

STABILITY ON LOCAL UNCONDITIONAL STRUCTURE AND THE GORDON-LEWIS PROPERTY IN TENSOR PRODUCTS

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ABSTRACT. We show that if X is an \mathcal{L}_1 -space (resp. \mathcal{L}_∞ -space) and Y a Banach space, then $X \hat{\otimes} Y$ (resp. $X \check{\otimes} Y$) has the local unconditional structure (l.u.st.) property (or GL-property) if Y does.

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1. Introduction. The idea of extending the local unconditional structure (l.u.st.) from two Banach spaces to their injective and projective tensor products is already in the fundamental paper of Y. Gordon and D.R. Lewis [6]. In particular one of the results they get is the following: if E and F are \mathcal{L}_p -spaces ($1 < p < \infty$), then none of $E \check{\otimes} F$, $E \hat{\otimes} F$, $(E \check{\otimes} F)^*$, $(E \hat{\otimes} F)^*$, $(E \check{\otimes} F)^{**}$, $(E \hat{\otimes} F)^{**}$, etc... has l.u.st. In this paper we study the case $p = 1$ and $p = \infty$. In fact we show that if X is an \mathcal{L}_1 -space (resp. \mathcal{L}_∞ -space) and Y a Banach space, then $X \hat{\otimes} Y$ (resp. $X \check{\otimes} Y$) has the l.u.st. property (or GL-property) if Y does.

In the same paper of Gordon-Lewis ([6]) one can find the proof that other tensor products fail to have l.u.st.: $\ell_\infty \hat{\otimes} \ell_p$ for $1 < p \leq \infty$ and $\ell_1 \check{\otimes} \ell_p$ for $1 \leq p < \infty$ do not have l.u.st. and so it is impossible to prove the theorem even for $\mathcal{L}_\infty \hat{\otimes} X$ and $\mathcal{L}_1 \check{\otimes} X$.

Actually the l.u.st. property was first introduced in a paper by E. Dubinsky, A. Pelczynski and H.P. Rosenthal ([5]) in a manner that is equivalent to: X has l.u.st. iff there exists a constant $\lambda > 1$ such that, for each $E \in \mathcal{F}_X$, there is an $F \in \mathcal{F}_X$ with $E \subseteq F$ and $ub(F) < \lambda$. In the sequel we will consider the definition due to Y. Gordon and D.R. Lewis ([6]).

There is a vast literature on the subject, including papers of Junge, Gordon, Lewis, Maurey, N.J. Nielsen, Pisier, Tomczak-Jaegermann, who have published intensively on the *gl* and *l.u.st* constants and summing operators (see [6], [7], [8], [12], [2], [13], to get other references there). A result related to the GL-property in injective tensor products of considerable interest is the following:

THEOREM 1. ([8]) *Let X and Y be Banach spaces. Then $gl(X_k \overset{\vee}{\otimes} Y_k) \xrightarrow{k \rightarrow \infty} \infty$ for every increasing sequence $\{X_k\}_{k=1}^{\infty}$ and $\{Y_k\}_{k=1}^{\infty}$ of finite-dimensional subspaces of X and Y respectively, if, and only if, X and Y do not contain subspaces uniformly isomorphic to the ℓ_{∞}^n spaces (i.e., X and Y have finite cotype).*

We also make reference to [4] for definitions and notations, and [9] for tensor product notions.

2. Preliminaries. Let X and Y be a Banach spaces. Throughout this paper we denote by B_X the ball of the Banach space X , $\mathcal{L}(X, Y)$ (resp. $\mathcal{K}(X, Y)$) the space of bounded (resp. compact) linear operators from X to Y , and with $\mathcal{B}(X, Y)$ the space of bilinear continuous maps from $X \times Y$ to the scalar field \mathbb{K} . We will denote also by \mathcal{F} the class of finite dimensional Banach spaces, and by \mathcal{F}_X the collection of all finite dimensional subspaces of X .

