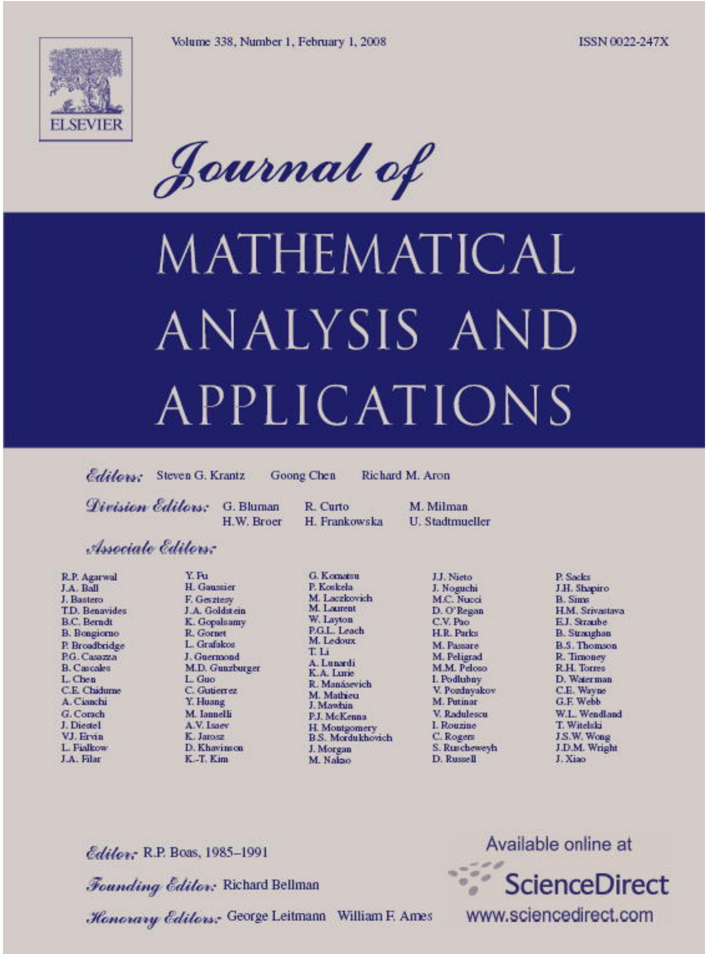
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# Bounded linear non-absolutely summing operators

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**Abstract**

We show that, in certain situations, we have lineability in the set of bounded linear and non-absolutely summing operators. Examples on lineability of the set $\Pi_p(E,F)\setminus I_p(E,F)$ are also presented and some open questions are proposed.


*Keywords:* $p$-Summing operators; $p$-Integral operators; Lineability

## 0. Preliminaries and background

Let $E$ be a Banach space. A subset $M$ of $E$ is said to be lineable [1,5,9] if there exists an infinite-dimensional vector space, $V\subset M\cup\{0\}$. Following [7, p. 55] we say that, given two Banach spaces $E$ and $F$, an operator $T\in\mathcal{L}(E,F)$ is *absolutely summing* if for each unconditionally convergent series $\sum_{i=1}^{\infty}x_i$ in $E$, the series $\sum_{i=1}^{\infty}T(x_i)$ is absolutely convergent in $F$. As usual, we denote by $\Pi(E,F)$ to the space of absolutely summing operators from $E$ to $F$ (also called 1-summing). In a more general definition, take any $1\leqslant p\leqslant\infty$ and $u\in\mathcal{L}(E,F)$. We say that $u$ is *p-summing* if there is a probability measure $\mu$ and bounded linear operators $a:L_p(\mu)\to\ell_\infty^{B_{F^*}}$ and $b:E\to L_\infty(\mu)$ giving rise to the commutative diagram

$$\begin{tikzcd} E \arrow[r, "u"] \arrow[d, "b"'] & F \arrow[r, "j_F"] & \ell_\infty^{B_{F^*}} \\ L_\infty(\mu) \arrow[rr, "i_p"'] & & L_p(\mu) \arrow[u, "a"'] \end{tikzcd}$$

