - f(0)\right)_n$ defines a linear operator $p : C([0,1]) \longrightarrow c_0$. Finally, we define a symmetric bilinear map

$$V : C([0,1]) \times C([0,1]) \longrightarrow \ell_2$$

given by $V(f,g) := p(f) \cdot q\left(g|_{[\frac{1}{2},1]}\right) + p(g) \cdot q\left(f|_{[\frac{1}{2},1]}\right)$, where for $a \in c_0$ and $b \in \ell_2$, $a \cdot b \in \ell_1$ is defined by $(a \cdot b)_n = a_n b_n$. It is clear that V is weakly compact. We claim that V is not jointly w-right-norm continuous. Indeed, let us pick a sequence (x_n) of mutually orthogonal continuous functions in $C([0,1])$ satisfying $\|x_n\| = x_n\left(\frac{1}{6n}\right) = 1$. By definition, (x_n) is a w-right-null sequence in $C([0,1])$ (compare [35, Lemma 13]), while (g_n) is a bounded sequence in $C([0,1])$. Thus, if V were jointly w-right-norm continuous, then Proposition 3.11 in [37], would imply that

$$1 = \|e_n\| = \|V(x_n, g_n)\| \longrightarrow 0,$$

which is impossible.

Let us recall that an operator is said to be *pseudo weakly compact* if it is sequentially w-right-norm continuous. A Banach space X is called *sequentially right* if every pseudo weakly compact operator from X to another Banach space is weakly compact. C*-algebras, JB*-triples and Banach spaces satisfying Pelczynski's Property (V) are examples of sequentially right spaces (compare [35]).

It is also known that a bounded linear operator $T : X \longrightarrow Y$ is weakly compact if and only if its bitranspose remains Y-valued. In the multilinear setting, a similar question has been recently considered in [37]. We first recall the following definition introduced in [37]: Given Banach spaces $X_1, \ldots, X_k, X$, a multilinear operator $T : X_1 \times \cdots \times X_k \longrightarrow X$ is *right quasi completely continuous* (RQCC) if for arbitrary w-right Cauchy sequences $\left(x_i^n\right)_n \subset X_i$ $(1 \leq i \leq k)$, the sequence $\left(T\left(x_1^n, \ldots, x_k^n\right)\right)_n$ converges in norm, equivalently, for every sequence $\left(x_i^n\right) \subset X_i$ which is w-right-convergent to $x_i \in X_i$ $(1 \leq i \leq k)$ we have

$$\lim_n \|T(x_1^n, \ldots, x_k^n) - T(x_1, \ldots, x_k)\| = 0,$$

that is T is jointly sequentially w-right-norm continuous. The following result follows from Proposition 3.3 and Theorem 3.8 in [37]. Let $X_1, \ldots, X_k$ be non zero sequentially right Banach spaces and let $T : X_1 \times \cdots \times X_k \longrightarrow X$

be a multilinear operator. Then T is RQCC if and only if all of the Aron-Berner extensions of T are X-valued if and only if T has an X-valued Aron-Berner extension. We shall study this equivalence in the case of holomorphic mappings between complex Banach spaces.

We now consider weakly compact holomorphic mappings. Let X, Y be two Banach spaces, a mapping $f : X \longrightarrow Y$ is said to be a *holomorphic map* if for each $x \in X$ there exists a sequence of polynomials

$$\hat{d}^n f(x) \in \mathcal{P}(^n X, Y)$$

and a neighborhood V_x of x such that the series

$$\sum_{n=0}^{\infty} \frac{1}{n!} \hat{d}^n f(x)(y - x)$$

converges uniformly to $f(y)$ for every $y \in V_x$.

A holomorphic function $f : X \longrightarrow Y$ is said to be of *bounded type* if it is bounded on all bounded subsets of X. The polynomial series at zero $f(y) = \sum_{n=0}^{\infty} \frac{1}{n!} \hat{d}^n f(0)(y)$ of such a function have infinite radius of uniform convergence, i.e.: $\limsup \|\frac{1}{n!} \hat{d}^n f(0)\|^{\frac{1}{n}} = 0$ (compare [16, Section 6.2]).