We briefly recall the notion of projective and injective tensor products of Banach spaces (see [9] for a complete reference). Let X and Y be Banach spaces. The *projective tensor product* $X \overset{\wedge}{\otimes} Y$ of X and Y is the completion of the algebraic tensor product $X \otimes Y$ in the projective norm

$$\|u\|_{\wedge} = \inf \left\{ \sum_{i=1}^n \|x_i\| \|y_i\| : u = \sum_{i=1}^n x_i \otimes y_i \right\} \quad u \in X \otimes Y$$

where the infimum is taken over all possible representations of u . Grothendieck ([9]) described the projective tensor product of X and Y in the following way: an element $u \in X \overset{\wedge}{\otimes} Y$ has the representation

$$u = \sum_{n=1}^{\infty} x_n \otimes y_n \quad \text{with} \quad \sum_{n=1}^{\infty} \|x_n\| \|y_n\| < \infty$$

and the projective tensor norm of u as

$$\|u\|_{\wedge} = \inf \left\{ \sum_{n=1}^{\infty} \|x_n\| \|y_n\| : u = \sum_{n=1}^{\infty} x_n \otimes y_n \right\}.$$

The *injective tensor product* $X \overset{\vee}{\otimes} Y$ of X and Y is the completion of the algebraic tensor product $X \otimes Y$ in the injective norm

$$\|u\|_{\vee} = \sup \{ |x^* \otimes y^*(u)| : x^* \in B_{X^*}, y^* \in B_{Y^*} \}.$$

Note that this is the natural way to view $X \overset{\vee}{\otimes} Y$ as a subspace of $\mathcal{B}(X^*, Y^*)$.

A bounded linear operator $u : X \rightarrow Y$ is called 1-factorizable if there exist a measure space (Ω, Σ, μ) and bounded linear operators $b : L_1(\mu) \rightarrow Y^{**}$, $a : X \rightarrow L_1(\mu)$ such that

$$k_Y u : X \xrightarrow{a} L_1(\mu) \xrightarrow{b} Y^{**}$$

where $k_Y : Y \rightarrow Y^{**}$ is the natural embedding. We write

$$\gamma_1(u) = \inf \|a\| \|b\|$$

where the infimum runs over all the possible factorizations of $k_Y u$ we have indicated. The collection of all 1-factorable operators from X to Y is denoted by $\Gamma_1(X, Y)$, which is a Banach space with the norm γ_1 .

DEFINITION 2. A Schauder basis $(x_n)_{n \in \mathbb{N}}$ for a Banach space X is said to be an *unconditional basis* for the space if, calling $(x_n^*)_{n \in \mathbb{N}}$ the biorthogonal sequence in X^* of the basis, there exists a constant $\lambda \geq 1$ so that $\sum_{n=1}^{\infty} t_n \langle x_n^*, x \rangle x_n$ converges for every $(t_n)_{n \in \mathbb{N}} \in \ell_{\infty}$ and

$$\left\| \sum_{n=1}^{\infty} t_n \langle x_n^*, x \rangle x_n \right\| \leq \lambda \left\| \sum_{n=1}^{\infty} \langle x_n^*, x \rangle x_n \right\| \quad \forall (t_n)_{n \in \mathbb{N}} \in B_{\ell_{\infty}}.$$

If we label $\lambda_{(x_n)_{n \in \mathbb{N}}}$ the smallest of such λ , we can define the *unconditional basis constant* of the space X :

$$ub(X) := \sup \{ \lambda_{(x_n)_{n \in \mathbb{N}}} : (x_n)_{n \in \mathbb{N}} \text{ is an unconditional basis of } X \}.$$

3. The local unconditional structure.

DEFINITION 3. A Banach space X has *local unconditional structure (l.u.st.)* if there exists a constant $\Lambda \geq 1$ such that, for every $E \in \mathcal{F}_X$, the canonical embedding $E \hookrightarrow X$ factors through a Banach space Y with unconditional basis via two operators $E \xrightarrow{v} Y \xrightarrow{u} X$ satisfying $\|u\| \|v\| ub(Y) \leq \Lambda$. The smallest of these constants is called the *l.u.st. constant* of X and is denoted by $\Lambda(X)$.