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where $i_p : L_\infty(\mu) \to L_p(\mu)$ is the formal identity, and $j_F : F \to \ell_\infty^{B_{F^*}}$ is the canonical isometric embedding. We denote

$$\pi_p(u) = \inf\{\|a\| \cdot \|b\|\},$$

and this infimum is extended over all measures $\mu$ and operators $a, b$ as above. $\Pi_p(E, F)$ denotes the Banach space of all $p$-summing operators from $E$ to $F$, which is a linear subspace of $\mathcal{L}(E, F)$. Also, $\pi_p$ defines a norm in $\Pi_p(E, F)$ with $\|u\| \leqslant \pi_p(u)$ for every $u \in \Pi_p(E, F)$. In a similar manner, we shall say that a linear mapping $u : E \to F$ between Banach spaces is a $p$-integral operator ($1 \leqslant p \leqslant \infty$) if there is a probability measure $\mu$ and bounded linear operators $a : L_p(\mu) \to F^{**}$ and $b : E \to L_\infty(\mu)$ giving rise to the commutative diagram

$$\begin{tikzcd} E \arrow[r, "u"] \arrow[d, "b"'] & F \arrow[r, "k_F"] & F^{**} \\ L_\infty(\mu) \arrow[rr, "i_p"'] & & L_p(\mu) \arrow[u, "a"'] \end{tikzcd}$$

where $i_p : L_\infty(\mu) \to L_p(\mu)$ is the formal identity, and $k_F : F \to F^{**}$ is the canonical isometric embedding. The Banach space of all $p$-integral operators from $E$ to $F$ is denoted $I_p(E, F)$. Analogously, for each $u \in I_p(E, F)$ we associate its $p$-integral norm,

$$\iota_p(u) = \inf\{\|a\| \cdot \|b\|\},$$

and this infimum is extended over all measures $\mu$ and operators $a, b$ as above. The interested reader can refer to [4] for a complete study of these classes of operators.

**Definition 0.1.** A Banach space $E$ is said to have the "*two series property*" provided there exist unconditionally convergent series $\sum_{i=1}^{\infty} f_i$ in $E^*$ and $\sum_{i=1}^{\infty} x_i$ in $E$ such that

$$\sum_{i=1}^{\infty} \left[ \sum_{j=1}^{\infty} \frac{|f_j(x_i)|^2}{\|f_j\|} \right]^{\frac{1}{2}} = +\infty.$$

**Definition 0.2.** A Banach space $E$ is said to be "*sufficiently Euclidean*" if there exist a positive constant $C$ and sequences of operators $\{J_n\}_{n\in\mathbb{N}}$, $\{P_n\}_{n\in\mathbb{N}}$ such that

$$\ell_2^n \xrightarrow{J_n} E \xrightarrow{P_n} \ell_2^n$$

with $P_n \circ J_n = I_n$ (the identity operator on $\ell_2^n$) and $\|P_n\| = 1$, $1 \leqslant \|J_n\| \leqslant C$, for every $n \in \mathbb{N}$.

Stegall and Retherford [11] proved that every sufficiently Euclidean Banach space $E$ has the two series property, and every $\mathcal{L}_p$-space ($1 < p < \infty$) is sufficiently Euclidean.

This note is divided in 2 main sections. In Section 1 we show that the set $\mathcal{L}(E, F^*) \setminus \Pi_1(E, F^*)$ is lineable, where $E$ is any Banach space with the two series property. Section 2 shows that the set $\Pi_p(E, F) \setminus I_p(E, F)$ is also lineable for every $p \geqslant 1$ ($p \neq 2$). We also propose some problems and give some directions to study the lineability of certain sets of operators. Some results due to Dvoretzky and Pełczyński, some summability techniques, and some set theoretical considerations are used.

## 1. The set $\mathcal{L}(E, F^*) \setminus \Pi_1(E, F^*)$

We start this section by introducing a basic lemma about divergent series.

**Lemma 1.1.** *Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of positive real numbers. If $\sum_{n=1}^{\infty} a_n = +\infty$ then there exists $\{A_i\}_{i\in\mathbb{N}} \subseteq \mathbb{N}$ such that*:

(i) $|A_i| = \omega_0$ *for each* $i \in \mathbb{N}$, *where* $|A|$ *denotes the cardinality of* $A$, *and* $\omega_0 = |\mathbb{N}|$,
(ii) $A_i \cap A_j = \emptyset$ *if* $i \neq j$, *and*
(iii) $\sum_{m \in A_i} a_m = +\infty$ *for each* $i \in \mathbb{N}$.