If $f : X \longrightarrow Y$ is a holomorphic function of bounded type and $f(y) = \sum_{n=0}^{\infty} \frac{1}{n!} \hat{d}^n f(0)(y)$ $(y \in X)$ is its Taylor series at 0, it follows by [21, Section 2] or [16, Proposition 6.16] that the assignment

$$y \longmapsto AB(f)(y) = \sum_{n=0}^{\infty} \frac{1}{n!} AB(\hat{d}^n f(0))(y), \quad (y \in X^{**})$$

defines a holomorphic function of bounded type, $AB(f) : X^{**} \longrightarrow Y^{**}$, called the Aron-Berner extension of f.

A holomorphic map $f : X \longrightarrow Y$ is said to be *weakly compact* if for every $x \in X$ there exists a neighborhood V_x of x such that $f(V_x)$ is a relatively weakly compact set of Y. See [28] or [17] for details about holomorphic maps. The Examples 4.1, and 4.2 show that weak compactness is not the correct property to guarantee Aron-Berner extensions valued in the same codomain space.

We shall now show that w-right-norm continuity of a holomorphic mapping f implies w-right-norm continuity of its derivatives at every point.

Proposition 4.3 *Let $f : X \longrightarrow Y$ be a holomorphic mapping between two Banach spaces. Then the following statements hold*:

(a) *If f is w-right-norm continuous* (*respectively, strong*-norm continuous*)*, then the polynomial $\hat{d}^n f(x)$ is w-right-norm continuous* (*respectively, strong*-norm continuous*) *for every $n \in \mathbb{N}$ and every $x \in X$.*

(b) *If f is sequentially w-right-norm continuous* (*respectively, strong*-norm continuous*)*, then the polynomial $\hat{d}^n f(x)$ is sequentially w-right-norm continuous* (*respectively, strong*-norm continuous*) *for every $n \in \mathbb{N}$ and every $x \in X$.*

P r o o f. We shall only include here the proof of the statements concerning the w-right topology, the proofs of those affirmations concerning the strong* topology follow similarly.

(a) Let us fix $x \in X$. By hypothesis, there exist reflexive spaces $R_1, \ldots, R_k$, bounded linear operators $T_i : X \longrightarrow R_i$ $(i \in \{1, \ldots, k\})$ and $\delta > 0$ satisfying that $f(W) \in f(x) + B(Y)$, where

$$W = \{y \in X : \|x - y\|_{T_i} < \delta, \forall i \in \{1, \ldots, k\}\}.$$

Since $W_0 := \{y \in X : \|y\|_{T_i} < \delta, \forall i \in \{1, \ldots, k\}\}$ is a balanced set, it follows by [41, Lemma 3.1] (compare also the proof of [6, Proposition 3.4]), that

$$\frac{1}{n!} \hat{d}^n f(x)(W_0) \subset \overline{co} f(x + W_0) \subseteq \overline{co}(f(x) + B(Y)),$$

where $\overline{co} A$ denotes the convex balanced hull of A. In particular there exists a constant $M_n > 0$ satisfying that

$$\|\hat{d}^n f(x)(y)\| \leq M_n, \quad \text{for all} \quad y \in W_0.$$

Taking $R = \oplus^{\ell_2} R_i$ and $T : X \longrightarrow R$, $x \longmapsto (\delta/2)^{-1}(T_i(x))$, it can be easily seen that, for each $y \in X$ with $T(y) \neq 0$, we have $\frac{y}{\|y\|_T} \in W_0$, and hence,

$$\|\hat{d}^n f(x)(y)\| \leq M_n \|y\|_T^n .$$

For each $y \in \ker(T)$, and $t > 0$, ty lies in W_0, thus $t^n \|\hat{d}^n f(x)(y)\| = \|\hat{d}^n f(x)(ty)\| \leq M_n$, which implies that $\hat{d}^n f(x)(y) = 0$. We have then shown that

$$\|\hat{d}^n f(x)(y)\| \leq M_n \|y\|_T^n ,$$

for all $y \in X$. This proves that $\hat{d}^n f(x)$ is w-right-norm continuous at 0, which gives the desired statement.