Now we start to recall some preliminary lemmas with easy proofs.

LEMMA 4. If $E \in \mathcal{F}$ then $ub(E) = ub(E^*)$.

LEMMA 5. If $F \in \mathcal{F}$ then $ub(\ell_{\infty}^n \overset{\vee}{\otimes} F) \leq ub(F)$.

An easy consequence of the previous lemmas is the following:

COROLLARY 6. If $F \in \mathcal{F}$ then $ub(\ell_1^n \overset{\wedge}{\otimes} F) \leq ub(F)$.

The following easy lemma will be very useful; we include the proof for the sake of completeness.

LEMMA 7. *Let X be a normed space and \tilde{X} its completion. If X has l.u.st. then \tilde{X} has l.u.st.*

Proof. For the lemma's proof we need the following result ([4] Lemma 17.3).

Let $\{x_1, \dots, x_n\}$ be a basis for the finite dimensional normed space E . Given $0 < \epsilon < 1$ there is a $\delta > 0$ such that if X is a Banach space containing E and if $\tilde{x}_1, \dots, \tilde{x}_n \in X$ satisfy $\|\tilde{x}_k - x_k\| \leq \delta$ ($1 \leq k \leq n$), then there exists an operator $u \in \mathcal{L}(X, X)$ such that

$$(a) \quad u(\tilde{x}_k) = x_k \quad (1 \leq k \leq n)$$

$$(b) \quad (1 - \epsilon)\|x\| \leq \|u(x)\| \leq (1 + \epsilon)\|x\|.$$

Finally, from (b) it follows that u is invertible with $\|u\| \leq 1 + \epsilon$ and $\|u^{-1}\| \leq (1 - \epsilon)^{-1}$.

Let \tilde{G} be a finite dimensional subspace of \tilde{X} . Suppose that $\tilde{G} = \text{span}\{u_1, \dots, u_n\}$. Now, fix $0 < \epsilon < 1$ and we consider δ as above. Since X is dense in \tilde{X} , we choose $u_k \in X$ so that $\|u_k - \tilde{u}_k\| < \epsilon$. Let $G = \text{span}\{u_1, \dots, u_n\}$. Then G is a finite dimensional subspace of X . Therefore there exist a Banach space F with unconditional basis, $a \in \mathcal{L}(G, F)$ and $b \in \mathcal{L}(F, X)$ so that

$$i = b \circ a \quad , \quad \|a\| \|b\| \text{ub}(F) \leq \Lambda(X)$$

where we are denoting with i the natural map from G to X .

Let $u \in \mathcal{L}(\tilde{X}, \tilde{X})$ so that

$$(a) \quad u(u_k) = \tilde{u}_k \quad (1 \leq k \leq n)$$

$$(b) \quad \|u\| \leq 1 + \epsilon \text{ and } \|u^{-1}\| \leq (1 - \epsilon)^{-1}.$$

Then $u|_X \circ b \circ a \circ u^{-1}|_{\tilde{G}}$ is just the natural map from \tilde{G} to \tilde{X} with $\|a \circ u^{-1}|_{\tilde{G}}\| \|u|_X \circ b\| \text{ub}(F) \leq \Lambda(X)$. \square

In the case of finite dimensions the two preceding computations lead us to the following two theorems.

THEOREM 8. *Let X be a Banach space and K a compact Hausdorff space. Then $C(K, X)$ has l.u.st. if X has and $\Lambda(C(K, X)) \leq \Lambda(X)$.*

Proof. By the preceding lemma it is enough to show that $C(K) \otimes X$ has l.u.st when equipped with the injective norm $\|\cdot\|_{\vee}$.