**Proof.** Since $\sum_{n=1}^{\infty} a_n = +\infty$, and $a_n \geqslant 0$, $\forall n \in \mathbb{N}$, by Cauchy's criterion there exists $\varepsilon > 0$ such that

$$\forall \nu \in \mathbb{N} \quad \exists p \in \mathbb{N}\colon \quad a_\nu + \cdots + a_{\nu+p} \geqslant \varepsilon.$$

Then, fixed $\nu_1 \in \mathbb{N}$ there is $p_1 \in \mathbb{N}$ with $a_{\nu_1} + \cdots + a_{\nu_1+p_1} \geqslant \varepsilon$. Now, for each $k \in \mathbb{N}$, consider $\nu_k \in \mathbb{N}$ with $\nu_k \geqslant \nu_{k-1} + p_{k-1}$. Then there exists $p_k \in \mathbb{N}$ such that $a_{\nu_k} + \cdots + a_{\nu_k+p_k} \geqslant \varepsilon$. Next, let

$$B_1 = \bigcup_{k \in \mathbb{N}} \{\nu_{2k}, \ldots, \nu_{2k+p_{2k}}\}$$

and

$$A_1 = \bigcup_{k \in \mathbb{N}} \{\nu_{2k+1}, \ldots, \nu_{2k+1+p_{2k+1}}\}.$$

We now have: $B_1, A_1 \subseteq \mathbb{N}$ with $|B_1| = |A_1| = \omega_0$, $B_1 \cap A_1 = \emptyset$ and $\sum_{n \in B_1} a_n = \sum_{n \in A_1} a_n = +\infty$.

Working on $B_1$ and for the same reason as above we have two countably infinite disjoint subsets $B_2, A_2$ of $B_1$ such that $\sum_{n \in B_2} a_n, \sum_{n \in A_2} a_n = +\infty$. Following this inductive procedure we obtain the desired sequence, $\{A_i\}_{i \in \mathbb{N}}$. $\square$

**Lemma 1.2.** *Let* $E$ *be a Banach space enjoying the two series property. Let* $\sum_{i=1}^{\infty} f_i$ *in* $E^*$ *and* $\sum_{i=1}^{\infty} x_i$ *in* $E$ *such that* $\sum_{i=1}^{\infty} [\sum_{j=1}^{\infty} \frac{|f_j(x_i)|^2}{\|f_j\|}]^{\frac{1}{2}} = +\infty$. *Then, there exists* $\{A_n\}_{n \in \mathbb{N}} \subseteq \mathbb{N}$ *such that*:

(i) $|A_n| = \omega_0$ *for each* $n \in \mathbb{N}$,
(ii) $A_n \cap A_m = \emptyset$ *if* $n \neq m$, *and*
(iii) $\sum_{i=1}^{\infty} [\sum_{j \in A_n} \frac{|f_j(x_i)|^2}{\|f_j\|}]^{\frac{1}{2}} = +\infty$ *for each* $n \in \mathbb{N}$.

**Proof.** To simplify, let us denote by $a_{i,j} := \frac{|f_j(x_i)|}{\|f_j\|^{1/2}}$ for each $i, j \in \mathbb{N}$. Now, notice that one can assume that $a_i = (a_{i,j})_{j \in \mathbb{N}} \in \ell_2$ for each $i \in \mathbb{N}$ (otherwise the conclusion follows, directly, from the previous lemma). Thus, for some positive elements $r_i = (r_{i,j})_{j \in \mathbb{N}}$ of the unit sphere of $\ell_2$, we have

$$\sum_{i=1}^{\infty} \left[ \sum_{j=1}^{\infty} \frac{|f_j(x_i)|^2}{\|f_j\|} \right]^{\frac{1}{2}} = \sum_{i=1}^{\infty} \left[ \sum_{j=1}^{\infty} a_{i,j}^2 \right]^{\frac{1}{2}} = \sum_{i=1}^{\infty} \|a_i\|_{\ell_2} = \sum_{i=1}^{\infty} \langle a_i, r_i \rangle = \sum_{i=1}^{\infty} \left[ \sum_{j=1}^{\infty} a_{i,j} r_{i,j} \right] = \sum_{j=1}^{\infty} \left[ \sum_{i=1}^{\infty} a_{i,j} r_{i,j} \right].$$