(b) We assume that f is sequentially w-right-norm continuous. Let (y_k) be a sequence in X converging in the w-right topology to $y \in X$. Let us fix $x \in X$ and φ in the closed unit ball of Y^*. Defining $g_k(\lambda) := \varphi f(x + \lambda y_k)$ and $g(\lambda) := \varphi f(x + \lambda y)$, it follows by Cauchy's integral formula that

$$\begin{aligned} \left| \frac{1}{n!} \varphi\big(\hat{d}^n f(x)(y_k) - \hat{d}^n f(x)(y)\big) \right| &= \left| \left(g_k^{(n)}(0) - g^{(n)}(0) \right) / n! \right| \\ &\leq \sup\{ |(g_k - g)(\lambda)| : |\lambda| = 1 \} \\ &\leq \sup\{ \|f(x + \lambda y_k) - f(x + \lambda y)\| : |\lambda| = 1 \}. \end{aligned}$$

Taking supreme over all φ in the closed unit ball of Y^*, we have

$$\|\hat{d}^n f(x)(y_k) - \hat{d}^n f(x)(y)\| \leq n! \sup\{ \|f(x + \lambda y_k) - f(x + \lambda y)\| : |\lambda| = 1 \}.$$

Finally, since f is sequentially w-right-norm continuous, it can be easily seen that

$$\lim_{k \longrightarrow \infty} \sup\{ \|f(x + \lambda y_k) - f(x + \lambda y)\| : |\lambda| = 1 \} = 0.$$

□

Theorem 4.4 *Let X be a sequentially right space, Y a Banach space and let $f : X \longrightarrow Y$ be a holomorphic function of bounded type. Then f is sequentially w-right-norm continuous if and only if $AB(f)$ is Y-valued.*

Proof. Let $f(y) = \sum_{n=0}^{\infty} \frac{1}{n!} \hat{d}^n f(0)(y)$, $(y \in X)$ and $AB(f)(y) = \sum_{n=0}^{\infty} \frac{1}{n!} AB(\hat{d}^n f(0))(y)$, $(y \in X^{**})$ be the Taylor series of f and $AB(f)$ at zero, respectively. If f is sequentially w-right-norm continuous, then Proposition 4.3 $b)$ implies that, for each natural n, $\hat{d}^n f(0)$ is sequentially w-right-norm continuous. The polarization formula (4.1) implies that, for each natural n, the generating multilinear operator of $\hat{d}^n f(0)$ is jointly sequentially w-right-norm continuous or RQCC. Theorem 3.8 in [37] guarantees that $AB(\hat{d}^n f(0))$ is Y-valued for all natural n. The uniform convergence of the Taylor series at zero of the function $AB(f)$ assures that $AB(f)(X^{**}) \subseteq Y$.

Assume now that $AB(f)(X^{**}) \subseteq Y$. Since X^{**} is a balanced set, it follows by [41, Lemma 3.1] (compare also the proof of [6, Proposition 3.4]), that

$$\frac{1}{n!} AB(\hat{d}^n f(0))(X^{**}) \subset \overline{co} AB(f)(X^{**}) \subseteq Y.$$

It follows again from Theorem 3.8 in [37] that $\hat{d}^n f(0)$ is sequentially w-right-norm continuous. The desired statement will finally follow from the uniform convergence of the Taylor series. □

Acknowledgements The second and third named authors were partially supported by project I+D MCYT No. MTM2008-02186, and Junta de Andalucía grants FQM0199 and FQM3737. Part of the paper was written while D. Puglisi was visiting the Departamento de Análisis Matemático at Universidad de Granada (Spain); he would like to thank sincerely A. M. Peralta for his warm hospitality. We are also grateful to an anonymous referee for pointing out a gap in an earlier stage of this paper and to Professor M. Maestre for fruitful discussions and comments.

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