$C(K)$ is an \mathcal{L}_∞ -space, so if $\varepsilon > 0$ is given and $E \in \mathcal{F}_{C(K)}$, there are an $\tilde{E} \in \mathcal{F}_{C(K)}$, $E \subseteq \tilde{E}$, and an isomorphism $u: \tilde{E} \rightarrow \ell_\infty^{\dim \tilde{E}}$ with $\|u\| \|u^{-1}\| < 1 + \varepsilon$.

X has l.u.st., so, given $F \in \mathcal{F}_X$, there are an $\tilde{F} \in \mathcal{F}_X$, $F \subseteq \tilde{F}$, and $v: F \rightarrow \tilde{F}$, $w: \tilde{F} \rightarrow X$ with $w \circ v = i_F$ such that $\|v\| \|w\| \text{ub}(\tilde{F}) \leq \Lambda(X)$.

If $G \in \mathcal{F}_{C(K) \otimes X}$ we can find an $E \in \mathcal{F}_{C(K)}$ and an $F \in \mathcal{F}_X$ such that $G \subseteq E \otimes F \subseteq C(K) \otimes X$, where the two inclusions are isometries when each of the two tensor products is equipped with the injective norm, because the injective norm is injective.

Let us name our isometries:

$$I: G \hookrightarrow E \otimes F, \quad i: E \hookrightarrow \tilde{E}, \quad j: \tilde{E} \hookrightarrow C(K).$$

Now put them together:

$$G \xrightarrow{I} E \otimes F \xrightarrow{i \otimes id_F} \tilde{E} \otimes F \xrightarrow{u \otimes v} \ell_\infty^{\dim \tilde{E}} \otimes \tilde{F} \xrightarrow{u^{-1} \otimes w} \tilde{E} \otimes X \xrightarrow{j \otimes id_X} C(K) \otimes X$$

and call $U = (u \otimes v) \circ (i \otimes i_F) \circ I$ and $V = (j \otimes id_X) \circ (u^{-1} \otimes w)$. Notice that $V \circ U$ is the natural inclusion of G into $C(K) \otimes X$ and, taking into account our previous Lemma 4,

$$\begin{aligned} & \|U\| \|V\| \text{ub}(\ell_\infty^{\dim \tilde{E}} \otimes \tilde{F}) \\ & \leq \|u\| \|v\| \|i\| \|i_F\| \|I\| \|j\| \|id_X\| \|u^{-1}\| \|w\| \text{ub}(\tilde{F}) \\ & = \|u\| \|u^{-1}\| \|v\| \|w\| \text{ub}(\tilde{F}) \leq (1 + \varepsilon) \Lambda(X). \end{aligned}$$

Because $\varepsilon > 0$ is arbitrary, we get: $C(K, X)$ has l.u.st. and $\Lambda(C(K, X)) \leq \Lambda(X)$. \square

The proof of the next theorem is similar to the previous one and for that reason we will not include the proof.

THEOREM 9. *If (Ω, Σ, μ) is a measure space and X is a Banach space with l.u.st., then $L_1(\mu, X)$ has l.u.st.*

4. The Gordon-Lewis property.

DEFINITION 10. A Banach space X has the *Gordon Lewis property* (or X is a *GL-space*) if every 1-summing operator from X to ℓ_2 is 1-factorable. This is equivalent to requiring the existence of a constant c such that

$$\gamma_1(u) \leq c \pi_1(u)$$

for every $u \in \Pi_1(X, \ell_2)$. The smallest of these constants will be denoted by $gl(X)$.

Remark. Analogous to Lemma 7 above we have: Let X be a normed space with completion \tilde{X} . If X is a GL-space then \tilde{X} is a GL-space.

This is clear since, if $u \in \Pi_1(\tilde{X}, \ell_2)$ then $u|_X \in \Pi_1(X, \ell_2) = \Gamma_1(X, \ell_2)$. That means $u \in \Gamma_1(\tilde{X}, \ell_2)$.

THEOREM 11. $C(K, X)$ has the GL-property if X does.