Next, let

$$I := \left\{ j \in \mathbb{N}\colon \sum_{i=1}^{\infty} a_{i,j} r_{i,j} = +\infty \right\}.$$

We need to consider two possible cases:

(1) $I$ is infinite. In this case the assertion follows trivially.
(2) $I$ is finite. In this case we can suppose that $I = \emptyset$ since, considering permutations of index $\sigma(i)$ and $\rho(j)$, we can work with $\sum_{j=1}^{\infty} f_{\rho(j)}$ in $E^*$ and $\sum_{i=1}^{\infty} x_{\sigma(i)}$ in $E$ (still unconditionally convergent) in order to have $I$ empty. Then, in this case, we can apply the previous lemma to the sequence $s_j = \sum_{i=1}^{\infty} a_{i,j} r_{i,j}$ (note that $a_{i,j} r_{i,j} \geqslant 0$ for every $i, j \in \mathbb{N}$). Thus, there exists $\{A_k\}_{k \in \mathbb{N}} \subseteq \mathbb{N}$ such that:
   (i) $|A_k| = \omega_0$ for each $i \in \mathbb{N}$,
   (ii) $A_k \cap A_s = \emptyset$ for $k \neq s$, and
   (iii) $\sum_{j \in A_n} \sum_{i=1}^{\infty} a_{i,j} r_{i,j} = +\infty$ for each $n \in \mathbb{N}$.

Finally,

$$+\infty = \sum_{j\in A_n}\sum_{i=1}^{\infty} a_{i,j} r_{i,j} = \sum_{i\in\mathbb{N}}\sum_{j\in A_n} a_{i,j} r_{i,j} \leqslant \sum_{i=1}^{\infty} \big\|(a_{i,j})_{j\in A_n}\big\|_{\ell_2} \quad \forall n\in\mathbb{N},$$

and we are done. □

**Theorem 1.3.** *Let $E$ be a Banach space with the two series property. Then*

$$\mathcal{L}(E,\ell_2)\setminus \Pi_1(E,\ell_2)$$

*is lineable.*

**Proof.** By hypothesis, we know that there exist unconditionally convergent series $\sum_{i=1}^{\infty} f_i$ in $E^*$ and $\sum_{i=1}^{\infty} x_i$ in $E$ such that

$$\sum_{i=1}^{\infty}\left(\sum_{j=1}^{\infty}\frac{|f_j(x_i)|^2}{\|f_j\|}\right)^{\frac{1}{2}} = +\infty.$$

Consider the sequence $\{A_n\}_{n\in\mathbb{N}}$ as in the previous lemma.

For each $n\in\mathbb{N}$, let us define

$$T_n : E \longrightarrow \ell_2$$

by

$$T_n(x) = \sum_{k\in A_n}\frac{f_k(x)}{\|f_k\|^{\frac{1}{2}}} e_k.$$

Since $\sum_{j=1}^{\infty} f_j$ is unconditionally convergent in $E^*$, we have that $\sup_{\|x\|=1}\sum_{j=1}^{\infty}|f_j(x)|$ is finite, call it $C$. Moreover, for each $x\in B_X$ (the closed unit ball of $X$)

$$\big\|T_n(x)\big\|_{\ell_2} = \left\|\sum_{k\in A_n}\frac{f_k(x)}{\|f_k\|^{\frac{1}{2}}} e_k\right\|_{\ell_2} = \left(\sum_{k\in A_n}\frac{|f_k(x)|^2}{\|f_k\|}\right)^{\frac{1}{2}} \leqslant \left(\sum_{k\in\mathbb{N}}\frac{|f_k(x)|^2}{\|f_k\|}\right)^{\frac{1}{2}} \leqslant C^{\frac{1}{2}}.$$

This means that $T_n$ is well-defined and $T_n\in\mathcal{L}(E,\ell_2)$ for every $n\in\mathbb{N}$.