Proof. Let $u : C(K, X) \rightarrow \ell_2$ be a 1-summing operator. By a result of C. Swartz [14] there exists a vector measure $m : \mathcal{B}(K) \rightarrow \Pi_1(X, \ell_2)$ from the Borel sets of K of bounded variation such that

$$u(f) = \int_K f(k) dm(k) \quad \forall f \in C(K, X).$$

Now, since X has the GL-property, we know that for each $A \in \mathcal{B}(K)$ there are bounded linear operators $b_A : X \rightarrow L_1(\mu_A)$ and $a_A : L_1(\mu_A) \rightarrow \ell_2$ such that the following diagram commutes

$$\begin{array}{ccc} X & \xrightarrow{m(A)} & \ell_2 \\ b_A \searrow & & \nearrow a_A \\ & L_1(\mu_A) & \end{array}$$

Let $|m|$ be the variation of the vector measure m , and define

$$(*) \quad \tilde{u} : L_1(|m|, X) \rightarrow \ell_2$$

by

$$\tilde{u}(f) = \int_K f d|m| \quad \forall f \in L_1(|m|, X).$$

By definition \tilde{u} is a bounded linear operator. Let us define

$$\tilde{b} : L_1(|m|, X) \rightarrow \left(\bigoplus_{A \in \mathcal{B}(K)} L_1(\mu_A) \right)_{\ell_1}$$

by

$$\tilde{b}\left(\sum_{i=1}^n \chi_{A_i} \otimes x_i\right) = \begin{cases} b_{A_i}(x_i), & A = A_i; \\ 0, & \text{otherwise.} \end{cases}$$

Of course we are defining \tilde{b} on a dense subset of $L_1(|m|, X)$, and by continuity we can extend it on whole space. Let us also define

$$a : \left(\bigoplus_{A \in \mathcal{B}(K)} L_1(\mu_A) \right)_{\ell_1} \rightarrow \ell_2$$

by

$$a\left(\sum_{A \in \mathcal{B}(K)} (f_A)_{A \in \mathcal{B}(K)}\right) = \sum_{A \in \mathcal{B}(K)} a_A(f_A).$$

Note that the operators \tilde{b} and a are well defined since m is a measure of finite variation. Finally, by (*) we have

$$\begin{aligned} \tilde{u}\left(\sum_{i=1}^n \chi_{A_i} \otimes x_i\right) &= \sum_{i=1}^n m(A_i)(x_i) \\ &= \sum_{i=1}^n a_{A_i} \circ b_{A_i}(x_i) \\ &= a \circ \tilde{b}\left(\sum_{i=1}^n \chi_{A_i} \otimes x_i\right) \end{aligned}$$

so that $\tilde{u} = a \circ \tilde{b}$. Since $\|\cdot\|_{L_1(|m|, X)} \leq \|\cdot\|_{C(K, X)}$ we have that

$$b = \tilde{b}|_{C(K, X)} : C(K, X) \longrightarrow \left(\bigoplus_{A \in \mathcal{B}(K)} L_1(\mu_A) \right)_{\ell_1}$$

is still a bounded linear operator and, of course, we get

$$u = a \circ b,$$

so that u factors through an L_1 -space. \square

Using a duality argument it is also easy to show that

THEOREM 12. $L_1(\mu, X)$ has the GL-property if X does.

5. Generalization to \mathcal{L}_1 and \mathcal{L}_∞ spaces. To extend the above results on \mathcal{L}_1 and \mathcal{L}_∞ -spaces we need the following lemmas :

LEMMA 13. $X^{**} \overset{\vee}{\otimes} Y$ is a closed subspace of $(X \overset{\vee}{\otimes} Y)^{**}$.