Also, for every $n\in\mathbb{N}$

$$\sum_{i=1}^{\infty}\big\|T_n(x_i)\big\| = \sum_{i=1}^{\infty}\left(\sum_{k\in A_n}\frac{|f_k(x_i)|^2}{\|f_k\|}\right)^{\frac{1}{2}} = +\infty$$

from the choice of the sequence $\{A_n\}_{n\in\mathbb{N}}$. Since $\sum_{i=1}^{\infty} x_i$ is unconditionally convergent in $E$, and by definition of absolutely summing operator, we have that $T_n\notin\Pi_1(E,\ell_2)$ for each $n\in\mathbb{N}$.

Now, from (ii) of the previous lemma, and from the definition, we also have that the sequence $\{T_n\}_{n\in\mathbb{N}}$ is linearly independent in $\mathcal{L}(E,\ell_2)$.

Now, we have to show that every bounded linear operator belonging to the linear span of $(T_n)_n$ is not absolutely summing. For simplicity, we consider the linear combination of two elements (the general case follows similarly). Thus, let $\lambda_1,\lambda_2\in\mathbb{K}$ and $n_1,n_2\in\mathbb{N}$, by definition

$$\lambda_1 T_{n_1}(x) + \lambda_2 T_{n_2}(x) = \sum_{k\in A_{n_1}}\frac{f_k(\lambda_1 x)}{\|f_k\|^{\frac{1}{2}}} e_k + \sum_{k\in A_{n_2}}\frac{f_k(\lambda_2 x)}{\|f_k\|^{\frac{1}{2}}} e_k.$$

We can also assume, without loss of generality, that $\lambda_1\neq 0$. Then

$$\sum_{i=1}^{\infty}\left\|\lambda_1 T_{n_1}\left(\frac{x_i}{\lambda_1}\right) + \lambda_2 T_{n_2}\left(\frac{x_i}{\lambda_1}\right)\right\|_{\ell_2} = \sum_{i=1}^{\infty}\left[\sum_{k\in A_{n_1}}\frac{|f_k(x_i)|^2}{\|f_k\|} + \sum_{k\in A_{n_2}}\frac{|f_k(\frac{\lambda_2}{\lambda_1}x_i)|^2}{\|f_k\|}\right]^{\frac{1}{2}} \geqslant \sum_{i=1}^{\infty}\left[\sum_{k\in A_{n_1}}\frac{|f_k(x_i)|^2}{\|f_k\|}\right]^{\frac{1}{2}}$$
$$= +\infty.$$

Since $\sum_{i=1}^{\infty} \frac{x_i}{\lambda_1}$ is an unconditional convergent series in $E$ we obtain that

$$\operatorname{span}\{T_n \colon n \in \mathbb{N}\} \subseteq \mathcal{L}(E, F) \setminus \Pi_1(E, F) \cup \{0\},$$

and we are done. □

Let us recall a couple of well-known results, that we will need later [2,8].

**Theorem 1.4** *(Dvoretzky). For each $\varepsilon > 0$ and each $n \in \mathbb{N}$, there exists $n(\varepsilon) \in \mathbb{N}$ such that if $E$ is a Banach space of dimension greater that $n(\varepsilon)$, then there exists a subspace $F$ of $E$ with $d(F, \ell_2^n) \leqslant 1 + \varepsilon$.*

**Proposition 1.5.** *Let $X, Y$ be any two Banach spaces, and $T : X \to Y$ a bounded linear operator. Then there exists a constant $C$ so that $T \in \Pi_1(X, Y)$ and $\pi_1(T) \leqslant 1$ if and only if for each finite subspace $X_0$ of $X$ the restriction $T|_{X_0} \in \Pi_1(X_0, Y)$ and $\pi(T|_{X_0}) \leqslant C$.*

Moreover, we know that, for any two Banach spaces $E, F$, we have $\mathcal{L}(E, F^*) \cong \mathcal{L}(F, E^*)$, and $\Pi_1(E, F^*) \cong \Pi_1(F, E^*)$ (here $\cong$ denotes an isometric isomorphism).

Thus, using the proposition above and Dvoretzky's result, if $T \in \mathcal{L}(E, \ell_2)$ but $T$ is not absolutely summing, then $T$ is not absolutely summing when seen as an operator in $\mathcal{L}(\ell_2, E^*)$. Again, by the above proposition there exists $n \in \mathbb{N}$ such that $T|_{\ell_2^n} \notin \Pi_1(\ell_2^n, E^*)$. Now, by Dvoretzky's result, there exists $F_0$ subspace of $F$ and $S_0 \colon F_0 \to \ell_2^n$ with $\|S_0\| \leqslant 1$, $\|S_0^{-1}\| \leqslant 1 + \epsilon$. Therefore $T \circ S_0 \notin \Pi_1(F_0, E^*)$ (otherwise, and by the ideal property, we would have $T = S_0^{-1} S_0 T \in \Pi_1(\ell_2^n, E^*)$). Finally, and using an argument about the projections, we can find $\widetilde{T} \in \mathcal{L}(F, E^*)$ and non-absolutely summing. Following in this way, given two Banach spaces $E, F$, with $E$ enjoying the two series property, we can construct a sequence $\widetilde{T}_n \in \mathcal{L}(E, F^*)$ non-absolutely summing from the sequence $T_n \in \mathcal{L}(E, \ell_2)$ made in Theorem 1.3. Finally, notice that $\widetilde{T}_n$ preserves the linear independence (because in our case $(T_n)_n \subseteq \mathcal{L}(\ell_2, E^*)$ have disjoint support, and to construct the sequence $(\widetilde{T}_n)_n$ we used bijections and projections).

As a consequence we have

**Corollary 1.6.** *Let $E, F$ be two Banach spaces, with $E$ having the two series property. Then*

$$\mathcal{L}\big(E, F^*\big) \setminus \Pi_1\big(E, F^*\big)$$

*is lineable.*

## 2. The set $\Pi_p(E, F) \setminus I_p(E, F)$: An example

We will need the following lemma in this section. Its proof is well known.

**Lemma 2.1.** *There exists a family $\mathcal{F}$ of subsets of $\mathbb{N}$ with:*

(1) *$|\mathcal{F}| = c$, where $c = |\mathbb{R}|$,*
(2) *for every $\Lambda \in \mathcal{F}$, $|\Lambda| = \omega_0$, and*
(3) *if $\Lambda \neq \Lambda' \in \mathcal{F}$, then $\Lambda \cap \Lambda'$ is finite.*

By means of the previous lemma we will construct a basis of an infinite-dimensional vector space every non-zero element of which belongs to $\Pi_p(E, F) \setminus I_p(E, F)$, $p \geqslant 1$, $p \neq 2$. We start by letting $\mathcal{F} = \{\Lambda_\alpha \colon \alpha \in I\}$ be a family as in the previous lemma. Let us also fix $N \in \mathbb{N}$ and consider, for every $\alpha \in I$, $\mathcal{U}_\alpha \subset \Lambda_\alpha$ such that $|\mathcal{U}_\alpha| = N$. It is well known that in $\mathcal{C}(\mathcal{U}_\alpha)$ the elements

$$e_n(k) = e^{\frac{2\pi k n}{N}}$$

are a basis of $\mathcal{C}(\mathcal{U}_\alpha)$ for $1 \leqslant n \leqslant N$. Then, each $f \in \mathcal{C}(\mathcal{U}_\alpha)$ can be written, uniquely, as

$$f = \sum_n \hat{f}(n) e_n.$$

Next, consider the measure $\mu$, on $\mathcal{U}_\alpha$, given by $\mu(\{k\}) = \frac{1}{N}$ for every $k \in \mathcal{U}_\alpha$. Now, for each $A_\alpha \subset \mathcal{U}_\alpha$ let us define the following subspaces:

$$\mathcal{C}_{A_\alpha} = \left\{ f \in \mathcal{C}(\mathcal{U}_\alpha)\colon\ f = \sum_{n \in A_\alpha} \hat{f}(n) e_n \right\},$$

and

$$L_{p,A_\alpha} = \left\{ f \in L_p(\mu)\colon\ f = \sum_{n \in A_\alpha} \hat{f}(n) e_n \right\}.$$