Proof. Denote the space of integral bilinear maps (see [9] or [3]) by $B^\wedge(X, Y)$, and note that we can consider it to be a closed subspace of $B^\wedge(X^{**}, Y)$. We can then view an element $x^{**} \otimes y \in X^{**} \overset{\vee}{\otimes} Y$ as element in $(X \overset{\vee}{\otimes} Y)^{**}$ as follows:

$$x^{**} \otimes y(\phi) = \phi(x^{**}, y) \quad \text{for all } \phi \in (X \overset{\vee}{\otimes} Y)^* = B^\wedge(X, Y).$$

Hence we have

$$|\phi(x^{**} \otimes y)| \leq \|\phi\|_\wedge \|x^{**} \otimes y\|_\vee$$

where we denote by $\|\phi\|_\wedge$ the norm of ϕ as element of $B^\wedge(X, Y)$. It follows that $x^{**} \otimes y \in (X \overset{\vee}{\otimes} Y)^{**}$.

Now we have to show that $\|\sum_{i=1}^p x_i^{**} \otimes y_i\|_\vee = \|\sum_{i=1}^p x_i^{**} \otimes y_i\|_{(X \overset{\vee}{\otimes} Y)^{**}}$. The inequality $\|\sum_{i=1}^p x_i^{**} \otimes y_i\|_\vee \leq \|\sum_{i=1}^p x_i^{**} \otimes y_i\|_{(X \overset{\vee}{\otimes} Y)^{**}}$ follows easily from the definition of \vee and Goldstine's theorem. For the converse

$$\begin{aligned} \left\| \sum_{i=1}^p x_i^{**} \otimes y_i \right\|_{(X \overset{\vee}{\otimes} Y)^{**}} &= \sup \left\{ \left| \left(\sum_{i=1}^p x_i^{**} \otimes y_i \right) (\phi) \right| : \phi \in B^\wedge(X, Y), \|\phi\|_\wedge \leq 1 \right\} \\ &\leq \sup \left\{ \left| \left(\sum_{i=1}^p x_i^{**} \otimes y_i \right) (\phi) \right| : \phi \in B^\wedge(X^{**}, Y), \|\phi\|_\wedge \leq 1 \right\} \\ &= \left\| \sum_{i=1}^p x_i^{**} \otimes y_i \right\|_\vee. \end{aligned}$$

Since the elements of the type $\sum_{i=1}^p x_i \otimes y_i$ are dense in $X^{**} \overset{\vee}{\otimes} Y$, we are done. \square

LEMMA 14. *Let X and Y be Banach spaces such that X^{**} or Y has the bounded approximation property. Then $X^{**} \hat{\otimes} Y$ is isomorphic to a closed subspace of $(X \hat{\otimes} Y)^{**}$.*

Proof. The trace duality $\Phi : X^{**} \hat{\otimes} Y \rightarrow (X \hat{\otimes} Y)^{**} = (\mathcal{L}(X, Y^*))^*$ given by

$$\Phi(z)(\phi) = \langle \phi, z \rangle$$

has norm 1. The canonical map $I : X^{**} \hat{\otimes} Y \rightarrow (X^* \check{\otimes} Y^*)^* = (K(X, Y^*))^*$ is an isometric embedding since X^{**} or Y^* has B.A.P. If $\alpha : (\mathcal{L}(X, Y^*))^* \rightarrow (K(X, Y^*))^*$ is the restriction map, we have $\alpha\Phi = I$. Then Φ is an isometric embedding as well. \square

THEOREM 15. *Let X and Y be Banach spaces. We have*

1. *If X is an \mathcal{L}_∞ -space, then $X \check{\otimes} Y$ has the l.u.st. property if Y does.*
2. *If X is an \mathcal{L}_1 -space, then $X \hat{\otimes} Y$ has the l.u.st. property if Y does.*