Also, denote by $u_{A_\alpha}$ the natural inclusion $\mathcal{C}_{A_\alpha} \hookrightarrow L_{p,A_\alpha}$. Clearly, $\pi_p(u_{A_\alpha}) \leqslant 1$. Now, we want to estimate the value of $\iota_p(u_{A_\alpha})$. We will follow a similar construction as in [4, pp. 103–104], sketching only the parts we need to change. In order to do that, consider the $p$-integral factorization (see [4]) given by

$$\mathcal{C}_{A_\alpha} \xrightarrow{i_\alpha} \mathcal{C}(\mathcal{U}_\alpha) \xrightarrow{j_{p,\alpha}} L_p(\nu) \xrightarrow{\tilde{u}_\alpha} L_{p,A_\alpha}$$

where $\nu$ is a probability measure on $\mathcal{U}_\alpha$. Since $|\varLambda_\alpha| = \omega_0$, there is a bijection $I_\alpha : \varLambda_\alpha \leftrightarrow \mathbb{N}$. Without loss of generality we can assume that $I_\alpha(\mathcal{U}_\alpha) = \{1, 2, \ldots, N\}$. For every $f \in \mathcal{C}(\mathcal{U}_\alpha)$, $k \in \mathbb{Z}$, define

$$f_k(n) = f\bigl(I_\alpha^{-1}\bigl(I_\alpha(n) + I_\alpha(k)\bigr)\bigr), \quad n \in \mathcal{U}_\alpha, \tag{1}$$

where $I_\alpha(n) + I_\alpha(k)$ is taken modulo $N$. Let us now define the following measure $\nu_k$, given by

$$\langle \nu_k, f \rangle = \langle \nu, f_k \rangle.$$

It is easy to see that $\mu = \frac{1}{N} \cdot \sum_{k=1}^N \nu_k$. Next, consider $w_\alpha : \mathcal{C}(\mathcal{U}_\alpha) \to L_{p,A_\alpha}$ defined as

$$w_\alpha(f) = \frac{1}{N} \cdot \sum_{k=1}^N (\tilde{u}_\alpha j_{p,\alpha} f_k)_{-k}. \tag{2}$$

Now, using the same argument as the ones from [4], one can obtain that

$$\sup_{A_\alpha} \iota_p(u_{A_\alpha}) \geqslant K_p \cdot N^{|\frac{1}{p} - \frac{1}{2}|}.$$

Defining now the spaces $X = (\bigoplus_N \mathcal{C}_{A_\alpha})_2$ and $Y = (\bigoplus_N L_{p,A_\alpha})_2$, and denoting

$$\tilde{u}_N^\alpha : X \xrightarrow{P_N} \mathcal{C}_{A_\alpha} \xrightarrow{u_{A_\alpha}} L_p(A_\alpha) \xrightarrow{J_N} Y,$$

we obtain that $\pi_p(\tilde{u}_N^\alpha) \leqslant 1$ for all $N \in \mathbb{N}$, and $\iota_p(\tilde{u}_N^\alpha) \xrightarrow{N \to \infty} \infty$. From the fact that $\varPi_p(X, Y)$ is a dual space (see, e.g. [10]) and using Cantor's diagonalization process, we can find $(n_k)_k \subset \mathbb{N}$ such that

$$\tilde{u}^\alpha = \omega^* - \lim_k \tilde{u}_{n_k}^\alpha$$

is well-defined. Thus, we obtain that $\tilde{u}^\alpha$ is $p$-summing but not $p$-integral. Also, and since the intersection of the supports of any finite number of these operators $\tilde{u}^\alpha$ is finite, we obtain that the family $\{\tilde{u}^\alpha\colon\ \alpha \in I\}$ is linearly independent. Now, it is not difficult to check that the family $\{\tilde{u}^\alpha\colon\ \alpha \in I\}$ spans a vector space in $(\varPi_p(X, Y) \setminus I_p(X, Y)) \cup \{0\}$. Indeed, let $\lambda_1, \lambda_2 \in \mathbb{K}$ and $\alpha_1, \alpha_2 \in I$ (as we did earlier, we will consider the linear combination of just two elements, but, as the reader can check, the general case follows similarly). By the above construction,

$$\lambda_1 \tilde{u}_N^{\alpha_1} + \lambda_2 \tilde{u}_N^{\alpha_2} = J_N \circ (\lambda_1 u_{A_{\alpha_1}} + \lambda_2 u_{A_{\alpha_2}}) \circ P_N$$

where $P_N$ and $J_N$ are the natural projection and embedding, respectively, and $\lambda_1 u_{A_{\alpha_1}} + \lambda_2 u_{A_{\alpha_2}} : C_{A_{\alpha_1} \times A_{\alpha_2}} \to L_p(A_{\alpha_1} \times A_{\alpha_2})$ is "almost" the natural map; the reader can see that we also obtain that

$$\sup_{A_{\alpha_1} \times A_{\alpha_2}} \{\lambda_1 u_{A_{\alpha_1}} + \lambda_2 u_{A_{\alpha_2}}\} \geqslant K_p\bigl(|\lambda_1|, |\lambda_2|\bigr) N^{|\frac{1}{p} - \frac{1}{2}|}$$

where the constant depends on $|\lambda_1|$ and $|\lambda_2|$. Thus, $\lambda_1 \tilde{u}^{\alpha_1} + \lambda_2 \tilde{u}^{\alpha_2}$ is not $p$-integral. We can summarize the previous construction as follows:

**Example 2.2.** Let $X$ and $Y$ be the Banach spaces constructed above. Then $\Pi_p(X,Y) \setminus I_p(X,Y)$ is lineable for every $1 \leqslant p < \infty$, $p \neq 2$. Moreover, there exists a vector space $V \subset (\Pi_p(X,Y) \setminus I_p(X,Y)) \cup \{0\}$ such that $\dim(V) = c$.

The following remark is also worth mentioning.

**Remark 2.3.** In the previous example we need $p \neq 2$ since $\Pi_2(X,Y) = I_2(X,Y)$ for every pair of Banach spaces $X, Y$ (see [4, Corollary 5.9]).

We finish by proposing some problems related to the lineability of certain subsets of operators in Banach spaces.

**Question 2.4.** *In* [3] *Davis and Johnson proved that there exists a bounded linear operator that is not absolutely summing under certain conditions, namely the set $\mathcal{L}(E,F) \setminus \Pi_p(E,F)$ is non-empty whenever $E$ is superreflexive and $F$ is any Banach space. One could ask whether this set is lineable. We believe that this is the case.*

**Question 2.5.** *In* [6] *Figiel and Johnson proved that, if a Banach space $X$ has the approximation property but fails the bounded approximation property, and $X^*$ is separable, then there exists a non-nuclear operator $T$ on $X$ such that $T^*$ is nuclear. Therefore, is the set of non-nuclear operators* (*and with nuclear adjoint*) *lineable?*

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## References

[1] R.M. Aron, V.I. Gurariy, J.B. Seoane-Sepúlveda, Lineability and spaceability of sets of functions on $\mathbb{R}$, Proc. Amer. Math. Soc. 133 (2005) 795–803.
[2] A. Dvoretzky, Some results on convex bodies and Banach spaces, in: Proc. Internat. Sympos. Linear Spaces, Jerusalem, 1960, Jerusalem Academic Press, 1960, pp. 123–160.
[3] W.J. Davis, W.B. Johnson, Compact, non-nuclear operators, Studia Math. 51 (1974) 81–85.
[4] J. Diestel, H. Jarchow, A. Tonge, Absolutely Summing Operators, Cambridge University Press, 1995.
[5] P. Enflo, V.I. Gurariy, On lineability and spaceability of sets in function spaces, preprint.
[6] T. Figiel, W.B. Johnson, The approximation property does not imply the bounded approximation property, Proc. Amer. Math. Soc. 41 (1973) 197–200.
[7] A. Grothendieck, Produits tensoriels topologiques et espaces nucléaries, Mem. Amer. Math. Soc. 16 (1955).
[8] R.C. James, Superreflexive spaces, Canad. J. Math. 24 (1972) 896–904.
[9] F.J. García-Pacheco, N. Palmberg, J.B. Seoane-Sepúlveda, Lineability and algebrability of pathological phenomena in analysis, J. Math. Anal. Appl. 326 (2007) 929–939.
[10] R. Ryan, Introduction to Tensor Products of Banach Spaces, Springer Monogr. Math., Springer-Verlag London, Ltd., London, 2002.
[11] C.P. Stegall, J.R. Retherford, Fully nuclear and completely nuclear operators with applications to $\mathcal{L}_1$- and $\mathcal{L}_\infty$-spaces, Trans. Amer. Math. Soc. 163 (1972) 457–492.