Proof. 1. Let S be a finite dimensional subspace of $X \check{\otimes} Y$. Since X is an \mathcal{L}_∞ -space, X^{**} has the metric approximation property and so $X^{**} \check{\otimes} Y = K_{w^*}(Y^*, X^{**})$ (= the Banach space of $w^* - w$ continuous compact operators from Y^* into X^{**}); furthermore, X^{**} is complemented in some $C(K)$ space and so $K_{w^*}(Y^*, X^{**})$ is complemented in $C(K, Y)$ (see [1]) by a projection P . We have

$$S \xrightarrow{i} X \check{\otimes} Y \xrightarrow{i_X \otimes id_Y} X^{**} \check{\otimes} Y \xrightarrow{\Psi} C(K, Y)$$

where i is the canonical embedding, and Ψ is the natural inclusion. Then $\Psi \circ i_X \otimes id_Y \circ i$ is the canonical embedding from S into $C(K, Y)$. Since $C(K, Y)$ has l.u.st., there is a finite-dimensional Banach space Z with unconditional basis such that

$$S \xrightarrow{v} Z \xrightarrow{u} C(K, Y)$$

$$\Psi \circ i_X \otimes id_Y \circ i = u \circ v.$$

Moreover, $P \circ u(Z)$ is a finite dimensional subspace of $X^{**} \check{\otimes} Y$. Hence, from Lemma 13 we have that $P \circ u(Z)$ is a finite dimensional subspace of $(X \check{\otimes} Y)^{**}$.

By the Principle of local reflexivity there exists an injective operator $s : P \circ u(Z) \rightarrow X \check{\otimes} Y$ such that

$$s(e) = e \quad \forall e \in (P \circ u(Z)) \cap (X \check{\otimes} Y).$$

Then $\tilde{u} = s \circ P \circ u$ is such that

$$\begin{aligned}\tilde{u} : Z &\longrightarrow X \overset{\vee}{\otimes} Y, \\ v \circ \tilde{u} &= i\end{aligned}$$

and

$$\|v\|\|\tilde{u}\|ub(Z) \leq \|P\|\Lambda(C(K, Y)).$$

2. The proof is the same as that in 1, but we consider an $L_1(\mu, Y)$ -space instead of $C(K, Y)$ and Lemma 14 instead of Lemma 13. \square

THEOREM 16. *Let X and Y be Banach spaces. We have*

1. *If X is an \mathcal{L}_∞ -space, then $X \overset{\vee}{\otimes} Y$ has the GL property if Y does.*
2. *If X is an \mathcal{L}_1 -space, then $X \overset{\wedge}{\otimes} Y$ has the GL property if Y does.*

Proof. 1. Let

$$u : X \overset{\vee}{\otimes} Y \rightarrow l_2$$

be a 1-summing operator, then $u^{**} \in \Pi_1((X \overset{\vee}{\otimes} Y)^{**}, l_2)$. We consider $\tilde{u} = u^{**}|_{X^{**} \overset{\vee}{\otimes} Y}$ (we note that from Lemma 13 it follows that $X^{**} \overset{\vee}{\otimes} Y$ is a closed subspace of $(X \overset{\vee}{\otimes} Y)^{**}$ so $\tilde{u} \in \Pi_1(X^{**} \overset{\vee}{\otimes} Y, l_2)$). As in the proof above $X^{**} \overset{\vee}{\otimes} Y$ is complemented in $C(K, Y)$. Since $C(K, Y)$ has the GL property, we have

$$\tilde{u} \circ P \in \Gamma_1(C(K, Y), l_2).$$

Then $u = \tilde{u} \circ P|_{X \overset{\vee}{\otimes} Y}$ is in $\Gamma_1(X \overset{\vee}{\otimes} Y, l_2)$, with

$$\gamma_1(u) \leq \|P\|gl(C(K, Y))\pi_1(u).$$

2. The proof is the same as the proof of 1 but we use Lemma 14 instead of Lemma 13.

This completes the proof. \square

We end this note with two natural questions:

- (a) If X is a Banach space so that $X \overset{\wedge}{\otimes} Y$ has l.u.st. (GL-property) whenever Y does, is X an \mathcal{L}_1 -space?

- (b) If X is a Banach space so that $X \overset{\vee}{\otimes} Y$ has l.u.st. (GL-property) whenever Y does, is X an \mathcal{L}_∞ -space?

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